GATE 2018

Section: General Aptitude

Q1. – Q5. carry one mark each.

Q1. "When she fell down the __________, she received many ________ but little help".

The words that best fill the blanks in the above sentence are

(a) stairs, stares (b) stairs, stairs

(c) stares, stairs (d) stares, stares

Ans. : (a)

Solution: stairs means steps while stares means to look someone continuously.

Q2. "In spite of being warned repeatedly, he failed to correct his___________ behaviour"

The word that best fills the blank in the above sentence is

(a) rational (b) reasonable (c) errant (d) good

Ans. : (c)

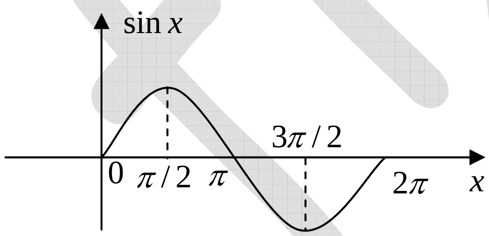
Solution: The most suitable option is errant as errant means irregular.

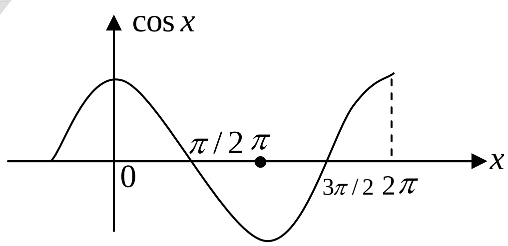
Q3. For $0 \le x \le 2\pi$, $\sin x$ and $\cos x$ are both decreasing functions in the interval________

(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \pi\right)$ (c) $\left(\pi, \frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2}, 2\pi\right)$

Ans. : (b)

Solution: Graph of $\sin x$ and $\cos x$ is shown in the figure below





From the graph we see that $\sin x$ and $\cos x$ are both decreasing function in the interval $\left(\frac{\pi}{2}, \pi\right)$

Q4. The area of an equivalent triangle is $\sqrt{3}$. What is the perimeter of the triangle?

(a) 2 (b) 4 (c) 6 (d) 8

Ans. : (c)

Solution: Let the side of equilateral triangle $=a$, then the area $=\frac{\sqrt{3}}{4}a^2$

or $\frac{\sqrt{3}}{4}a^2=\sqrt{3}$ or $a^2=4$ or $a=2$

Hence, the perimeter of the equilateral triangle $=3a=3\times 2=6$

Q5. Arrange the following three-dimensional objects in the descending order of their volumes:

(i) A cuboid with dimensions $10\,cm$, $8\,cm$ and $6\,cm$

(ii) A cube of side $8\,cm$

(iii) A cylinder with base radius $7\,cm$ and height $7\,cm$

(iv) A sphere of radius $7\,cm$

(a) (i), (ii), (iii), (iv)
(b) (ii), (i), (iv), (iii)
(c) (iii), (ii), (i), (iv)
(d) (iv), (iii), (ii), (i)

Ans. : (d)

Solution: The value of cuboid $=10cm\times 8cm\times 6cm=480cm^3$

The volume of cube $=8cm\times 8cm\times 8cm=512cm^3$

The volume of cylinder $=\pi r^2h=\frac{22}{7}\times 7\times 7\times 7cm^3=1078cm^3$

The value of sphere $=\frac{4}{3}\pi r^3=\frac{4}{3}\times\frac{22}{7}\times 7\times 7\times 7=1437.3cm^3$

Hence the descending orders of volume will be $1437.3cm^3$, $1078cm^3, 512cm^3$ and $480cm^3$

Q6. – Q10. carry two marks each.

Q6. An automobile travels from city A to city B and returns to city A by the same route. The speed of the vehicle during the onward and return journeys were constant at $60\,km/h$ and $90\,km/h$, respectively. What is the average speed in km/h for the entire journey?

(a) 72
(b) 73
(c) 74
(d) 75

Ans. : (a)

Solution: Let the distance between A and B is $x km$. Then

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}} = \frac{2x km}{\left(\frac{x}{60}+\frac{x}{90}\right)\text{ hour}} = 2x \times \frac{360}{10x} km/h = 72 km/h$$

Q7. A set of 4 parallel lines intersect with another set of 5 parallel lines. How many parallelograms are formed?

(a) 20 (b) 48 (c) 60 (d) 72

Ans. (c)

Solution: Any two lines in one direction and any two parallel line in the other direction can form a parallelograms.

So, number of parallelogram formed

$$= 5C_2 \times 4C_2 = \frac{5!}{2!3!} \times \frac{4!}{2!2!} = \frac{4\times 5}{2} \times \frac{3\times 4}{2} = 10 \times 6 = 60$$

Q8. To pass a test, a candidate needs to answer at least 2 out of 3 questions correctly. A total of 6,30,000 candidates appeared for the test. Question A was correctly answered by 3,30,000 candidates. Question B was answered correctly by 2,50,000 candidates. Question C was answered correctly by 2,60,000 candidates. Both questions A and B were answered correctly by 1,00,000 candidates. Both questions B and C were answered correctly by 90,000 candidates. Both questions were A and C were answered correctly by 80,000 candidates. If the number of students answering all questions correctly is the same as the number answering none, how many candidates failed to clear the test?

(a) 30,000 (b) 2,70,000 (c) 3,90,000 (d) 4,20,000

Ans. (d)

Solution: Let $n(0)$ denote the number of students answering none of the questions and $n(3)$ be the number of students answering all questions, then

$$n(A\cup B\cup C) - n(0) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(3)$$

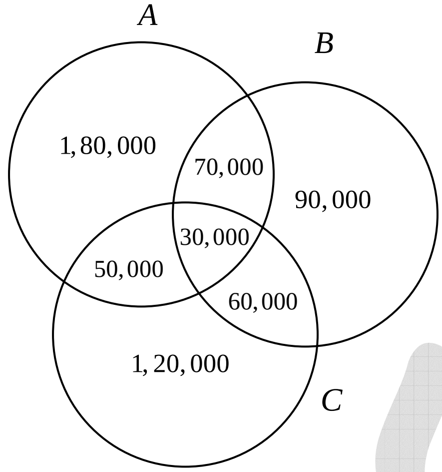
$$6,30,000 - n(0) = 3,30,000 + 2,50,000 + 2,60,000 - 1,00,000 - 90,000 - 80,000 + n(3)$$

$$\Rightarrow 6,30,000 - n(0) = 5,70,000 + n(3)$$

Since, $n(0) = n(3)$

Hence, $2n(0) = 60,000 \Rightarrow n(0) = 30,000$

Using this fact and the information given, one fill the Venn-diagram.



It is obvious that the number of failed students will be, the sum of number the students who only passed in one subject and the number of student answering none of the question. Hence, the number of students failed to clear the test

$$=1,80,000+1,20,000+90,000+30,000=4,20,000$$

Q9. If $x^2+x-1=0$, what is the value of $x^4+\frac{1}{x^4}$?

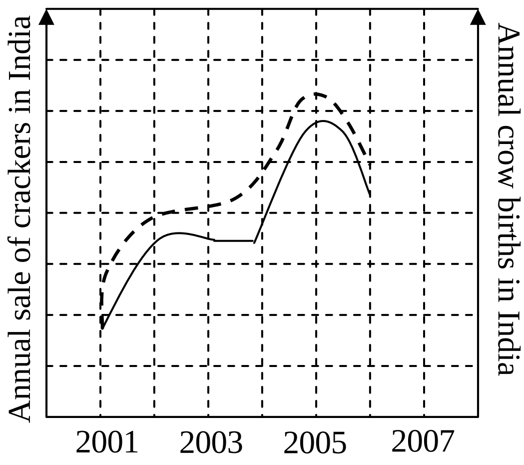
(a) 1 (b) 5 (c) 7 (d) 9

Ans. : (c)

Solution: Given that $x^2+x-1=0 \Rightarrow x(1+x)=1 \Rightarrow 1+x=\frac{1}{x} \Rightarrow x-\frac{1}{x}=-1,$

$$x^2+\frac{1}{x^2}=3 \Rightarrow x^4+\frac{1}{x^4}=9-2=7$$

Q10. In a detailed study of annual crow births in India, it was found that there was relatively no growth during the period 2002 to 2004 and a sudden spike from 2004 to 2005. In another unrelated study, it was found that the revenue from cracker sales in India which remained fairly flat from 2002 to 2004, saw a sudden spike in 2005 before declining again in 2006. The solid line in



the graph below refers to annual sale of crackers and the dashed line refers to the annual crow births in India. Choose the most appropriate inference from the above data.

(a) There is a strong correlation between crow birth and cracker sales

(b) Cracker usage increases crow birth rate

(c) If cracker sale declines, crow birth will decline

(d) Increased birth rate of crows will cause an increase in the sale of crackers

Ans.: (a)

Solution: The growth pattern of crows and the growth in annual sales of fire crackers in nearly the same. The two graphs are almost parallel to each other. Hence there is strong correlation between crow birth and crackers sales.

Section: Physics

Q1. – Q25. carry one mark each.

Q1. The eigenvalues of a Hermitian matrix are all

(a) real (b) imaginary

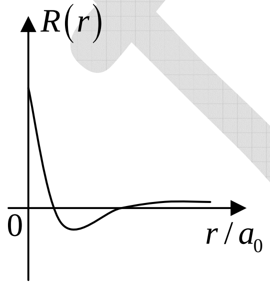
(c) of modulus one (d) real and positive

Ans. : (a)

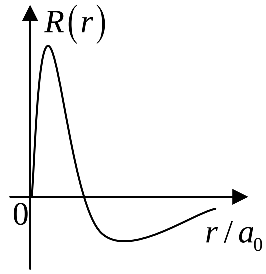
Solution: Eigenvalue of Hermitian matrix must be real.

Q2. Which one of the following represents the $3p$ radial wave function of hydrogen atom? (a_0 is the Bohr radius)

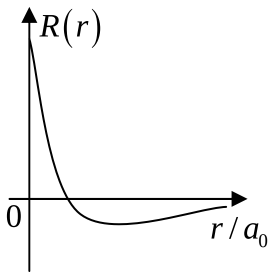
(a)



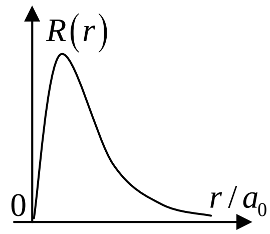
(b)



(c)



(d)



Ans. : (b)

Solution: $3p$ radial wave function is $R_{31}\alpha\ r\left(1-\frac{r}{6a_0}\right)e^{-\frac{r}{3a_0}}$

Q3. Given the following table,

Group I	Group II
P: Stern-Gerlach experiment	1: Wave nature of particles
Q: Zeeman effect	2: Quantization of energy of electrons in the atoms
R: Frank-Hertz experiment	3: Existence of electron spin
S: Davisson-Germer experiment	4: Space quantization of angular momentum

Which one of the following correctly matches the experiments from Group I to their inferences in Group II?

(a) P-2, Q-3, R-4, S-1 (b) P-1, Q-3, R-2, S-4

(c) P-3, Q-4, R-2, S-1 (d) P-2, Q-1, R-4, S-3

Ans. : (c)

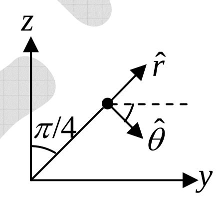
Q4. In spherical polar coordinates (r,θ,ϕ), the unit vector $\hat{\theta}$ at $(10,\pi/4,\ \pi/2)$ is

(a) $\hat{k}$ (b) $\frac{1}{\sqrt{2}}\left(\hat{j}+\hat{k}\right)$ (c) $\frac{1}{\sqrt{2}}\left(-\hat{j}+\hat{k}\right)$ (d) $\frac{1}{\sqrt{2}}\left(\hat{j}-\hat{k}\right)$

Ans. : (d)

Solution: $\hat{\theta}=\cos 45^0\,\hat{j}-\sin 45^0\,\hat{k}$

$\Rightarrow \hat{\theta}=\frac{1}{\sqrt{2}}\left(\hat{j}-\hat{k}\right)$



Q5. The scale factors corresponding to the covariant metric tensor g_{ij} in spherical polar coordinates are

(a) $1, r^2, r^2\sin^2\theta$ (b) $1, r^2, \sin^2\theta$ (c) 1,1,1 (d) $1, r, r\sin\theta$

Ans. : (d)

Q6. In the context of small oscillations, which one of the following does NOT apply to the normal coordinates?

(a) Each normal coordinate has an eigen-frequency associated with it

(b) The normal coordinates are orthogonal to one another

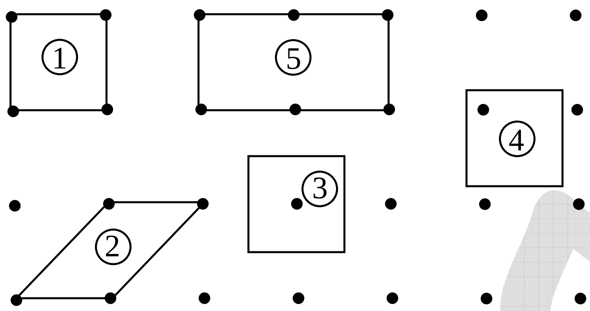
(c) The normal coordinates are all independent

(d) The potential energy of the system is a sum of squares of the normal coordinates with constant coefficients

Ans. : (b)

Solution: Normal co-ordinate must be independent. It is not necessary that it should orthogonal.

Q7. For the given unit cells of a two dimensional square lattice, which option lists all the primitive cells?



(a) (1) and (2)

(b) (1), (2) and (3)

(c) (1), (2), (3) and (4)

(d) (1), (2), (3), (4) and (5)

Ans. : (c)

Solution: For primitive cell, $N_{eff} = 1$

In cell (1), (2), (3) and (4) $N_{eff} = 1$, these are primitive cell

Whereas in cell (5), $N_{eff} = 2$, this is non-primitive cell.

Q8. Among electric field ($\vec{E}$), magnetic field ($\vec{B}$), angular momentum ($\vec{L}$) and vector potential ($\vec{A}$), which is/are **odd** under parity (space inversion) operation?

(a) $\vec{E}$ only

(b) $\vec{E}$ and $\vec{A}$ only

(c) $\vec{E}$ and $\vec{B}$ only

(d) $\vec{B}$ and $\vec{L}$ only

Ans. : (b)

Solution: Under parity operation $r \to -r$

$$E = -\frac{\partial V}{\partial r} \quad ; \quad E : P \to -E$$

$$B = \vec{I} \times \vec{r} \quad ; \quad B : P \to +B$$

$$L = \vec{r} \times \vec{p} \quad ; \quad L : P \to +L$$

$$E = -\frac{\partial A}{\partial t} \quad ; \quad A : P \to -A$$

Q9. The expression for the second overtone frequency in the vibrational absorption spectra of a diatomic molecule in terms of the harmonic frequency ω_e and anharmonicity constant x_e is

(a) $2\omega_e(1-x_e)$ (b) $2\omega_e(1-3x_e)$ (c) $3\omega_e(1-2x_e)$ (d) $3\omega_e(1-4x_e)$

Ans. : (d)

Solution: $\varepsilon_V = \omega_e\left(v+\frac{1}{2}\right)-\omega_e x_e\left(v+\frac{1}{2}\right)^2$

Second overtone $v=0 \rightarrow v=3$

$$\therefore\ \bar{v} = \varepsilon_{v=3}-\varepsilon_{v=0} = \frac{7}{2}\omega_e - \omega_e x_e\left(\frac{7}{2}\right)^2 - \frac{\omega_e}{2} + \omega_e x_e\left(\frac{1}{2}\right)^2 = 3\omega_e - 12\omega_e x_e = 3w_e(1-4x_e)$$

Q10. Match the physical effects and order of magnitude of their energy scales given below, where $\alpha = \frac{e^2}{4\pi\in_0 \hbar c}$ is fine structure constant; m_e and m_p are electron and proton mass, respectively.

Group I	Group II
P: Lamb shift	1: $\sim O(\alpha^2 m_e c^2)$
Q: Fine structure	2: $\sim O(\alpha^4 m_e c^2)$
R: Bohr energy	3: $\sim O(\alpha^4 m_e^2 c^2 / m_p)$
S: Hyperfine structure	4: $\sim O(\alpha^5 m_e c^2)$

(a) P-3, Q-1, R-2, S-4 (b) P-2, Q-3, R-1, S-4

(c) P-4, Q-2, R-1, S-3 (d) P-2, Q-4, R-1, S-3

Ans. : (c)

Solution:- Bohr energy $\Delta E \propto \alpha^2 m_e c^2$

Fine structure $\Delta E \propto \alpha^4 m_e c^2$

Lamb straight $\Delta E \propto \alpha^5 m_e c^2$

Hyperfine structure $\Delta E \propto \frac{\alpha^4 m_e c^2}{m_p}$

Q11. The logic expression $\bar{A}BC+\bar{A}\bar{B}C+AB\bar{C}+A\bar{B}\bar{C}$ can be simplified to

(a) A XOR C (b) A AND C (c) 0 (d) 1

Ans. : (a)

Solution: $Y=\bar{A}BC+\bar{A}\bar{B}C+AB\bar{C}+A\bar{B}\bar{C}=\bar{A}C\left(B+\bar{B}\right)+A\bar{C}\left(B+\bar{B}\right)$

$\Rightarrow Y=\bar{A}C+A\bar{C}=A$ XOR C

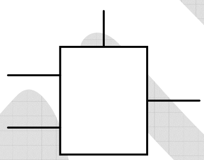
Q12. At low temperatures (T), the specific heat of common metals is described by (with α and β as constants)

(a) $\alpha T+\beta T^3$ (b) βT^3 (c) $\exp\left(-\alpha/T\right)$ (d) $\alpha T+\beta T^5$

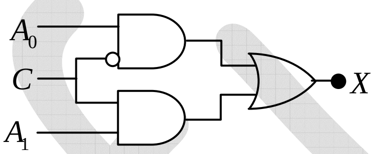
Ans. : (a)

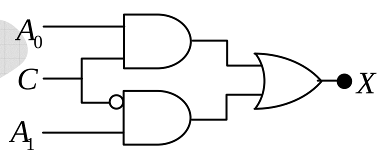
Solution: $C=C_e+C_{pn}=\alpha T+\beta T^3$

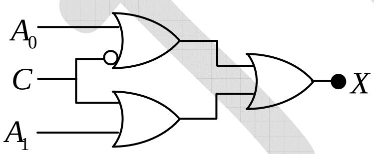
Q13. In a 2-to-1 multiplexer as shown below, the output $X=A_0$ if $C=0$ and $X=A_1$ if $C=1$.

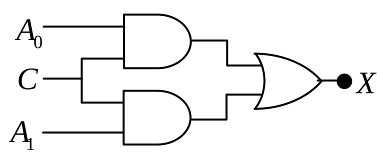


Which one of the following is the correct implementation of this multiplexer?

(a)


(b)


(c)


(d)


Ans. : (a)

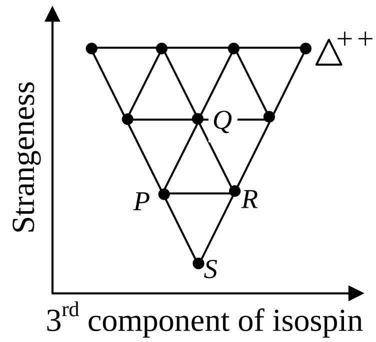
Solution: Check option (a),

$X=A_0\bar{C}+A_1C$

If $C=0\Rightarrow X=A_0$

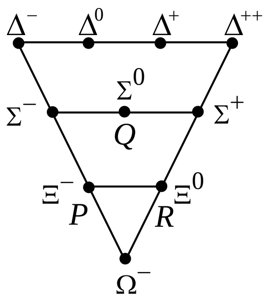
If $C=1\Rightarrow X=A_1$

Q14. The elementary particle Ξ^0 is placed in the baryon decuplet, shown below, at



(a) P (b) Q (c) R (d) S

Ans. : (c)



Q15. The intrinsic/permanent electric dipole moment in the ground state of hydrogen atom is (a_0 is the Bohr radius)

(a) $-3ea_0$ (b) zero (c) ea_0 (d) $3ea_0$

Ans. : (b)

Solution: For dipole moment energy is $-eEr\cos\theta$

$E_1^1 = \langle -eEr\cos\theta \rangle = eE\langle r \rangle \langle \cos\theta \rangle = 0$ $\quad [\because \langle \cos\theta \rangle = 0]$

Q16. The high temperature magnetic susceptibility of solids having ions with magnetic moments can be described by $\chi \propto \frac{1}{T+\theta}$ with T as absolute temperature and θ as constant. The three behaviours i.e., paramagnetic, ferromagnetic and anti-ferromagnetic are described, respectively, by

(a) $\theta < 0,\ \theta > 0,\ \theta = 0$ (b) $\theta > 0,\ \theta < 0,\ \theta = 0$

(c) $\theta = 0,\ \theta < 0,\ \theta > 0$ (d) $\theta = 0,\ \theta > 0,\ \theta < 0$

Ans. : (c)

Solution: Paramagnetism: $\chi = \frac{C}{T}$

Ferromagnetism: $\chi = \frac{C}{T - T_C}$

Anti-ferromagnetism: $\chi = \frac{C}{T + T_C}$

Q17. Which one of the following is an allowed electric dipole transition?

(a) ${}^1S_0 \rightarrow {}^3S_1$ (b) ${}^2P_{3/2} \rightarrow {}^2D_{5/2}$ (c) ${}^2D_{5/2} \rightarrow {}^2P_{1/2}$ (d) ${}^3P_0 \rightarrow {}^5D_0$

Ans. : (b)

Solution: For electric dipole transition

$\Delta L = 0, \pm 1 \quad (0 \not\rightarrow 0),\ \Delta J = 0.\pm 1,\ \Delta S = 0$

Only option (b) satisfies above selection rules

Q18. In the decay, $\mu^+ \rightarrow e^+ + \nu_e + X$, what is X?

(a) γ (b) $\bar{\nu}_e$ (c) ν_μ (d) $\bar{\nu}_\mu$

Ans. : (d)

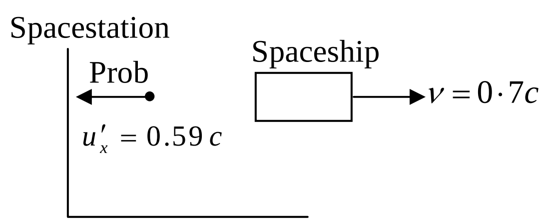
Solution:- $u^+ \rightarrow e^+ + \nu_e + \bar{\nu}_u$

L_u : -1 0 0 -1

L_e : 0 -1 $+1$ 0

Q19. A spaceship is travelling with a velocity of $0.7c$ away from a space station. The spaceship ejects a probe with a velocity $0.59c$ opposite to its own velocity. A person in the space station would see the probe moving at a speed Xc, where the value of X is ____________ (up to three decimal places).

Ans.: $0.187c$

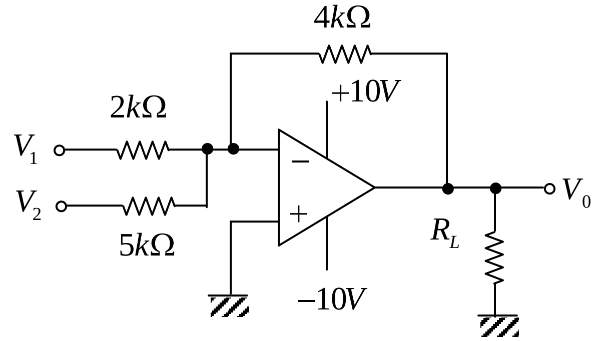


Solution: $v = 0 \cdot 7c$, $u_x' = -0 \cdot 59c$,

$$u_x = \frac{u_x' + v}{1 + \frac{u_x' v}{c_2}}$$

$$u_x = \frac{-0 \cdot 59c + 0 \cdot 7c}{1 - 0 \cdot 7 \times 0 \cdot 59} = \frac{0.11c}{1 - 0.413} = \frac{0.11c}{0.587} = 0.187\,c$$

Q20. For an operational amplifier (ideal) circuit shown below,



If $V_1 = 1V$ and $V_2 = 2V$, the value of V_0 is __________ V (up to one decimal place).

Ans. : –3.6

Solution: $V_0 = V_{01} + V_{02} = -\frac{4}{2}\times 1V - \frac{4}{5}\times 2V$

$V_0 = -2 - 1.6 = -3.6V$

Q21. An infinitely long straight wire is carrying a steady current I. The ratio of magnetic energy density at distance r_1 to that at $r_2 (= 2r_1)$ from the wire is ___________.

Ans. : 4

Solution: $u_B = \frac{B^2}{2\mu_0} \propto \frac{1}{r^2} \Rightarrow \frac{u_{B1}}{u_{B2}} = \frac{r_2^2}{r_1^2} = \frac{(2r_1)}{r_1^2} = 4$

Q22. A light beam of intensity I_0 is falling normally on a surface. The surface absorbs 20% of the intensity and the rest is reflected. The radiation pressure on the surface is given by $X\, I_0 / c$, where X is __________ (up to one decimal place). Here c is the speed of light.

Ans. : 1.8

Solution: Radiation pressure $= \frac{I_0}{c} - \left(-0.8\frac{I_0}{c}\right) = 1.8\frac{I_0}{c}$

Q23. The number of independent components of a general electromagnetic field tensor is__________

Ans. : 6

Solution: In Cartesian co-ordinate, three Independent coordinate for electric field, (E_x, E_y, E_z) and three Independent co-ordinate for magnetic field (B_x, B_y, B_z).

Q24. If X is the dimensionality of a free electron gas, the energy (E) dependence of density of states is given by $E^{\frac{1}{2}X-Y}$, where Y is_________.

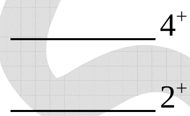
Ans. : 1

Solution: $\rho \propto E^{\left(\frac{d}{2}-1\right)}$

Q25. For nucleus ^{164}Er, a $J^{\pi}=2^{+}$ state is at 90 keV. Assuming ^{164}Er to be a rigid rotor, the energy of its 4^{+} state is ___________ keV (up to one decimal place)

Ans. : 300

Solution: $E_J = hcBJ\left(J+1\right)$



$E_{2^+}=hc\ B\ 2\left(2+1\right)$ and $E_{4^+}=hc\ B\ 4\left(4+1\right)$

Then, $\dfrac{E_{4^+}}{E_{2^+}}=\dfrac{20}{6} \Rightarrow \ E_{4^+}=\dfrac{20}{6}\times 90\ keV = 300\,keV$

Q26. – Q55. carry two marks each.

Q26. Given $\vec{V}_1=\hat{i}-\hat{j}$ and $\vec{V}_2=-2\hat{i}+3\hat{j}+2\hat{k}$, which one of the following $\vec{V}_3$ makes $\left(\vec{V}_1,\vec{V}_2,\vec{V}_3\right)$ a complete set for a three dimensional real linear vector space?

(a) $\vec{V}_3=\hat{i}+\hat{j}+4\hat{k}$ (b) $\vec{V}_3=2\hat{i}-\hat{j}+2\hat{k}$

(c) $\vec{V}_3=\hat{i}+2\hat{j}+6\hat{k}$ (d) $\vec{V}_3=2\hat{i}+\hat{j}+4\hat{k}$

Ans. : (d)

Solution: Let A be the matrix formed by taking $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ as column matrix i.e.,

$$A=\left[V_1 \quad V_2 \quad V_3\right]=\begin{bmatrix} 1 & -2 & 2 \\ -1 & 3 & 1 \\ 0 & 2 & 4 \end{bmatrix} \Rightarrow |A|=-2 \text{. Here } V_3=\left(2\hat{i}+\hat{j}+4\hat{k}\right)$$

Since, $|A|\neq 0$, hence, $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ form a three dimensional real vector space.

Hence, option (d) is correct.

Q27. An interstellar object has speed v at the point of its shortest distance R from a star of much larger mass M. Given $v^2=2GM/R$, the trajectory of the object is

(a) circle (b) ellipse (c) parabola (d) hyperbola

Ans. : (c)

Solution: At shortest distance $E = \frac{J^2}{2mR^2} - \frac{GMm}{R}$

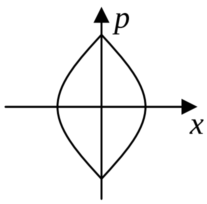
Since, $mvR = J \Rightarrow J^2 = m^2 v^2 R^2$

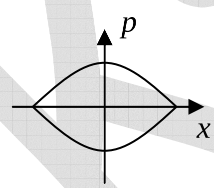
Now, $J^2 = m^2 2GMR = 2GMm^2 R$ (Given that $v^2 = \frac{2GM}{R}$)

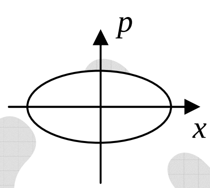
$$E = \frac{2GMm^2R}{2mR^2} - \frac{GMm}{R} = \frac{GMm}{R} - \frac{GMm}{R} = 0$$

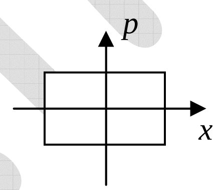
For Kepler's potential, if energy is zero, then the shape is parabola.

Q28. A particle moves in one dimension under a potential $V(x) = \alpha|x|$ with some non-zero total energy. Which one of the following best describes the particle trajectory in the phase space?

(a) 

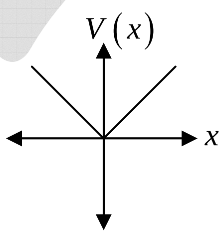
(b) 

(c) 

(d) 

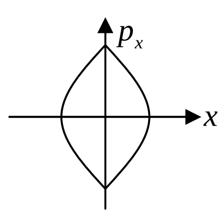
Ans.: (a)

Solution: $E = \frac{p^2}{2m} + \alpha|x|$



For $x > 0$, $E = \frac{p^2}{2m} + \alpha x$

$\Rightarrow p^2 = 2m(E - \alpha x)$



For $x < 0$, $E = \frac{p^2}{2m} - \alpha x$

$\Rightarrow p^2 = 2m(E + \alpha x)$

Q29. Consider an infinitely long solenoid with N turns per unit length, radius R and carrying a current $I(t)=\alpha\cos\omega t$, where α is a constant and ω is the angular frequency. The magnitude of electric field at the surface of the solenoid is

(a) $\frac{1}{2}\mu_0 NR\omega\alpha\sin\omega t$ (b) $\frac{1}{2}\mu_0\omega NR\cos\omega t$

(c) $\mu_0 NR\omega\alpha\sin\omega t$ (d) $\mu_0\omega NR\cos\omega t$

Ans. : (a)

Solution: $\vec{B}=\begin{cases}\mu_0 NI(t)\hat{z}, & \text{inside}\\ 0, & \text{outside}\end{cases}$

Since, $\oint_{line}\vec{E}\cdot d\vec{l}=-\int\frac{\partial\vec{B}}{\partial t}\cdot d\vec{a}$

$\Rightarrow\left|\vec{E}\right|\times 2\pi R=-\mu_0 N(-\alpha\omega\sin\omega t)\times\pi R^2$

$\Rightarrow\left|\vec{E}\right|=\frac{1}{2}\mu_0 NR\omega\alpha\sin\omega t$

Q30. A constant and uniform magnetic field $\vec{B}=B_0\hat{k}$ pervades all space. Which one of the following is the correct choice for the vector potential in Coulomb gauge?

(a) $-B_0(x+y)\hat{i}$ (b) $B_0(x+y)\hat{j}$ (c) $B_0 x\hat{j}$ (d) $-\frac{1}{2}B_0(x\hat{i}-y\hat{j})$

Ans. : (c)

Solution: Check option (c),

$\vec{\nabla}\cdot\vec{A}=0,\ \vec{B}=\vec{\nabla}\times\vec{A}=B_0\hat{k}$

Q31. If H is the Hamiltonian for a free particle with mass m, the commutator $[x,[x,H]]$ is

(a) $\hbar^2/m$ (b) $-\hbar^2/m$ (c) $-\hbar^2/(2m)$ (d) $\hbar^2/(2m)$

Ans. : (b)

Solution: For free particle, potential is zero.

$\Rightarrow H=\frac{P_x^2}{2m}$

Now, $\left[x,H\right]=\left[x,\frac{P_x^2}{2m}\right]=\frac{2i\hbar}{2m}P_x$

$$\left[x,\left[x,H\right]\right]=\frac{2i\hbar}{2m}\left[x,P_x\right]=\frac{i\hbar}{m}\left(i\hbar\right)=-\frac{\hbar^2}{m}$$

Q32. A long straight wire, having radius a and resistance per unit length r, carries a current I. The magnitude and direction of the Poynting vector on the surface of the wire is

(a) $I^2r/2\pi a$, perpendicular to axis of the wire and pointing inwards

(b) $I^2r/2\pi a$, perpendicular to axis of the wire and pointing outwards

(c) $I^2r/\pi a$, perpendicular to axis of the wire and pointing inwards

(d) $I^2r/\pi a$, perpendicular to axis of the wire and pointing outwards

Ans. : (a)

Solution: $\left|\vec{S}\right|=\frac{1}{\mu_0}\left|\left(\vec{E}\times\vec{B}\right)\right|=\frac{1}{\mu_0}\frac{V}{l}\times\frac{\mu_0 I}{2\pi a}=\frac{IR}{l}\times\frac{I}{2\pi a}$

$\because V=IR,\ r=\frac{R}{l}\Rightarrow\left|\vec{S}\right|=\frac{I^2r}{2\pi a}$

Q33. Three particles are to be distributed in four non-degenerate energy levels. The possible number of ways of distribution: (i) for distinguishable particles, and (ii) for identical Boson, respectively, is

(a) (i) 24, (ii) 4   (b) (i) 24, (ii) 20   (c) (i) 64, (ii) 20   (d) (i) 64, (ii) 16

Ans. : (c)

Solution: Number of particles, $N=3$

Number of state, $g=4$

For distinguishable particle, $w=g^N=4^3=64$

For identical Bosons, $w=\frac{\lfloor N+g-1}{\lfloor N\lfloor g-1}=\frac{\lfloor 6}{\lfloor 3\lfloor 3}=\frac{6\times5\times4}{3\times2}=20$

Q34. The term symbol for the electronic ground state of oxygen atom is

(a) $^1S_0$ (b) $^1D_2$ (c) $^3P_0$ (d) $^3P_2$

Ans. : (d)

Solution: $O:\ \ 1s^2,2s^2,2p^4$

$M_L=-1$	0	$+1$
↑↓	↑	↑

Here, $S=1,\ L=2$

According to Hund's rule, for ground state energy

$J = (L+S) = 2 \qquad \therefore \quad {}^{2S+1}L_J = {}^3P_2$

Q35. The energy dispersion for electrons in one dimensional lattice with lattice parameter a is given by $E(k) = E_0 - \frac{1}{2}W\cos ka$, where W and E_0 are constants. The effective mass of the electron near the bottom of the band is

(a) $\frac{2\hbar^2}{Wa^2}$ (b) $\frac{\hbar^2}{Wa^2}$ (c) $\frac{\hbar^2}{2Wa^2}$ (d) $\frac{\hbar^2}{4Wa^2}$

Ans. : (a)

Solution: $E(k) = E_0 - \frac{1}{2}W\cos(ka)$

$$\frac{dE}{dk} = \frac{aW}{2}\sin(ka) \Rightarrow \frac{d^2E}{dk^2} = \frac{a^2W}{2}\cos(ka)$$

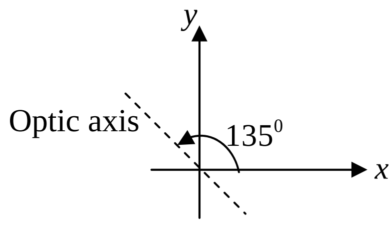
$$\therefore \quad m^* = \frac{\hbar^2}{\frac{d^2E}{dk^2}} = \frac{\hbar^2}{\frac{a^2W}{2}\cos(ka)} = \frac{2\hbar^2}{Wa^2}$$ [At bottom of the band, $k=0$]

Q36. Amongst electrical resistivity (ρ), thermal conductivity (κ), specific heat (C), Young's modulus (Y) and magnetic susceptibility (χ), which quantities show a sharp change at the superconducting transition temperature?

(a) ρ, κ, C, Y (b) ρ, C, χ (c) ρ, κ, C, χ (d) κ, Y, χ

Ans. : (b)

Q37. A quarter wave plate introduces a path difference of $\lambda/4$ between the two components of polarization parallel and perpendicular to the optic axis. An electromagnetic wave with $\vec{E} = (\hat{x}+\hat{y})E_0e^{i(kz-\omega t)}$ is incident normally on a quarter wave plate which has its optic axis making an angle 135^0 with the x- axis as shown.



The emergent electromagnetic wave would be

(a) elliptically polarized

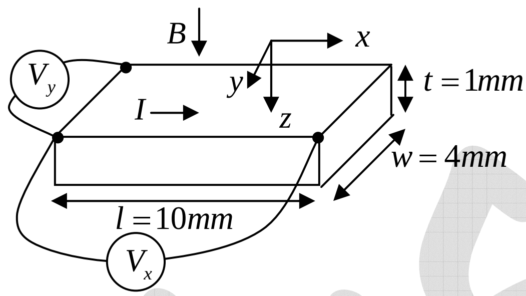
(b) circularly polarized

(c) linearly polarized with polarization as that of incident wave

(d) linearly polarized but with polarization at 90^0 to that of the incident wave

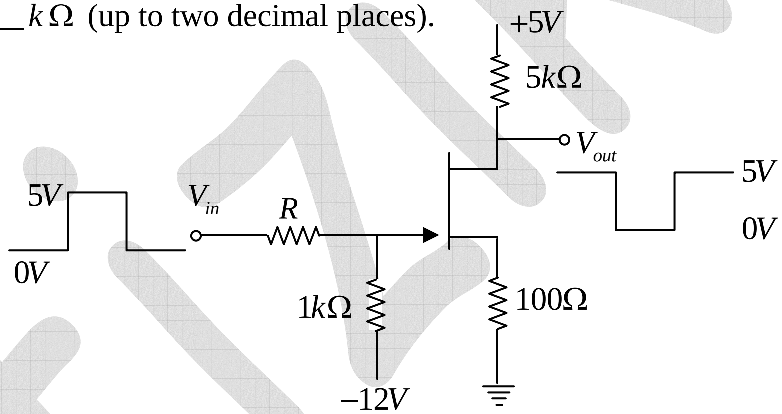
Ans. : (c)

Q38. A p - doped semiconductor slab carries a current $I = 100\,mA$ in a magnetic field $B = 0.2\,T$ as shown. One measures $V_y = 0.25\ mV$ and $V_x = 2\,mV$. The mobility of holes in the semiconductor is___________ $m^2V^{-1}s^{-1}$ (up to two decimal places)



Ans. : 1.55

Q39. An n - channel FET having Gate-Source switch-off voltage $V_{GS(\text{OFF})} = -2V$ is used to invert a $0-5V$ square-wave signal as shown. The maximum allowed value of R would be _________ $k\Omega$ (up to two decimal places).



Ans. : 0.70

Q40. Inside a large nucleus, a nucleon with mass $939\,MeVc^{-2}$ has Fermi momentum 1.40 fm^{-1} at absolute zero temperature. Its velocity is Xc, where the value of X is__________ (up to two decimal places).

($\hbar c = 197\,MeV$-fm)

Ans. : 0.29

Solution: Here, fermi – momentum or fermi radius, $k_F = 1.40\,fm^{-1}$ and $\hbar c = 197$ Mev – fm

Now, Fermi velocity –

$$V_F = \frac{P}{m} = \frac{\hbar k_F}{m} = \frac{(\hbar c)k_F \cdot c}{mc_2} = \frac{(197)\times 1\cdot 40\times c}{939} = \frac{275\cdot 8c}{939} = 0.29c$$

Q41. $4MeV$ γ- rays emitted by the de-excitation of ^{19}F are attributed, assuming spherical symmetry, to the transition of protons from $1d_{3/2}$ state to $1d_{5/2}$ state. If the contribution of spin-orbit term to the total energy is written as $C\langle \vec{l}\cdot\vec{s}\rangle$, the magnitude of C is ______ MeV (up to one decimal place).

Ans. : 1.6

Solution: $l=1, s=\frac{1}{2}, \hat{j}_1=\frac{3}{2}, \hat{j}_2=\frac{5}{2}$

$$\bar{j}=\left(\bar{l}+\bar{s}\right)\Rightarrow j^2=l^2+s^2+2\vec{l}\cdot\vec{s}\Rightarrow \vec{l}\cdot\vec{s}=\frac{\left(j^2+l^2-s^2\right)}{2}$$

$$\langle \vec{l}\cdot\vec{s}\rangle=\frac{\left[j\left(j+1\right)-\left(l+1\right)-s\left(s+1\right)\right]\hbar^2}{2}$$

$$\Delta E=\alpha\left[\langle \vec{l}\cdot\vec{s}\rangle_{5/2}-\langle \vec{l}\cdot\vec{s}\rangle_{3/2}\right]=\alpha\left[\frac{5}{2}\cdot\frac{7}{2}-\frac{3}{2}\cdot\frac{5}{2}\right]\frac{\hbar^2}{2}=\alpha\cdot\left(\frac{20}{8}\right)\hbar^2=\frac{20}{8}\cdot C$$

$$\Delta E=\frac{20}{8}C=4MeV\Rightarrow C=\frac{32}{20}MeV, C=1.6MeV.$$

Q42. An α particle is emitted by a $^{230}_{90}Th$ nucleus. Assuming the potential to be purely Coulombic beyond the point of separation, the height of the Coulomb barrier is________ MeV (up to two decimal places).

$\left(\frac{e^2}{4\pi\in_0}=1.44\,MeV\text{-fm}, r_0=1.30\,\text{fm}\right)$

Ans. : 25.995

Solution: The height of coulomb barrier for α particle from

$${}_{90}Th^{230}\rightarrow{}_{88}X^{226}+2He^4 \ (\alpha\text{ - particle})$$

$$V_C=\frac{1}{4\pi\in_0}\left(\frac{2ze^2}{R}\right)$$

Here, $R_0=1.3\ fm$, $\frac{e^2}{4\pi\in_0}=1.44\ MeV\ fm$

And $R=R_0A^{1/3}$

Here, we consider pure Coulombic interection

$A_{Th}^{1/3} = A_X^{1/3} + A_\alpha^{1/3} = (226)^{1/3} + (4)^{1/3} = (6.09 + 1.58) = 7.67$

$R = R_0 A_{Th}^{1/3} = 1.3(7.67)$

Hence, $V_C = \left(\frac{e^2}{4\pi \in_0}\right)\frac{2\times 90}{1.3(7.67)} = \frac{180\times 1.44}{1.3\times 7.67}\frac{MeV}{fm}$

$V_C = 25.995\ MeV$

Q43. For the transformation

$$Q = \sqrt{2q}\, e^{-1+2\alpha} \cos p, P = \sqrt{2q}\, e^{-\alpha-1} \sin p$$

(where α is a constant) to be canonical, the value of α is _________.

Ans. : 2

Solution: $Q = \sqrt{2q}e^{-1+2\alpha} \cos p, P = \sqrt{2q}e^{-\alpha-1}.\sin p$

Since, $[Q, P] = 1$

$$\Rightarrow \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = 1$$

$$\Rightarrow \left(\frac{1}{2}\sqrt{2}q^{-\frac{1}{2}}.e^{-1+2\alpha}\cos p\right)\left(\sqrt{2q}e^{-\alpha-1}\cos p\right) - \sqrt{2q}e^{-1+2\alpha}(-\sin p).\frac{\sqrt{2}}{2}q^{-\frac{1}{2}}e^{-\alpha.1}\sin p = 1$$

$$\Rightarrow e^{\alpha-2}.\left[\cos^2 p + \sin^2 p\right] = 1 = e^0$$

$$\Rightarrow \alpha = 2$$

Q44. Given

$$\frac{d^2 f(x)}{dx^2} - 2\frac{df(x)}{dx} + f(x) = 0,$$

and boundary conditions $f(0) = 1$ and $f(1) = 0$, the value of $f(0.5)$ is __________ (up to two decimal places) .

Ans. : 0.81

Solution: $\frac{d^2 f(x)}{dx^2} - 2\frac{df(x)}{dx} + f(x) = 0$

Auxiliary equation is,

$(m^2 - 2m + 1) = 0$

$(m-1)^2 = 0 \Rightarrow m = 1,1$

Hence, the solution is

$f(x) = (c_1 + c_2 x)e^x$

using boundary condition,

$f(0) = c_1 e^0 \Rightarrow c_1 = 1$ (i)

$f(1) = (c_1 + c_2)e = 0$ (ii)

From (i) and (ii), $c_2 = -1$

Hence, $f(x) = (1-x)e^x \Rightarrow f(0.5) = (1-0.5)e^{0.5} = 0.81$

Q45. The absolute value of the integral

$$\int \frac{5z^3 + 3z^2}{z^2 - 4} dz ,$$

over the circle $|z-1.5| = 1$ in complex plane, is __________ (up to two decimal places).

Ans. : 81.64

Solution: $f(z) = \dfrac{5z^3 + 3z^2}{(z-2)(z+2)}$

Pole, $z = 2, -2$

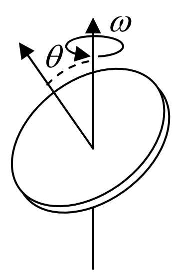
$z = -2$ is outside the center

$|-2-1.5| > 1$ So, will not be considered

Now, $\operatorname{Re} s(2) = \lim_{z \to 2} (z-2) \dfrac{(5z^3 + 3z^2)}{(z-2)(z+2)} = \dfrac{52^3 + 32^2}{4} = \dfrac{40+12}{4} = 13$

$I = 2\pi i \times residue = 2\pi i \times 13 = 26 \times 3.14 \Rightarrow I = 81.64$

Q46. A uniform circular disc of mass m and radius R is rotating with angular speed ω about an axis passing through its centre and making an angle $\theta = 30^0$ with the axis of the disc. If the kinetic energy of the disc is $\alpha m \omega^2 R^2$, the value of α is__________ (up to two decimal places).



Ans. : 0.21

Solution: The kinetic energy of the disc is,

$T = \frac{1}{2}\vec{L}\cdot\vec{\omega}$

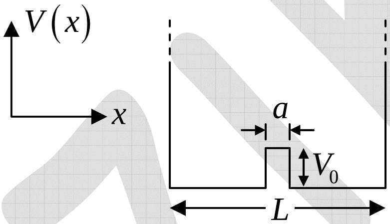
Where $\vec{L}$ is angular momentum and ω is angular velocity

$T = \frac{1}{2}|\vec{L}||\vec{\omega}|\cos 30^0 = \frac{1}{2}I\omega\cdot\omega\frac{\sqrt{3}}{2} = \frac{1}{2}\left(\frac{mR^2}{2}\right)\omega^2\times\frac{\sqrt{3}}{2}$

$T = \frac{\sqrt{3}}{8}m\omega^2R^2 = 0.21m\omega^2R^2 \Rightarrow \alpha m\omega^2R^2 = 0.21m\omega^2R^2$

Hence, $\alpha = 0.21$

Q47. The ground state energy of a particle of mass m in an infinite potential well is E_0. It changes to $E_0\left(1+\alpha\times10^{-3}\right)$, when there is a small potential pump of height $V_0 = \frac{\pi^2\hbar^2}{50\,mL^2}$ and width $a = L/100$, as shown in the figure. The value of α is ________ (up to two decimal places).



Ans. : 0.81

Solution: $\alpha_1 = \left(\frac{L}{2}-\frac{a}{2}\right)$, $\alpha_2 = \left(\frac{L}{2}+\frac{a}{2}\right)$, $a = \frac{L}{100}$

$$E_1 = V_0\int_{\alpha_1}^{\alpha_2}\left(\sqrt{\frac{2}{L}}\right)^2\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{V_0}{L}\int_{\alpha_1}^{\alpha_2}\left[1-\cos\frac{2\pi x}{L}\right]dx = \frac{V_0}{L}\left[x-\frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_{\alpha_1}^{\alpha_2}$$

$$= \frac{V_0}{L}\left[a-\frac{L}{2\pi}\left(\sin\frac{2\pi(L+a)}{2L}-\sin\frac{2\pi(L-a)}{2L}\right)\right]$$

$$= \frac{V_0}{L}\left[\frac{L}{100}-\frac{L}{2\pi}\left(\sin\left(\pi+\frac{\pi a}{L}\right)-\sin\left(\pi-\frac{\pi a}{L}\right)\right)\right]$$

$$= V_0\left[\frac{1}{100}+\frac{1}{2\pi}(0.0314+0.0314)\right]$$

$$=V_0\times10^{-3}(10+10)=E_0\times10^{-3}\times\left(\frac{20}{25}\right)\Rightarrow\alpha E_0\times10^{-3}=0.81\times E_0\times10^{-3}$$

Hence, $\alpha=0.81$

Q48. An electromagnetic plane wave is propagating with an intensity $I=1.0\times10^{5}\,Wm^{-2}$ in a medium with $\in=3\in_0$ and $\mu=\mu_0$. The amplitude of the electric field inside the medium is __________$\times10^{3}\,Vm^{-1}$ (up to one decimal place).

($\in_0=8.85\times10^{-12}C^2N^{-1}m^{-2}$, $\mu_0=4\pi\times10^{-7}NA^{-2}, c=3\times10^{8}ms^{-1}$)

Ans. : 6.6

Solution: $I=\frac{1}{2}v\in E^2\Rightarrow E^2=\frac{2I}{v\in}=\frac{2I}{\frac{1}{\sqrt{\mu\in}}\in}=2I\sqrt{\frac{\mu}{\in}}$

$$\Rightarrow E^2=2\times10^{5}\sqrt{\frac{\mu_0}{3\in_0}}=2\times10^{5}\sqrt{\frac{4\pi\times10^{-7}}{3\times8.8\times10^{-12}}}\approx4363.4\times10^{4}$$

$$\Rightarrow E\approx66\times10^{2}\approx6.6\times10^{3}\,V/m$$

Q49. A microcanonical ensemble consists of 12 atoms with each taking either energy 0 state, or energy $\in$ state. Both states are non-degenerate. If the total energy of this ensemble is $4\in$, its entropy will be __________ k_B (up to one decimal place), where k_B is the Boltzmann constant.

Ans. : 6.204

Solution: The number of ways having total energy $4\in$, out of 12 atom is

$$={}^{12}C_4=\frac{\lfloor 12}{\lfloor 4\lfloor 8}=\frac{12\times11\times10\times9}{4\times3\times2}=495$$

Hence, entropy, $S=k_B\ln w=k_B\ln(495)=k_B(6.204)=6.204\,k_B$

Q50. A two-state quantum system has energy eigenvalues $\pm\in$ corresponding to the normalized states $|\psi_\pm\rangle$. At time $t=0$, the system is in quantum state $\frac{1}{\sqrt{2}}\left[|\psi_+\rangle+|\psi_-\rangle\right]$. The probability that the system will be in the same state at $t=h/(6\in)$ is __________ (up to two decimal places).

Ans. : 0.25

Solution: $\left|\psi(0)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\psi_+\right\rangle + \left|\psi_-\right\rangle\right]$

And $\left|\psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\psi_+\right\rangle e^{-\frac{i\in t}{\hbar}} + \left|\psi_-\right\rangle e^{\frac{i\in t}{\hbar}}\right]$

At $t = \frac{\hbar}{6\in}$,

$$\left|\psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left[\left|\psi_+\right\rangle e^{-\frac{i\in h\times 2\pi}{6\in h}} + \left|\psi\right\rangle e^{\frac{i\in h\times 2\pi}{6\in h}}\right] = \frac{1}{\sqrt{2}}\left[\left|\psi_+\right\rangle e^{\frac{-i\pi}{3}} + \left|\psi_-\right\rangle e^{\frac{i\pi}{3}}\right]$$

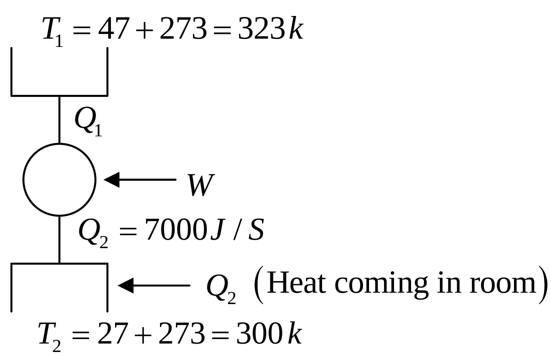
Now, probability in same state

$$P = \frac{\left|\left\langle\psi(t)\middle|\psi(0)\right\rangle\right|^2}{\left\langle\psi\middle|\psi\right\rangle} = \frac{1}{4}\left|e^{-i\pi/3} + e^{i\pi/3}\right|^2 = \frac{1}{4}\left|2\cos\frac{\pi}{3}\right|^2 = \frac{1}{4}\times\left|2\times\frac{1}{2}\right|^2 = 0.25$$

Q51. An air-conditioner maintains the room temperature at $27^0 C$ while the outside temperature is $47^0 C$. The heat conducted through the walls of the room from outside to inside due to temperature difference is $7000\ W$. The minimum work done by the compressor of the air-conditioner per unit time is__________ W.

Ans. : 466.67

Solution: $Q_2 + W = Q_1$



Coefficient of performance of refrigerator (AC) $= \frac{Q_2}{W}$

Also, coefficient of performance of refrigerator, $= \frac{T_2}{T_1 - T_2}$

$$\Rightarrow \frac{300}{47-27} = \frac{7000}{W}$$

$$\Rightarrow W = \frac{7000\times 20}{300} J/s = \frac{1400}{3} = 466.67\ W$$

Q52. Two solid spheres A and B have same emissivity. The radius of A is four times the radius of B and temperature of A is twice the temperature of B. The ratio of the rate of heat radiated from A to that from B is __________.

Ans. : 256

Solution: $\frac{\text{Rate of heat radiation from solid sphere}(A)}{\text{Rate of heat radiation from solid sphere}(B)} = \frac{4\pi R_A^2 T_A^4}{4\pi R_B^2 T_B^4}$

$\because\ R_A = 4R_B$ and $T_A = 2T_B$

$$= \frac{4\pi R_A^2 T_A^4}{4\pi R_B^2 T_B^4} = \frac{(4R_B)^2 \times (2T_B)^4}{(R_B)^2 \times (T_B)^4} = 16 \times 16 = 256$$

Q53. The partition function of an ensemble at a temperature T is

$$Z = \left(2\cosh\frac{\varepsilon}{k_B T}\right)^N$$

where k_B is the Boltzmann constant. The heat capacity of this ensemble at $T = \frac{\varepsilon}{k_B}$ is $X\ Nk_B$, where the value of X is __________ (up to two decimal places).

Ans. : 0.42

Solution: The partition function, $z = \left[2\cosh\left(\frac{\varepsilon}{k_B T}\right)\right]^N$

The average energy, $\langle E\rangle = k_B T^2 \frac{\partial(\ln z)}{\partial T}$

$$= \frac{Nk_B T^2\left[2\sinh\left(\frac{\varepsilon}{k_B T}\right)\right]\left(\frac{-\varepsilon}{k_B T^2}\right)}{2\cosh\left(\frac{\varepsilon}{k_B T}\right)} = -N\varepsilon\tanh\left(\frac{\varepsilon}{k_B T}\right)$$

$$C = \frac{d\langle E\rangle}{dT} = -N\varepsilon\,\mathrm{sec}\,h^2\left(\frac{\varepsilon}{k_B T}\right).\left(\frac{-\varepsilon}{k_B T^2}\right)$$

At $T = \frac{\varepsilon}{k}$, $C = \frac{N\varepsilon^2}{k\cdot(\varepsilon^2/k^2)}\mathrm{sec}\,h^2(1) = Nk\,\mathrm{sec}\,h^2(1) = 0.42Nk_B$

Q54. An atom in its singlet state is subjected to a magnetic field. The Zeeman splitting of its 650 nm spectral line is 0.03 nm. The magnitude of the field is ___________ Tesla (up to two decimal places).

$(e = 1.60\times10^{-19}C, m_e = 9.11\times10^{-31}kg, c = 3.0\times10^8 ms^{-1})$

Ans. : 1.52

Solution: $\Delta\lambda = \frac{\lambda^2}{c}\times\frac{eB}{4\pi m}$

$$\Rightarrow B = \frac{c}{\lambda^2}\cdot\frac{4\pi m}{e}\Delta\lambda = \frac{3\times10^{8}}{\left(650\times10^{-9}\right)^2}\cdot\frac{4\pi\times9.1\times10^{-31}}{1.6\times10^{-19}}\cdot\left(0.03\times10^{-9}\right) = 1.52T$$

Q55. The quantum effects in an ideal gas become important below a certain temperature T_Q when de Broglie wavelength corresponding to the root mean square thermal speed becomes equal to the inter-atomic separation. For such a gas of atoms of mass $2\times10^{-26}\,kg$ and number density $6.4\times10^{25}\,m^{-3}$, $T_Q =$ ____________ $\times10^{-3}K$ (up to one decimal place).

$(k_B = 1.38\times10^{-23}\,J/K, h = 6.6\times10^{-34}\,J\text{-s})$

Ans. : 84.2

Solution: $\lambda = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{3mkT}}$

At $T = T_Q,\ \lambda = a$

$$\therefore\ \frac{h}{\sqrt{3mkT_Q}} = a \Rightarrow T_Q = \frac{h^2}{3mka^2}$$

where $\frac{1}{a^3} = 6.4\times10^{25}\,m^{-3} \Rightarrow a = 2.5\times10^{-9}\ m$

$$\therefore\ T_Q = \frac{\left(6.6\times10^{-34}\,J\text{-}s\right)^2}{3\times2\times10^{-26}\,kg\times1.38\times10^{-23}\,J/k\times\left(2.5\times10^{-9}\,m\right)^2}$$

$= 0.0842\,K = 84.2\times10^{-3}\ K$