THERMODYNAMICS AND STATISTICAL PHYSICS SOLUTIONS

GATE 2010

Q1. A system of N non-interacting classical point particles is constrained to move on the two-dimensional surface of a sphere. The internal energy of the system is

(a) $\frac{3}{2}Nk_BT$ (b) $\frac{1}{2}Nk_BT$ (c) Nk_BT (d) $\frac{5}{2}Nk_BT$

Ans: (c)

Solution: There are 2 N degree of freedom.

The internal energy of the system is $\frac{Nk_BT}{2}+\frac{Nk_BT}{2}=Nk_BT$

Q2. Which of the following atoms cannot exhibit Bose-Einstein condensation, even in principle?

(a) $^1H_1$ (b) $^4H_2$ (c) $^{23}Na_{11}$ (d) $^{30}K_{19}$

Ans: (d)

Solution: For Bose-Einstein condensation:

Number of electron + number of proton + number of neutron = Even

For $^{30}K_{19}$

Number of proton = 19, Number of electron = 19, Number of neutron = 11.

19 + 19 + 11 = 49 this is odd. So it will not exhibit Bose-Einstein condensation.

Q3. For a two-dimensional free electron gas, the electronic density n, and the Fermi energy E_F, are related by

(a) $n=\frac{(2mE_F)^{3/2}}{3\pi^2\hbar^3}$ (b) $n=\frac{mE_F}{\pi\hbar^2}$

(c) $n=\frac{mE_F}{2\pi\hbar^2}$ (d) $n=\frac{(2mE_F)^{3/2}}{\pi\hbar}$

Ans: (c)

Solution: $n=\int_0^{E_F} g(E)f(E)dE$, $g(E)dE=\frac{2m}{h^2}dE$

At T=0, $f(E)=\begin{cases}1, & if\ E<E_F\\ 0, & if\ E>E_F\end{cases} \Rightarrow n=\frac{2mE_F}{h^2}=\frac{mE_F}{2\pi^2\hbar^2}$

Q4. Which among the following sets of Maxwell relations is correct? (U-internal energy, H-enthalpy, A-Helmholtz free energy and G-Gibbs free energy)

(a) $T=\left(\frac{\partial U}{\partial V}\right)_S$ and $P=\left(\frac{\partial U}{\partial S}\right)_V$ (b) $V=\left(\frac{\partial H}{\partial P}\right)_S$ and $T=\left(\frac{\partial H}{\partial S}\right)_P$

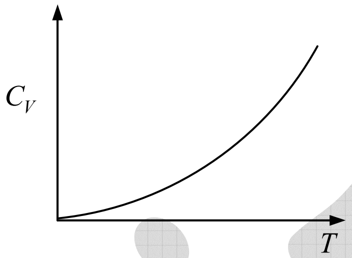
(c) $P=-\left(\frac{\partial G}{\partial V}\right)_T$ and $V=\left(\frac{\partial G}{\partial P}\right)_S$ (d) $P=-\left(\frac{\partial A}{\partial S}\right)_T$ and $S=\left(\frac{\partial A}{\partial P}\right)_V$

Ans: (b)

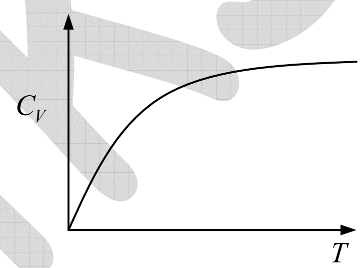
Solution: $dH=TdS+VdP \Rightarrow \left(\frac{\partial H}{\partial S}\right)_P=T, \left(\frac{\partial H}{\partial P}\right)_S=V$

Q5. Partition function for a gas of photons is given as, $\ln Z=\frac{\pi^2 V\left(k_B T\right)^3}{45\hbar^3 C^3}$. The specific heat of the photon gas varies with temperature as

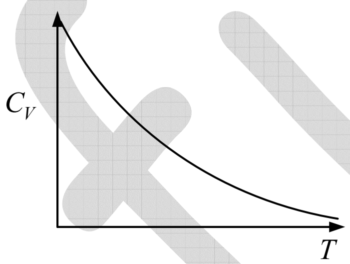
(a)



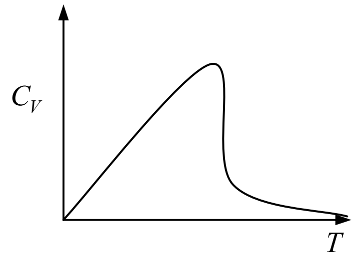
(b)



(c)



(d)



Ans: (a)

Solution: $\mathrm{U}=\mathrm{K_B T^2}\frac{\partial \ln z}{\partial \mathrm{T}}, \quad \mathrm{C_V}=\left(\frac{\partial \mathrm{U}}{\partial \mathrm{T}}\right)_{\mathrm{v}} \Rightarrow \mathrm{C_V} \propto \mathrm{T}^3$.

Q6. From Q. no. 5, the pressure of the photon gas is

(a) $\frac{\pi^2\left(k_B T\right)^3}{15\hbar^3 C^3}$ (b) $\frac{\pi^2\left(k_B T\right)^4}{8\hbar^3 C^3}$ (c) $\frac{\pi^2\left(k_B T\right)^4}{45\hbar^3 C^3}$ (d) $\frac{\pi^2\left(k_B T\right)^{3/2}}{45\hbar^3 C^3}$

Ans: (c)

Solution: Since, $P=-\frac{\partial F}{\partial V} \Rightarrow P=KT\left(\frac{\partial \ln z}{\partial V}\right)_T=\frac{\pi^2\left(k_0 T\right)^4}{45\hbar^3 C^3}$

GATE 2011

Q7. A Carnot cycle operates on a working substance between two reservoir at temperatures T_1 and T_2 with $T_1 > T_2$. During each cycle, an amount of heat Q_1 is extracted from the reservoir at T_1 and an amount Q_2 is delivered in the reservoir at T_2. Which of the following statements is **INCORRECT**?

(a) Work done in one cycle is $Q_1 - Q_2$

(b) $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

(c) Entropy of the hotter reservoir decreases

(d) Entropy of the universe (consisting of the working substance and the two reservoirs) increases

Ans: (c)

Solution: Entropy of hotter reservoirs decreases.

Q8. In a first order phase transition, at the transition temperature, specific heat of the system

(a) diverges and its entropy remains the same

(b) diverges and its entropy has finite discontinuity

(c) remains unchanged and its entropy has finite discontinuity

(d) has finite discontinuity and its entropy diverges

Ans: (b)

Q9. A system of N non-interacting and distinguishable particle of spin 1 is in thermodynamic equilibrium. The entropy of the system is

(a) $2k_B \ln N$ (b) $3k_B \ln N$ (c) $Nk_B \ln 2$ (d) $Nk_B \ln 3$

Ans: (d)

Solution: $S = k_B \sum_i \ln\Omega$, $\Omega = 3$ is number of microstate. $S = 1$; $S_z = -1,\ 0,\ 1$

The entropy of the system is $Nk_B \ln 3$.

Q10. A system has two energy levels with energies ε and 2ε. The lower level is 4-fold degenerate while the upper level is doubly degenerate. If there are N non-interacting classical particles in the system, which is in thermodynamic equilibrium at a temperature T, the fraction of particles in the upper level is

(a) $\frac{1}{1+e^{\varepsilon/k_BT}}$ (b) $\frac{1}{1+2e^{\varepsilon/k_BT}}$

(c) $\frac{1}{2e^{\varepsilon/k_BT}+4e^{2\varepsilon/k_BT}}$ (d) $\frac{1}{2e^{\varepsilon/k_BT}-4e^{2\varepsilon/k_BT}}$

Ans: (b)

Solution: Partition function $Z = 4e^{-\epsilon/kT} + 2e^{-\epsilon/kT} \Rightarrow P(2\varepsilon) = \frac{2e^{-2\epsilon/kT}}{4e^{-\epsilon/kT}+2e^{-2\epsilon/kT}} = \frac{1}{1+2e^{\epsilon/kT}}$

GATE 2012

Q11. The isothermal compressibility, κ of an ideal gas at temperatures T_0 and V_0 is given by

(a) $-\frac{1}{V_0}\frac{\partial V}{\partial P}\Big|_{T_0}$ (b) $\frac{1}{V_0}\frac{\partial V}{\partial P}\Big|_{T_0}$ (c) $-V_0\frac{\partial P}{\partial V}\Big|_{T_0}$ (d) $V_0\frac{\partial P}{\partial V}\Big|_{T_0}$

Ans: (c)

Solution: Isothermal compressibility $\kappa = -\mathrm{V}\left(\frac{\partial \mathrm{P}}{\partial \mathrm{V}}\right)_\mathrm{T}$

Q12. For an ideal Fermi gas in three dimensions, the electron velocity V_F at the Fermi surface is related to electron concentration n as,

(a) $V_F \propto n^{2/3}$ (b) $V_F \propto n$ (c) $V_F \propto n^{1/2}$ (d) $V_F \propto n^{1/3}$

Ans: (d)

Solution: $E_F = \frac{1}{2}mV_F^2 \quad \because E_F \propto n^{2/3} \Rightarrow V_F^2 \propto n^{2/3} \Rightarrow V_F \propto n^{1/3}$.

Q13. A classical gas of molecules, each of mass m, is in thermal equilibrium at the absolute temperature T. The velocity components of the molecules along the Cartesian axes are v_x, v_y and v_z. The mean value of $(v_x+v_y)^2$ is

(a) $\frac{k_BT}{m}$ (b) $\frac{3}{2}\frac{k_BT}{m}$ (c) $\frac{1}{2}\frac{k_BT}{m}$ (d) $\frac{2k_BT}{m}$

Ans: (d)

Solution: $\left\langle (V_x+V_y)^2\right\rangle = \left\langle v_x^2\right\rangle + \left\langle v_y^2\right\rangle + 2\left\langle v_x\cdot v_y\right\rangle = \left\langle v_x^2\right\rangle + \left\langle v_y^2\right\rangle + 2\left\langle v_x\right\rangle\cdot\left\langle v_y\right\rangle = \dfrac{2k_BT}{m}$

$\because \left\langle v_x\right\rangle = \left\langle v_y\right\rangle = 0$ and $\left\langle V_x^2\right\rangle + \left\langle V_y^2\right\rangle = \dfrac{2k_BT}{m}$.

Q14. The total energy, E of an ideal non-relativistic Fermi gas in three dimensions is given by $E \propto \dfrac{N^{5/3}}{V^{2/3}}$, where N is the number of particles and V is the volume of the gas. Identify the **CORRECT** equation of state (P being the pressure),

(a) $PV = \frac{1}{3}E$ (b) $PV = \frac{2}{3}E$ (c) $PV = E$ (d) $PV = \frac{5}{3}E$

Ans: (b)

Solution: $P = -\left(\dfrac{\partial E}{\partial V}\right)_N = \dfrac{2}{3}\left(\dfrac{N}{V}\right)^{\frac{5}{3}} \Rightarrow PV = \dfrac{2}{3}\dfrac{N^{5/3}}{V^{2/3}} = \dfrac{2}{3}E$.

Q15. Consider a system whose three energy levels are given by 0, ε and 2ε. The energy level ε is two-fold degenerate and the other two are non-degenerate. The partition function of the system with $\beta = \dfrac{1}{k_BT}$ is given by

(a) $1+2e^{-\beta\varepsilon}$ (b) $2e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}$ (c) $(1+e^{-\beta\varepsilon})^2$ (d) $1+e^{-\beta\varepsilon}+e^{-2\beta\varepsilon}$

Ans: (c)

Solution: $E_1 = 0,\ E_2 = \varepsilon, E_3 = 2\varepsilon$; $g_1 = 1,\ g_2 = 2, g_3 = 1$ where g_1, g_2 and g_3 are degeneracy.

The partition function $Z = g_1e^{-\beta\cdot E_1} + g_2e^{-\beta\cdot E_2} + g_3e^{-\beta\cdot E_3} = 1+2e^{-\beta\varepsilon} + e^{-\beta 2\varepsilon} = \left(1+e^{-\beta\varepsilon}\right)^2$

GATE 2013

Q16. If Planck's constant were zero, then the total energy contained in a box filled with radiation of all frequencies at temperature T would be (k is the Boltzmann constant and T is nonzero)

(a) zero (b) Infinite (c) $\frac{3}{2}kT$ (d) kT

Ans: (d)

Solution: If Planck's constant were zero, then the system behaved as a classical system and the energy is kT.

Q17. Across a first order phase transition, the free energy is

(a) proportional to the temperature

(b) a discontinuous function of the temperature

(c) a continuous function of the temperature but its first derivative is discontinuous

(d) such that the first derivative with respect to temperature is continuous

Ans: (c)

Q18. Two gases separated by an impermeable but movable partition are allowed to freely exchange energy. At equilibrium, the two sides will have the same

(a) pressure and temperature (b) volume and temperature

(c) pressure and volume (d) volume and energy

Ans: (a)

Q19. The entropy function of a system is given by $S(E) = aE(E_0 - E)$ where a and E_0 are positive constants. The temperature of the system is

(a) negative for some energies (b) increases monotonically with energy

(c) decreases monotonically with energy (d) Zero

Ans: (a)

Solution: From first and second law of thermodynamics

$$TdS = dU - PdV \Rightarrow dS = \frac{1}{T}(dU - PdV) \Rightarrow \left(\frac{\partial S}{\partial E}\right)_V = \frac{1}{T} \because E = U$$

$$S(E) = aE(E_0 - E) \Rightarrow \left(\frac{\partial S}{\partial E}\right)_V = a(E_0 - E) - aE = a(E_0 - 2E) \Rightarrow T = \frac{1}{a(E_0 - 2E)}.$$

Q20. Consider a linear collection of N independent spin ½ particles, each at a fixed location. The entropy of this system is (k is the Boltzmann constant)

(a) zero (b) Nk (c) $\frac{1}{2}Nk$ (d) $Nk\ln(2)$

Ans: (d)

Solution: There are two microstates possible for spin $\frac{1}{2}$ particle, so entropy is given by $Nk\ln(2)$.

Q21. Consider a gas of atoms obeying Maxwell-Boltzmann statistics. The average value of $e^{\vec{a}\cdot\vec{p}}$ over all the moments $\vec{p}$ of each of the particles (where $\vec{a}$ is a constant vector and a is the magnitude, m is the mass of each atom, T is temperature and k is Boltzmann's constant) is,

(a) one (b) zero (c) $e^{-\frac{1}{2}a^2mkT}$ (d) $e^{-\frac{3}{2}a^2mkT}$

Ans: (c)

Solution: $\langle e^{\vec{p}.\vec{a}}\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\left(p_x, p_y, p_z\right)e^{\vec{p}.\vec{a}}dp_x dp_y dp_z$ where $f\left(p_x, p_y, p_z\right)$ is Maxwell probability distribution at temperature T.

$$\langle e^{\vec{p}.\vec{a}}\rangle = \int_{-\infty}^{\infty} A_x e^{-\frac{p_x^2}{2mkT}} e^{p_x a_x} dp_x \int_{-\infty}^{\infty} A_y e^{-\frac{p_y^2}{2mkT}} e^{p_y a_y} dp_y \int_{-\infty}^{\infty} A_z e^{-\frac{p_z^2}{2mkT}} e^{p_z a_z} dp_z$$

$$\langle e^{\vec{p}.\vec{a}}\rangle = e^{\frac{-(a_x^2+a_y^2+a_z^2)mkT}{2}} \int_{-\infty}^{\infty} A_x e^{-\frac{(p_x-mkTa_x)^2}{2mkT}} dp_x \int_{-\infty}^{\infty} A_y e^{-\frac{(p_y-mkTa_y)^2}{2mkT}} dp_y \int_{-\infty}^{\infty} A_z e^{-\frac{(p_z-mkTa_z)^2}{2mkT}} dp_z$$

$$\langle e^{\vec{p}.\vec{a}}\rangle = e^{\frac{-(a_x^2+a_y^2+a_z^2)mkT}{2}}.1.1.1 \;=\; e^{-\frac{1}{2}a^2mkT}$$

Common Data for Questions 22 and 23: There are four energy levels $E,\ 2E, 3E$ and $4E$ (where $E > 0$). The canonical partition function of two particles is, if these particles are

Q22. Two identical fermions

(a) $e^{-2\beta E} + e^{-4\beta E} + e^{-6\beta E} + e^{-8\beta E}$ (b) $e^{-3\beta E} + e^{-4\beta E} + e^{-5\beta E} + e^{-6\beta E} + e^{-7\beta E}$

(c) $\left(e^{-\beta E} + e^{-2\beta E} + e^{-3\beta E} + e^{-4\beta E}\right)^2$ (d) $e^{-2\beta E} - e^{-4\beta E} + e^{-6\beta E} - e^{-8\beta E}$

Ans: (b)

Solution: The possible value of Energy for two Fermions

$$E_1 = 3E, E_2 = 4E, E_3 = 5E, E_4 = 6E, E_5 = 7E$$

The partition function is $Z = e^{-3\beta E} + e^{-4\beta E} + 2e^{-5\beta E} + e^{-6\beta E} + e^{-7\beta E}$, then the answer may be option (b).

Q23. Two distinguishable particles

(a) $e^{-2\beta E} + e^{-4\beta E} + e^{-6\beta E} + e^{-8\beta E}$ (b) $e^{-3\beta E} + e^{-4\beta E} + e^{-5\beta E} + e^{-6\beta E} + e^{-7\beta E}$

(c) $\left(e^{-\beta E} + e^{-2\beta E} + e^{-3\beta E} + e^{-4\beta E}\right)^2$ (d) $e^{-2\beta E} - e^{-4\beta E} + e^{-6\beta E} - e^{-8\beta E}$

Ans: (c)

Solution: When two particles are distinguishable then minimum value of Energy is $2E$ and maximum value is $8E$.

So from checking all four options $\left(Z = e^{-\beta E} + e^{-2\beta E} + e^{-3\beta E} + e^{-4\beta E}\right)^2$

GATE 2014

Q24. For a gas under isothermal condition its pressure P varies with volume V as $P \propto V^{-5/3}$. The bulk modules B is proportional to

(a) $V^{-1/2}$ (b) $V^{-2/3}$ (c) $V^{-3/5}$ (d) $V^{-5/3}$

Ans: (d)

Solution: $P = KV^{-5/3}$, $B = -V\frac{dP}{dV}$ $B \propto V^{-5/3}$

Q25. At a given temperature T, the average energy per particle of a non-interacting gas of two-dimensional classical harmonic oscillators is ________ $k_B T$

(k_B is the Boltzmann constant)

Ans: 2

Q26. Which one of the following is a fermion?

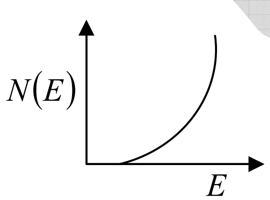
(a) α particle (b) ${}_4Be^2$ nucleus (c) Hydrogen atom (d) deuteron

Ans (d)

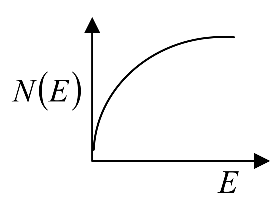
Solution: If total number of particles i.e., electron, proton and neutron is odd, then it is a fermions: $P + N + E = 3$

Q27. For a free electron gas in two dimensions the variations of the density of states. $N(E)$ as a function of energy E, is best represented by

(a)



(b)



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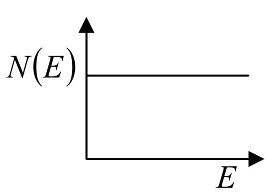
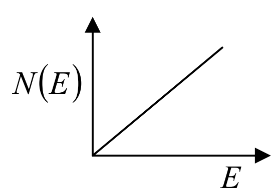


(d)



Ans. : (c)

Solution: $N(E) \propto E^0$

Q28. For a system of two bosons each of which can occupy any of the two energy levels 0 and ε. The mean energy of the system at temperature T with $\beta = \frac{1}{k_\beta T}$ is given by

(a) $\frac{\varepsilon e^{-\beta\varepsilon} + 2\varepsilon e^{-2\beta\varepsilon}}{1 + 2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$ (b) $\frac{1 + \varepsilon e^{-\beta\varepsilon}}{2e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$ (c) $\frac{2\varepsilon e^{-\beta\varepsilon} + \varepsilon e^{-2\beta\varepsilon}}{2 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$ (d) $\frac{\varepsilon e^{-\beta\varepsilon} + 2\varepsilon e^{-2\beta\varepsilon}}{2 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon}}$

Ans. : None of the options are matched.

Solution: If both particle will in ground state the energy will 0, which is non-degenerate. If one particle is in ground state and other is in first excited state then energy is ε and non degenerate. If both particles will in first excited state, then energy will 2ε, which is non-degenerate.

Then partition function is $Z = 1 + \exp(-\beta\varepsilon) + \exp(-2\beta\varepsilon)$

Average value of energy $= \frac{\exp - \beta\varepsilon + 2\varepsilon \exp - 2\beta\varepsilon}{1 + \exp - \beta\varepsilon + \exp - 2\beta\varepsilon}$

No one answer is correct, but answer may be (a).

Q29. Consider a system of 3 fermions which can occupy any of the 4 available energy states with equal probability. The entropy of the system is

(a) $k_B \ln 2$ (b) $2k_B \ln 2$ (c) $2k_B \ln 4$ (d) $3k_B \ln 4$

Ans: (b)

Solution: Number of ways that 3 fermions will adjust in 4 available energy is ${}^4C_3 = 4$ so entropy is $k_B \ln 4 = 2k_B \ln 2$

GATE 2015

Q30. In Boss-Einstein condensation, the particles

(a) have strong interparticle attraction

(b) condense in real space

(c) have overlapping wavefunctions

(d) have large and positive chemical potential

Ans.: (c)

Solution: In Bose- Einstein condensates, the particles have overlapping wave function.

Q31. For a black body radiation in a cavity, photons are created and annihilated freely as a result of emission and absorption by the walls of the cavity. This is because

(a) the chemical potential of the photons is zero

(b) photons obey Pauli exclusion principle

(c) photons are spin-1 particles

(d) the entropy of the photons is very large

Ans.: (a)

Solution: The chemical potential of photon is zero

Q32. Consider a system of N non-interacting spin $-\frac{1}{2}$ particles, each having a magnetic moment μ, is in a magnetic field $\vec{B} = B\hat{z}$. If E is the total energy of the system, then number of accessible microstates Ω is given by

(a) $\Omega = \dfrac{N!}{\frac{1}{2}\left(N - \frac{E}{\mu B}\right)!\ \frac{1}{2}\left(N + \frac{E}{\mu B}\right)!}$

(b) $\Omega = \dfrac{\left(N - \frac{E}{\mu B}\right)!}{\left(N + \frac{E}{\mu B}\right)!}$

(c) $\Omega = \frac{1}{2}\left(N - \frac{E}{\mu B}\right)!\ \frac{1}{2}\left(N + \frac{E}{\mu B}\right)!$

(d) $\Omega = \dfrac{N!}{\left(N + \frac{E}{\mu B}\right)!}$

Ans.: (a)

Solution: Number of microstate is ${}^{N}C_{n_1}$, where n_1 is number of particle in $+\frac{1}{2}$ state and

$n_2 = (N - n_1)$ is number of state in $-\frac{1}{2}$ state.

where $n_1 = \frac{1}{2}\left(N - \frac{E}{\mu B}\right)$, $n_2 = \frac{1}{2}\left(N + \frac{E}{\mu B}\right)$

So, number of microstate $= \dfrac{\underline{|N}}{\underline{\left|\frac{1}{2}\left(N - \frac{E}{\mu B}\right)\right.}\ \underline{\left|\frac{1}{2}\left(N + \frac{E}{\mu B}\right)\right.}}$

Q33. The average energy U of a one dimensional quantum oscillator of frequency ω and in contact with a heat bath at temperature T is given by

(a) $U=\frac{1}{2}\hbar\omega\coth\left(\frac{1}{2}\beta\hbar\omega\right)$ (b) $U=\frac{1}{2}\hbar\omega\sinh\left(\frac{1}{2}\beta\hbar\omega\right)$

(c) $U=\frac{1}{2}\hbar\omega\tanh\left(\frac{1}{2}\beta\hbar\omega\right)$ (d) $U=\frac{1}{2}\hbar\omega\cosh\left(\frac{1}{2}\beta\hbar\omega\right)$

Ans.: (a)

Solution: $\because Z=\sum e^{-\beta E_i}=\sum_{i=0}^{\infty}e^{-\beta\left(n+\frac{1}{2}\right)\hbar\omega}$ where $E=\left(n+\frac{1}{2}\right)\hbar\omega\Rightarrow Z=\frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}$

$$\because U=\frac{-\partial}{\partial\beta}\ln Z=-\frac{\partial}{\partial\beta}\ln\left[\frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)}\right]=\frac{\hbar\omega}{2}\coth\left(\frac{\beta\hbar\omega}{2}\right)$$

Q34. The entropy of a gas containing N particles enclosed in a volume V is given by $S=Nk_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)$, where E is the total energy, a is a constant and k_B is the Boltzmann constant. The chemical potential μ of the system at a temperature T is given by

(a) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-\frac{5}{2}\right]$ (b) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)-\frac{3}{2}\right]$

(c) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{3/2}}\right)-\frac{5}{2}\right]$ (d) $\mu=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{3/2}}\right)-\frac{3}{2}\right]$

Ans.: (a)

Solution: $\left(\frac{\partial G}{\partial T}\right)_P=-S=-Nk_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)\quad\because S=Nk_B\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)$

$$\Rightarrow G=-Nk_BT\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)+\ln A$$

$$\Rightarrow\mu=\left(\frac{\partial G}{\partial N}\right)=-\left[k_BT\ln\left(\frac{aVE^{3/2}}{N^{5/2}}\right)+Nk_BT\frac{N^{5/2}}{aVE^{3/2}}\cdot\frac{(-5/2)}{N^{7/2}}aVE^{3/2}\right]=-k_BT\left[\ln\left(\frac{aVE^{3/2}}{N^{\frac{5}{2}}}\right)-\frac{5}{2}\right]$$

GATE-2016

Q35. The total power emitted by a spherical black body of radius R at a temperature T is P_1. Let P_2 be the total power emitted by another spherical black body of radius $\frac{R}{2}$ kept at temperature $2T$. The ratio, $\frac{P_1}{P_2}$ is _______. (Give your answer upto two decimal places)

Ans.: 0.25

Solution: $P \propto AT^4 \Rightarrow \frac{P_1}{P_2} = \frac{R_1^2 T_1^4}{R_2^2 T_2^4} = \frac{R^2 T^4}{\left(\frac{R}{2}\right)^2 (2T)^4} = \frac{4}{16} = \frac{1}{4} = 0.25$

Q36. The entropy S of a system of N spins, which may align either in the upward or in the downward direction, is given by $S = -k_B N\left[p\ln p + (1-p)In(1-p)\right]$. Here k_B is the Boltzmann constant. The probability of alignment in the upward direction is *p*. The value of *p*, at which the entropy is maximum, is _______. (Give your answer upto one decimal place)

Ans.: 0.5

Solution: $S = -k_B N\left[p\ln p + (1-p)In(1-p)\right]$

For maximum entropy, $\frac{dS}{dp} = 0 \;\Rightarrow \ln p + p \times \frac{1}{p} - \ln(1-p) + (1-p) \times \frac{1}{1-p}(-1) = 0$

$$\ln p + 1 - \ln(1-p) - 1 = 0 \Rightarrow \ln\left(\frac{p}{1-p}\right) = 0 \Rightarrow p = 1 - p \Rightarrow p = 0.5$$

Q37. For a system at constant temperature and volume, which of the following statements is correct at equilibrium?

(a) The Helmholtz free energy attains a local minimum.

(b) The Helmholtz free energy attains a local maximum.

(c) The Gibbs free energy attains a local minimum.

(d) The Gibbs free energy attains a local maximum.

Ans.: (a)

Solution: $dF = -SdT - PdV$

Q38. N atoms of an ideal gas are enclosed in a container of volume V. The volume of the container is changed to $4V$, while keeping the total energy constant. The change in the entropy of the gas, in units of $Nk_B \ln 2$, is _______, where k_B is the Boltzmann constant.

Ans.: 2

Solution: $S_1 = -Nk_B \ln 1,\ S_2 = -Nk_B \ln \frac{1}{4} \Rightarrow \Delta S = S_2 - S_1 = Nk_B \ln 4 = 2Nk_B \ln 2$

Q39. Consider a system having three energy levels with energies 0, 2ε and 3ε ,with respective degeneracies of 2,2 and 3 . Four bosons of spin zero have to be accommodated in these levels such that the total energy of the system is $10\,\varepsilon$. The number of ways in which it can be done is ______.

Ans.: 18

Solution: The system have energy $10\,\varepsilon$, if out of four boson two boson are in energy level 2ε and two boson are in energy level 3ε and

$$W = \prod_i \frac{\lfloor n_i + g_i - 1}{\lfloor n_i \lfloor g_i - 1},\ n_1 = 2, g_1 = 2 \text{ and } n_2 = 2, g_2 = 3$$

$$W = \frac{\lfloor 2+2-1}{\lfloor 2 \lfloor 2-1} \times \frac{\lfloor 2+3-1}{\lfloor 2 \lfloor 3-1} = 3 \times 6 = 18$$

Q40. A two-level system has energies zero and E. The level with zero energy is non-degenerate, while the level with energy E is triply degenerate. The mean energy of a classical particle in this system at a temperature T is

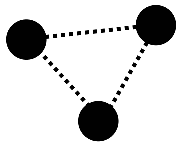
(a) $\dfrac{Ee^{\frac{-E}{k_BT}}}{1+3e^{\frac{-E}{k_BT}}}$ (b) $\dfrac{Ee^{\frac{-E}{k_BT}}}{1+e^{\frac{-E}{k_BT}}}$ (c) $\dfrac{3Ee^{\frac{-E}{k_BT}}}{1+e^{\frac{-E}{k_BT}}}$ (d) $\dfrac{3Ee^{\frac{-E}{k_BT}}}{1+3e^{\frac{-E}{k_BT}}}$

Ans.: (d)

Solution: $$\langle E \rangle = \frac{\sum_i g_i E_i e^{-\frac{E_i}{kT}}}{\sum_i g_i e^{-\frac{E_i}{kT}}} = \frac{0 \times e^{-\frac{0}{kT}} + 3 \times E \times e^{-\frac{E}{kT}}}{e^{-\frac{0}{kT}} + 3 \times e^{-\frac{E}{kT}}} = \frac{3Ee^{\frac{-E}{k_BT}}}{1+3e^{\frac{-E}{k_BT}}}$$

GATE 2017

Q41. Consider a triatomic molecule of the shape shown in the figure in three dimensions. The heat capacity of this molecule at high temperature (temperature much higher than the vibrational and rotational energy scales of the molecule but lower than its bond dissociation energies) is:



(a) $\frac{3}{2}k_B$ (b) $3k_B$ (c) $\frac{9}{2}k_B$ (d) $6k_B$

Ans. : (d)

Solution: If given molecules are at lower temperature i.e. atoms are attached to rigid rod then degree of freedom is 6, so internal energy is $\frac{6k_BT}{2}$, but at high temperature, vibration mode will active, so there are three extra vibration mode will active, so total energy

$$U = 3k_BT + 3k_BT = 6k_BT$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = 6k_B$$

Q42. A reversible Carnot engine is operated between temperatures T_1 and $T_2\left(T_2 > T_1\right)$ with a photon gas as the working substance. The efficiency of the engine is

(a) $1-\frac{3T_1}{4T_2}$ (b) $1-\frac{T_1}{T_2}$ (c) $1-\left(\frac{T_1}{T_2}\right)^{3/4}$ (d) $1-\left(\frac{T_1}{T_2}\right)^{4/3}$

Ans. : (b)

Solution: Efficiency of Carnot engine does not depends on nature of working substance rather depends on temperature of source and sink

$$\eta = 1-\frac{T_1}{T_2}$$

Q43. Water freezes at 0^0C at atmospheric pressure $\left(1.01\times10^5\ Pa\right)$. The densities of water and ice at this temperature and pressure are $1000\,kg/m^3$ and $934\,kg/m^3$ respectively. The latent heat of fusion is $3.34\times10^5\ J/kg$. The pressure required for increasing the melting temperature of ice by 10^0C is…………… GPa. (up to two decimal places)

Ans. : 0.01×10^{-2}

Solution: $\left(\frac{dP}{dT}\right)_V = \frac{L}{T(v_2 - v_1)} \Rightarrow \int_{P_1}^{P_2} dP = \frac{L}{(v_2 - v_1)} \int_{T_1}^{T_2} \frac{dT}{T} \Rightarrow P_2 - P_1 = \frac{L}{(v_2 - v_1)} \ln \frac{T_2}{T_1}$

$$\Rightarrow P_2 = P_1 + \frac{L}{(v_2 - v_1)} \ln \frac{T_2}{T_1} = 1 \times 10^5 \, Pa = 0.01 \times 10^{-2} \, GPa$$

Q44. Consider N non- interacting, distinguishable particles in a two-level system at temperature T. The energies of the levels are 0 and ε, where $\varepsilon > 0$. In the high temperature limit $(k_B T > \varepsilon)$, what is the population of particles in the level with energy ε?

(a) $\frac{N}{2}$ (b) N (c) $\frac{N}{4}$ (d) $\frac{3N}{4}$

Ans. : (a)

Solution: $P(\varepsilon) = \frac{\exp - \frac{\varepsilon}{kT}}{1 + \exp - \frac{\varepsilon}{kT}}$, population of particle in the level with energy ε is

$$NP(\varepsilon) = N \frac{\exp - \frac{\varepsilon}{kT}}{1 + \exp - \frac{\varepsilon}{kT}}, \text{ for } (k_B T > \varepsilon), \; NP(\varepsilon) = N \frac{\exp - \frac{\varepsilon}{kT}}{1 + \exp - \frac{\varepsilon}{kT}} = N \frac{1}{1+1} = \frac{N}{2}$$

Q45. The energy density and pressure of a photon gas are given by $u = aT^4$ and $P = \frac{u}{3}$. Where T is the temperature and a is the radiation constant. The entropy per unit volume is given by $\alpha a T^3$. The value of α is…………… (up to two decimal places)

Ans. : 1.33

Solution: $TdS = dU + PdV \Rightarrow T\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + P$

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T} = \frac{aT^4}{T} + \frac{aT^4}{3T} = \frac{4}{3} aT^3 = 1.33$$

Q46. Consider two particles and two non-degenerate quantum levels 1 and 2. Level 1 always contains a particle. Hence, what is the probability that level 2 also contains a particle for each of the two cases:

(i) when the two particles are distinguishable and (ii) when the two particles are bosons?

(a) (i) $\frac{1}{2}$ and (ii) $\frac{1}{3}$ (b) (i) $\frac{1}{2}$ and (ii) $\frac{1}{2}$

(c) (i) $\frac{2}{3}$ and (ii) $\frac{1}{2}$ (d) (i) 1 and (ii) 0

Ans. : (c)

Solution: (I): For distinguishable particle: $\frac{\underline{\ B\ }}{\underline{\ A\ }}$ $\frac{\underline{\ A\ }}{\underline{\ B\ }}$ $\frac{\underline{\quad}}{\underline{\ AB\ }}$, $P(2)=\frac{2}{3}$

(II): For indistinguishable particle (Bosons): $\frac{\underline{\ A\ }}{\underline{\ A\ }}$ $\frac{\underline{\quad}}{\underline{\ AA\ }}$, $P(2)=\frac{1}{2}$

GATE-2018

Q47. A microcanonical ensemble consists of 12 atoms with each taking either energy 0 state, or energy $\in$ state. Both states are non-degenerate. If the total energy of this ensemble is $4\in$, its entropy will be ________ k_B (up to one decimal place), where k_B is the Boltzmann constant.

Ans. : 6.204

Solution: The number of ways having total energy $4\in$, out of 12 atom is

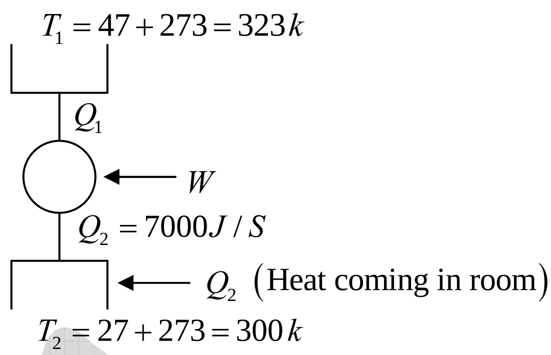
$$= {}^{12}C_4 = \frac{\lfloor 12}{\lfloor 4 \lfloor 8} = \frac{12\times 11\times 10\times 9}{4\times 3\times 2} = 495$$

Hence, entropy, $S = k_B \ln w = k_B \ln(495) = k_B(6.204) = 6.204\ k_B$

Q48. An air-conditioner maintains the room temperature at $27^0 C$ while the outside temperature is $47^0 C$. The heat conducted through the walls of the room from outside to inside due to temperature difference is $7000\ W$. The minimum work done by the compressor of the air-conditioner per unit time is__________ W.

Ans. : 466.67

Solution: $Q_2 + W = Q_1$



Coefficient of performance of refrigerator (AC) $= \dfrac{Q_2}{W}$

Also, coefficient of performance of refrigerator, $= \dfrac{T_2}{T_1 - T_2}$

$$\Rightarrow \frac{300}{47-27} = \frac{7000}{W}$$

$$\Rightarrow W = \frac{7000 \times 20}{300} J/s = \frac{1400}{3} = 466.67\ W$$

Q49. Two solid spheres A and B have same emissivity. The radius of A is four times the radius of B and temperature of A is twice the temperature of B. The ratio of the rate of heat radiated from A to that from B is ___________.

Ans. : 256

Solution: $\dfrac{\text{Rate of heat radiation from solid sphere}(A)}{\text{Rate of heat radiation from solid sphere}(B)} = \dfrac{4\pi R_A^2 T_A^4}{4\pi R_B^2 T_B^4}$

$\because\ R_A = 4R_B$ and $T_A = 2T_B$

$$= \frac{4\pi R_A^2 T_A^4}{4\pi R_B^2 T_B^4} = \frac{(4R_B)^2 \times (2T_B)^4}{(R_B)^2 \times (T_B)^4} = 16 \times 16 = 256$$

Q50. The partition function of an ensemble at a temperature T is

$$Z = \left(2\cosh\frac{\varepsilon}{k_B T}\right)^N$$

where k_B is the Boltzmann constant. The heat capacity of this ensemble at $T = \dfrac{\varepsilon}{k_B}$ is $X\,Nk_B$, where the value of X is ___________ (up to two decimal places).

Ans. : 0.42

Solution: The partition function, $z = \left[2\cosh\left(\dfrac{\varepsilon}{k_B T}\right)\right]^N$

The average energy, $\langle E \rangle = k_B T^2 \dfrac{\partial(\ln z)}{\partial T}$

$$= \frac{Nk_BT^2\left[2\sinh\left(\frac{\varepsilon}{k_BT}\right)\right]\left(\frac{-\varepsilon}{k_BT^2}\right)}{2\cosh\left(\frac{\varepsilon}{k_BT}\right)} = -N\varepsilon\tanh\left(\frac{\varepsilon}{k_BT}\right)$$

$$C = \frac{d\langle E\rangle}{dT} = -N\varepsilon\sec h^2\left(\frac{\varepsilon}{k_BT}\right)\cdot\left(\frac{-\varepsilon}{k_BT^2}\right)$$

At $T = \frac{\varepsilon}{k}$, $C = \frac{N\varepsilon^2}{k\cdot\left(\varepsilon^2/k^2\right)}\sec h^2(1) = Nk\sec h^2(1) = 0.42Nk_B$