Quantum Mechanics

JEST-2012

Q1. The ground state (apart from normalization) of a particle of unit mass moving in a one-dimensional potential V(x) is $\exp\left(-x^2/2\right)\cosh\left(\sqrt{2}x\right)$. The potential V(x), in suitable units so that h = 1, is (up to an addiative constant.)

(a) $\pi^2/2$ (b) $\pi^2/2-\sqrt{2}x\tanh\left(\sqrt{2}x\right)$

(c) $\pi^2/2-\sqrt{2}x\tan\left(\sqrt{2}x\right)$ (d) $\pi^2/2-\sqrt{2}x\coth\left(\sqrt{2}x\right)$

Ans. : (b)

Q2. Consider the Bohr model of the hydrogen atom. If α is the fine-structure constant, the velocity of the electron in its lowest orbit is

(a) $\frac{c}{1+\alpha}$ (b) $\frac{c}{1+\alpha^2}$ $or\,(1-\alpha)c$ (c) $\alpha^2 c$ (d) αc

Ans. : (d)

Solution: $mvr=n\hbar$

$$\frac{mv^2}{r}=\frac{1}{4\pi\in_0}\frac{ze^2}{r^2}\Rightarrow r=\frac{1}{4\pi\in_0}\frac{ze^2}{mr^2}$$

$$mv\cdot\frac{1}{4\pi\in_0}\frac{ze^2}{mv^2}=n\hbar$$

$v=\frac{ze^2}{4\pi\in_0 n\hbar}$ and fine structure constant $\alpha=\frac{e^2}{4\pi\in_0\hbar c}$

For lowest orbit, $v=\frac{ze^2}{4\pi\in_0\hbar}\Rightarrow v=\frac{ze^2\;c}{4\pi\in_0\hbar c}$

$$v=\alpha c$$

Q3. Define $\sigma_x=\left(f^\dagger+f\right)$, and $\sigma_y=-i\left(f^\dagger-f\right)$, where the σ' are Pauli spin matrices and $f,f^\dagger$ obey anti-commutation relations $\{f,f\}=0,\{f,f^\dagger\}=1$. Then σ_z is given by

(a) $f^\dagger f-1$ (b) $2f^\dagger f-1$ (c) $2f^\dagger f+1$ (d) $f^\dagger f$

Ans. : (c)

Solution: $\sigma_x\sigma_y=i\sigma_z$

$$i\sigma_z=\sigma_x\sigma_y$$

$$\sigma_z = \frac{1}{i}\sigma_x\sigma_y = \frac{-i}{i}\left(f^\dagger + f\right)\left(f^\dagger - f\right) = -\left[\left(f^\dagger\right)^2 - f^\dagger f + ff^\dagger - f^2\right]$$

$$= -\left[-f^\dagger f + \left(1 - f^\dagger.\ \ f\right)\right] = -\left[1 - 2f^\dagger f\right] = 2f^\dagger f - 1$$

Q4. Consider a system of two spin-$\frac{1}{2}$ particles with total spin $S = S_1 + S_2$, where S_1 and S_2 are in terms of Pauli matrices σ_i. The spin triplet projection operator is

(a) $\frac{1}{4} + S_1 \cdot S_2$ (b) $\frac{3}{4} - S_1 \cdot S_2$ (c) $\frac{3}{4} + S_1 \cdot S_2$ (d) $\frac{1}{4} - S_1 \cdot S_2$

Ans. : (c)

Solution: $\Rightarrow S = S_1 + S_2$ $\quad S^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2$

$$S^2 = \left(\frac{3}{4} + \frac{3}{4} + 2.S_1 \cdot S_2\right)\hbar^2 \qquad [\because S = 0,\ 1]$$

$$S^2 = 2\left[\frac{3}{4} + S_1 \cdot S_2\right]\hbar^2 \text{ for Triplet projection operator}$$

$$s(s+1)\hbar^2 = 2\left[\frac{3}{4} + S_1 \cdot S_2\right]\hbar^2 \qquad S = 1$$

$$1(1+1) = 2\left(\frac{3}{4} + S_1 \cdot S_2\right) \qquad \Rightarrow \frac{3}{4} + S_1 \cdot S_2 = I$$

Q5. Consider a spin-$\frac{1}{2}$ particle in the homogeneous magnetic field of magnitude B along z-axis which is prepared initially in a state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$ at time $t = 0$. At what time t will the particles be in the state $-|\psi\rangle$ (μ_B is Bohr magneton)?

(a) $t = \frac{\pi\hbar}{\mu_B B}$ (b) $t = \frac{2\pi\hbar}{\mu_B B}$ (c) $t = \frac{\pi\hbar}{2\mu_B B}$ (d) Never

Ans.: (a)

Solution: $\vec{E} = \mu_B \cdot B\hat{z}$ $\ |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$|\psi(x,t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}e^{-\frac{iEt}{\hbar}} \Rightarrow |\psi(x,t)\rangle = -|\psi\rangle$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}e^{\frac{-i\mu_B Bt}{\hbar}}=\frac{-1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$e^{\frac{-i\mu_B Bt}{\hbar}}=-1$$

$$\cos\left(\frac{\mu_B Bt}{\hbar}\right)=\cos\pi$$

$$\frac{\mu_B Bt}{\hbar}=\pi\Rightarrow t=\frac{\hbar\pi}{\mu_B B}$$

Q6. The ground state energy of 5 identical spin-$\frac{1}{2}$ particles which are subject to a one-dimensional simple harmonic oscillator potential of frequency ω is

(a) $\frac{15}{2}\hbar\omega$ (b) $\frac{13}{2}\hbar\omega$ (c) $\frac{1}{2}\hbar\omega$ (d) $5\hbar\omega$

Ans. : (b)

Solution: Degeneracy $=2s+1=2\times\frac{1}{2}+1=2$

$$E_{ground}=2\times\frac{1}{2}\hbar\omega+2\times\frac{3}{2}\hbar\omega+1\times\frac{5}{2}\hbar\omega=\frac{13}{2}\hbar\omega$$

Q7. The spatial part of a two-electron state is symmetric under exchange. If $|\uparrow\rangle$ and $|\downarrow\rangle$ represent the spin-up and spin-down states respectively of each particle, the spin-part of the two-particle state is

(a) $|\uparrow\rangle|\downarrow\rangle$ (b) $|\downarrow\rangle|\uparrow\rangle$

(c) $(|\downarrow\rangle|\uparrow\rangle-|\uparrow\rangle|\downarrow\rangle)/\sqrt{2}$ (d) $(|\downarrow\rangle|\uparrow\rangle+|\uparrow\rangle|\downarrow\rangle)/\sqrt{2}$

Ans. : (c)

Solution: Since, electrons are Fermions and Fermions have anti-symmetric wave function

$\because$ spatial part is symmetric then its spin part is antisymmetric to maintain antisymmtric wave function

$$\psi(x)=\frac{1}{\sqrt{2}}(|\downarrow\rangle|\uparrow\rangle-|\uparrow\rangle|\downarrow\rangle)$$

Q8. The wave function of a free particle in one dimension is given by $\psi(x) = A\sin x + B\sin 3x$. Then $\psi(x)$ is an eigenstate of

(a) the position operator
(b) the Hamiltonian
(c) the momentum operator
(d) the parity operator

Ans. : (d)

Solution: $\psi(-x) = \psi(x)$

$= -\psi(x)$ {parity (even and odd)

$\psi(-x) = A\sin(-x) + B\sin(-3x) = -[A\sin x + B\sin 3x]$

$\psi(-x) = -\psi(x) \Rightarrow$ negative parity i.e. parity operator

Q9. The quantum state $\sin x|\uparrow\rangle + \exp(i\phi)\cos x|\downarrow\rangle$, where $\langle\uparrow|\downarrow\rangle = 0$ and x, ϕ are, real, is orthogonal to:

(a) $\sin x|\uparrow\rangle$
(b) $\cos x|\uparrow\rangle + \exp(i\phi)\sin x|\downarrow\rangle$
(c) $-\cos x|\uparrow\rangle - \exp(i\phi)\sin x|\downarrow\rangle$
(d) $-\exp(-i\phi)\cos x|\uparrow\rangle + \sin x|\downarrow\rangle$

Ans.: (d)

Solution: $\langle\uparrow||\downarrow\rangle = 0$, $|\psi\rangle = \sin x|\uparrow\rangle + \exp(i\phi)\cos x|\downarrow\rangle$

$$\langle\psi'|\psi\rangle = -\exp(i\phi)\cos x\sin x\langle\uparrow|\uparrow\rangle - \exp(i\phi)\exp(i\phi)\cos x\langle\downarrow|\uparrow\rangle$$
$$+\sin^2 x\langle\downarrow|\uparrow\rangle + \exp(i\phi)\cos x\sin x\langle\downarrow|\downarrow\rangle$$

$= -\exp(i\phi)\cos x\sin x + \exp(i\phi)\cos x\sin x = 0$

JEST-2013

Q10. A particle of mass m is contained in a one-dimensional infinite well extending from $x=-\frac{L}{2}$ to $x=\frac{L}{2}$. The particle is in its ground state given by $\varphi_0(x)=\sqrt{2/L}\cos(\pi x/L)$. The walls of the box are moved suddenly to form a box extending from $x=-L$ to $x=L$. what is the probability that the particle will be in the ground state after this sudden expansion?

(a) $(8/3\pi)^2$ (b) 0 (c) $(16/3\pi)^2$ (d) $(4/3\pi)^2$

Ans.: (a)

Solution: Probability $|\langle\phi_0|\phi_1\rangle|^2, \phi_0=\sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}, \phi_1\sqrt{\frac{2}{2L}}\cos\frac{\pi x}{2L}$

Since the wall of box are moved suddenly then

$$\text{Probability} = \left|\int_{-L/2}^{L/2}\sqrt{\frac{2}{L}}\cdot\sqrt{\frac{1}{L}}\frac{\cos\pi x}{L}\cdot\frac{\cos\pi x}{2L}dx\right|^2 = \left|\frac{\sqrt{2}}{L}\frac{1}{2}\int_{-L/2}^{L/2}\frac{2\cos\pi x}{L}\cdot\frac{\cos\pi x}{2L}dx\right|^2$$

$$\Rightarrow\left|\frac{\sqrt{2}}{L}\cdot\frac{1}{2}\int_{-L/2}^{L/2}\left[\cos\left(\frac{3\pi x}{2L}\right)+\cos\left(\frac{\pi x}{2L}\right)\right]dx\right|^2 \Rightarrow\left|\frac{\sqrt{2}}{L}\cdot\frac{1}{2}\left[\frac{2L}{3\pi}\sin\frac{3\pi x}{2L}+\frac{2L}{\pi}\sin\frac{\pi x}{2L}\right]_{-L/2}^{L/2}\right|^2$$

$$\Rightarrow\left|\frac{\sqrt{2}}{L}\cdot\frac{1}{2}\left[\frac{2L}{3\pi}\left(\sin\frac{3\pi}{4}+\sin\frac{3\pi}{4}\right)+\frac{2L}{\pi}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{4}\right)\right]\right|^2 \Rightarrow\left|\frac{2}{3\pi}+\frac{2}{\pi}\right|^2=\left|\frac{8}{3\pi}\right|^2$$

Q11. A quantum mechanical particle in a harmonic oscillator potential has the initial wave function $\psi_0(x)+\psi_1(x)$, where ψ_0 and ψ_1 are the real wavefunctions in the ground and first excited state of the harmonic oscillator Hamiltonian. For convenience we take $m=\hbar=\omega=1$ for the oscillator. What is the probability density of finding the particle at x at time $t=\pi$?

(a) $(\psi_1(x)-\psi_0(x))^2$ (b) $(\psi_1(x))^2-(\psi_0(x))^2$

(c) $(\psi_1(x)+\psi_0(x))^2$ (d) $(\psi_1(x))^2+(\psi_0(x))^2$

Ans.: (a)

Solution: $\psi(x)=\psi_0(x)+\psi_1(x)$

$$\psi(x,t)=\psi_0(x)e^{-i\frac{E_0 t}{\hbar}}+\psi_1(x)e^{-i\frac{E_1 t}{\hbar}}$$

Now probability density at time t

$$|\psi(x,t)|^2=\psi^*(x,t)\psi(x,t)=|\psi_0(x)|^2+|\psi_1(x)|^2+2\operatorname{Re}\psi_0^*(x)\psi_1(x)\cos(E_1-E_0)\frac{t}{\hbar}$$

putting $t=\pi$

$$|\psi(x,t)|^2=|\psi_0(x)|^2+|\psi_1(x)|^2+2\operatorname{Re}\psi_0^*(x)\psi_1(x)\cos\pi \qquad [\because E_1-E_0=\hbar\omega=1]$$

$$|\psi(x,t)|^2=|\psi_0(x)|^2+|\psi_1(x)|^2-2\operatorname{Re}\psi_0^*(x)\psi_1(x)=[\psi_1(x)-\psi_0(x)]^2$$

Q12. If J_x, J_y and J_z are angular momentum operators, the eigenvalues of the operator $\frac{(J_x+J_y)}{\hbar}$ are:

(a) real and discrete with rational spacing

(b) real and discrete with irrational spacing

(c) real and continuous

(d) not all real

Ans.: (b)

Solution: $J_x=\frac{1}{2}(J_+ + J_-)$, $J_y=\frac{i}{2}(J_- - J_+)\Rightarrow J_+=\hbar\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$, $J_-=\hbar\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$

$$J_x=\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad J_y=\frac{i\hbar}{2}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\Rightarrow\frac{J_x+J_y}{\hbar}=\frac{1}{2}\begin{bmatrix}0 & 1-i\\ 1+i & 0\end{bmatrix}$$

eigen value $\frac{1}{2}\begin{pmatrix}-\lambda & 1-i\\ 1+i & -\lambda\end{pmatrix}\Rightarrow\lambda^2-2=0\Rightarrow\lambda=\pm\sqrt{2}$

Q13. A simple model of a helium-like atom with electron-electron interaction is replaced by Hooke's law force is described by Hamiltonian

$$\frac{-\hbar^2}{2m}(\nabla_1^2+\nabla_2^2)+\frac{1}{2}m\omega^2(r_1^2+r_2^2)-\frac{\lambda}{4}\ m\omega^2|\vec{r}_1-\vec{r}_2|^2.$$

What is the exact ground state energy?

(a) $E=\frac{3}{2}\hbar\omega\left(1+\sqrt{1+\lambda}\right)$

(b) $E=\frac{3}{2}\hbar\omega\left(1+\sqrt{\lambda}\right)$

(c) $E=\frac{3}{2}\hbar\omega\sqrt{1-\lambda}$

(d) $E=\frac{3}{2}\hbar\omega\left(1+\sqrt{1-\lambda}\right)$

Ans.: (b)

Q14. Consider the state $\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$ corresponding to the angular momentum $l=1$ in the L_z basis of states with $m=+1,\ 0,\ -1$. If L_z^2 is measured in this state yielding a result 1, what is the state after the measurement?

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1/\sqrt{3} \\ 0 \\ \sqrt{2/3} \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

Ans.: (d)

Solution: $L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Corresponding eigenvalue 1, 0, 1

Now state after measurement yielding $1 \Rightarrow |\phi_1\rangle + |\phi_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Q15. What are the eigenvalues of the operator $H = \vec{\sigma}\cdot\vec{a}$, where $\vec{\sigma}$ are the three Pauli matrices and $\vec{a}$ is a vector?

(a) $a_x + a_y$ and a_z (b) $a_x + a_z \pm i a_y$ (c) $\pm(a_x + a_y + a_z)$ (d) $\pm|\vec{a}|$

Ans.: (d)

Solution: $H = \vec{\sigma}\cdot\vec{a} = (\sigma_x.a_x + \sigma_y.a_y + \sigma_z.a_z)$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} a_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} a_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a_z = \begin{pmatrix} a_z & (a_x - i a_y) \\ (a_x + i a_y) & -a_z \end{pmatrix}$$

For eigen value,

$$\begin{pmatrix} (a_z - \lambda) & (a_x - i a_y) \\ (a_x + i a_y) & -(a_z + \lambda) \end{pmatrix} = 0 \Rightarrow -(a_z - \lambda)(a_z + \lambda) - (a_x - i a_y)(a_x + i a_y) = 0$$

$$\Rightarrow -a_z^2 + \lambda^2 - a_x^2 - a_y^2 = 0 \Rightarrow \lambda^2 = a_x^2 + a_y^2 + a_z^2 \Rightarrow \lambda = \pm|\vec{a}|$$

Q16. The hermitian conjugate of the operator $\left(\frac{-\partial}{\partial x}\right)$ is

(a) $\frac{\partial}{\partial x}$ (b) $-\frac{\partial}{\partial x}$ (c) $i\frac{\partial}{\partial x}$ (d) $-i\frac{\partial}{\partial x}$

Ans.: (a)

Solution: $\Rightarrow\left(\psi^*(x)-\frac{\partial}{\partial x}\psi(x)\right)^{\dagger}=\left(\frac{-\partial\psi^*(x)}{\partial x}\psi(x)\right)$

$$\Rightarrow\int_{-\infty}^{\infty}\psi^*(x)\left[-\frac{\partial}{\partial x}\psi(x)\right]dx=\psi^*(x)\psi(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}-\frac{\partial\psi^*(x)}{\partial x}\psi(x)dx$$

$$=\int_{-\infty}^{\infty}\frac{\partial\psi^*(x)}{\partial x}\psi(x)dx$$

Q17. If the expectation value of the momentum is $\langle p\rangle$ for the wavefunction $\psi(x)$, then the expectation value of momentum for the wavefunction $e^{ikx/\hbar}\psi(x)$ is

(a) k (b) $\langle p\rangle-k$ (c) $\langle p\rangle+k$ (d) $\langle p\rangle$

Ans.: (c)

Solution: $\int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)dx=\langle p\rangle$

Now

$$\int_{-\infty}^{\infty}e^{-\frac{ikx}{\hbar}}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)e^{\frac{ikx}{\hbar}}\psi(x)dx\Rightarrow\int_{-\infty}^{\infty}e^{\frac{-ikx}{\hbar}}\psi^*(x)(-i\hbar)\left[e^{\frac{ikx}{\hbar}}\frac{\partial}{\partial x}\psi(x)+\frac{ik}{\hbar}e^{\frac{ikx}{\hbar}}\psi(x)\right]$$

$$\Rightarrow\int_{-\infty}^{\infty}e^{-\frac{ikx}{\hbar}}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)e^{\frac{ikx}{\hbar}}+\int_{-\infty}^{\infty}-i\hbar.\frac{ik}{\hbar}e^{\frac{-ikx}{\hbar}}\psi^*(x)\psi(x)dx$$

$$\Rightarrow\int_{-\infty}^{\infty}\psi^*(x)\left[-i\hbar\frac{\partial}{\partial x}\psi(x)\right]+k\int_{-\infty}^{\infty}\psi^*(x)\psi(x)\Rightarrow\langle p\rangle+K$$

Q18. Two electrons are confined in a one dimensional box of length L. The one-electron states are given by $\psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$. What would be the ground state wave function $\psi(x_1,x_2)$ if both electrons are arranged to have the same spin state?

(a) $\psi(x_1,x_2)=\frac{1}{\sqrt{2}}\left[\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)+\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]$

(b) $\psi(x_1,x_2)=\frac{1}{\sqrt{2}}\left[\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)-\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]$

(c) $\psi(x_1,x_2)=\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)$

(d) $\psi(x_1,x_2)=\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)$

Ans.: (b)

Solution: Electrons are Fermions of spin $\frac{1}{2}$ and its wave functions are anti-symmetric

Since, spin part is symmetric, therefore, space part will be anti-symmetric (since as total wave function is anti-symmetric)

Then,

$$\psi(x_1,x_2)=\frac{1}{\sqrt{2}}\left[\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right).\sin\left(\frac{2\pi x_2}{L}\right)-\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right).\sin\left(\frac{\pi x_2}{L}\right)\right]$$

Q19. The operator $\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)$ is equivalent to

(a) $\frac{d^2}{dx^2}-x^2$

(b) $\frac{d^2}{dx^2}-x^2+1$

(c) $\frac{d^2}{dx^2}-x\frac{d}{dx}x^2+1$

(d) $\frac{d^2}{dx^2}-2x\frac{d}{dx}-x^2$

Ans.: (b)

Solution: $\Rightarrow\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)f(x)\Rightarrow\left(\frac{d}{dx}-x\right)\left[\frac{d}{dx}f(x)+xf(x)\right]$

$$\Rightarrow \frac{d}{dx}\left[\frac{d}{dx}f(x)+xf(x)\right]-x\frac{d}{dx}f(x)-x^2 f(x)$$

$$\Rightarrow \frac{d^2}{dx^2}f(x)+f(x)+x\frac{df(x)}{dx}-x\frac{d}{dx}f(x)-x^2 f(x)$$

$$\Rightarrow \frac{d^2}{dx^2}f(x)-x^2 f(x)+f(x)=\left(\frac{d^2}{dx^2}-x^2+1\right)f(x)$$

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Q20. Suppose a spin $1/2$ particle is in the state

$$|\psi\rangle = \frac{1}{\sqrt{6}}\begin{bmatrix}1+i\\2\end{bmatrix}$$

If S_x (x component of the spin angular momentum operator) is measured what is the probability of getting $+\hbar/2$?

(a) $1/3$ (b) $2/3$ (c) $5/6$ (d) $1/6$

Ans.: (c)

Solution: $S_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ with eigenvalues $\pm\frac{\hbar}{2}$ and eigenvector corresponding to $\frac{\hbar}{2}$ is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

Now probability getting $+\frac{\hbar}{2}$

$$p\left(\frac{\hbar}{2}\right) = \frac{|\langle\phi|\psi\rangle|}{\langle\psi|\psi\rangle} = \frac{\left|\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{6}}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1+i\\2\end{bmatrix}\right|^2}{\frac{1}{6}\begin{bmatrix}1-i & 2\end{bmatrix}\begin{bmatrix}1+i\\2\end{bmatrix}} = \frac{\frac{1}{12}|1+i+2|^2}{6\times\frac{1}{6}} = \frac{5}{6}$$

Q21. The Hamiltonian operator for a two-state system is given by

$$H = \alpha\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right),$$

where α is a positive number with the dimension of energy. The energy eigenstates corresponding to the larger and smaller eigenvalues respectively are:

(a) $|1\rangle - \left(\sqrt{2}+1\right)|2\rangle,\ |1\rangle + \left(\sqrt{2}-1\right)|2\rangle$

(b) $|1\rangle + \left(\sqrt{2}-1\right)|2\rangle,\ |1\rangle - \left(\sqrt{2}+1\right)|2\rangle$

(c) $|1\rangle + \left(\sqrt{2}-1\right)|2\rangle,\ \left(\sqrt{2}+1\right)|1\rangle - |2\rangle$

(d) $|1\rangle - \left(\sqrt{2}+1\right)|2\rangle,\ \left(\sqrt{2}-1\right)|1\rangle + |2\rangle$

Ans.: (b)

Solution: $H = \alpha\left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1|\right) \Rightarrow H|1\rangle = \alpha\left(|1\rangle + |2\rangle\right),\ H|2\rangle = \alpha\left(|1\rangle - |2\rangle\right)$

Lets check for option (b): $|1\rangle + \left(\sqrt{2}-1\right)|2\rangle,\ |1\rangle - \left(\sqrt{2}+1\right)|2\rangle$

Now $H|\psi\rangle = \alpha|\psi\rangle \Rightarrow H\left[|1\rangle + \left(\sqrt{2}-1\right)|2\rangle\right] = H|1\rangle + H\left(\sqrt{2}+1\right)|2\rangle$

$H\left[|1\rangle+\left(\sqrt{2}-1\right)|2\rangle\right] \Rightarrow H\left(|1\rangle\right)+\left(\sqrt{2}-1\right)H|2\rangle \Rightarrow \alpha\left(|1\rangle+|2\rangle\right)+\left(\sqrt{2}-1\right)\alpha\left(|1\rangle-|2\rangle\right)$

$\Rightarrow \alpha\left[1+\sqrt{2}-1\right]|1\rangle+\alpha\left[1-\left(\sqrt{2}-1\right)\right]|2\rangle \Rightarrow \alpha\sqrt{2}|1\rangle+\alpha\left(2-\sqrt{2}\right)|2\rangle$

$\Rightarrow \alpha\sqrt{2}\left[|1\rangle+\left(\sqrt{2}-1\right)|2\rangle\right]$

Now $H\left(|1\rangle-\sqrt{2}+1\right)|2\rangle \Rightarrow H\left[|1\rangle-\left(\sqrt{2}+1\right)|2\rangle\right] \Rightarrow H|1\rangle-H\left(\sqrt{2}+1\right)|2\rangle$

$\Rightarrow \alpha\left(|1\rangle+|2\rangle\right)-\alpha\left[\left(\sqrt{2}+1\right)\left(|1\rangle-|2\rangle\right)\right] \Rightarrow \alpha\left(1-\sqrt{2}-1\right)|1\rangle+\alpha\left(1+\sqrt{2}+1\right)|2\rangle$

$\Rightarrow -\sqrt{2}\alpha|1\rangle+\left(2+\sqrt{2}\right)\alpha|2\rangle \Rightarrow -\alpha\sqrt{2}\left[|1\rangle-\left(1+\sqrt{2}\right)|2\rangle\right]$

Q22. Consider an eigenstate of $\vec{L}^2$ and L_z operator denoted by $|l, m\rangle$. Let $A=\hat{n}\cdot\vec{L}$ denote an operator, where $\hat{n}$ is a unit vector parametrized in terms of two angles as $\left(n_x,n_y,n_z\right)=\left(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\right)$. The width ΔA in $|l, m\rangle$ state is:

(a) $\sqrt{\frac{l(l+1)-m^2}{2}}\hbar\cos\theta$ (b) $\sqrt{\frac{l(l+1)-m^2}{2}}\hbar\sin\theta$

(c) $\sqrt{l(l+1)-m^2}\hbar\sin\theta$ (d) $\sqrt{l(l+1)-m^2}\hbar\cos\theta$

Ans.: (c)

Solution: $A=\hat{n}\cdot\vec{L} \Rightarrow A=L_x\cdot\frac{x}{r}+L_y\cdot\frac{y}{r}+L_z\cdot\frac{z}{r}$

$\Rightarrow A=L_x\cdot\frac{r\sin\theta\cos\phi}{r}+L_y\cdot\frac{r\sin\theta\ \ \sin\phi}{r}+L_z\cdot\frac{r\cos\theta}{r}$

$\Rightarrow A=L_x\sin\theta\cos\phi+L_y\sin\theta\cdot\sin\phi+L_z\cos\theta$

Now $\Delta A=\sqrt{\left\langle A^2\right\rangle-\left\langle A\right\rangle^2}$

$\left\langle A\right\rangle=\left\langle L_x\right\rangle\sin\theta\cos\phi+\left\langle L_y\right\rangle\sin\theta\sin\phi+\left\langle L_z\right\rangle\cos\theta$

$\left\langle A\right\rangle=\left(m\hbar\right)\cos\theta \qquad \because \left\langle L_x\right\rangle=0,\left\langle L_y\right\rangle=0$

$\left\langle A^2\right\rangle=\left\langle L_x^2\right\rangle\sin^2\theta\cos^2\phi+\left\langle L_y^2\right\rangle\sin^2\theta\sin^2\phi+\left\langle L_z^2\right\rangle\cos^2\theta$

$=\left(\left\langle L_x^2\right\rangle+\left\langle L_y^2\right\rangle\right)\sin^2\theta+\left\langle L_z^2\right\rangle\cos^2\theta$

$$=\left(\langle L^2\rangle-\langle L_z^2\rangle\right)\sin^2\theta-\langle L_z^2\rangle\cos^2\theta$$

$$\Rightarrow\langle A^2\rangle=\left[l(l+1)-m^2\right]\hbar^2\sin^2\theta+m^2\hbar^2\cos^2\theta$$

$$\Rightarrow\langle A^2\rangle=\left[l(l+1)-m^2\right]\hbar^2\sin^2\theta+m^2\hbar^2\cos^2\theta$$

$$\Delta A=\sqrt{\langle A^2\rangle-\langle A\rangle^2}=\sqrt{\left(l(l+1)-m^2\right)\hbar^2\sin^2\theta+m^2\hbar^2\cos^2\theta-m^2\hbar^2\cos^2\theta}$$

$$\Delta A=\sqrt{\left[l(l+1)-m^2\right]}\hbar\sin\theta$$

Q23. Consider a three-state system with energies E, E and $E-3g$ (where g is a constant) and respective eigenstates $|\psi_1\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $|\psi_2\rangle=\frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$ and $|\psi_3\rangle=\frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

If the system is initially (at $t=0$), in state $|\psi_i\rangle=\begin{pmatrix}1\\0\\0\end{pmatrix}$

what is the probability that at a later time t system will be in state $|\psi_f\rangle=\begin{pmatrix}0\\0\\1\end{pmatrix}$

(a) 0

(b) $\frac{4}{9}\sin^2\left(\frac{3gt}{2\hbar}\right)$

(c) $\frac{4}{9}\cos^2\left(\frac{3gt}{2\hbar}\right)$

(d) $\frac{4}{9}\sin^2\left(\frac{E-3gt}{2\hbar}\right)$

Ans.: (b)

Q24. The lowest quantum mechanical energy of a particle confined in a one-dimensional box of size L is $2eV$. The energy of the quantum mechanical ground state for a system of three non-interacting spin $\frac{1}{2}$ particles is

(a) $6eV$ (b) $10eV$ (c) $12eV$ (d) $16eV$

Ans.: (c)

Solution: $E_1 = \frac{\pi^2\hbar^2}{2ml^2} = 2eV$, $E_2 = 4E_1 = 8\ eV$

Spin, spin is $\frac{1}{2}$, therefore, degeneracy $g_i = 2S+1 = 2\times\frac{1}{2}+1 = 2$

$\Rightarrow$ ground state energy $= 2\times 2\ eV + 1\times 8eV = 12\ eV$

Q25. A ball bounces off earth. You are asked to solve this quantum mechanically assuming the earth is an infinitely hard sphere. Consider surface of earth as the origin implying $V(0) = \infty$ and a linear potential elsewhere (i.e. $V(x) = -mgx$ for $x > 0$). Which of the following wave functions is physically admissible for this problem (with $k > 0$):

(a) $\psi = e^{-kx}/x$ (b) $\psi = xe^{-kx^2}$ (c) $\psi = -Axe^{kx}$ (d) $\psi = Ae^{-kx^2}$

Ans.: (b)

Solution: $\psi = xe^{-kx^2}$

For given potential, at $x = 0$ and $x = \infty$ wave function must vanish.

Q26. The operator A and B share all the eigenstates. Then the least possible value of the product of uncertainties $\Delta A \Delta B$ is

(a) $\hbar$ (b) 0 (c) $\hbar/2$ (d) Determinant (AB)

Ans.: (b)

Solution: $\Delta A \cdot \Delta B \geq \left|\frac{[AB]}{2}\right|$

$\Delta A \cdot \Delta B \geq 0$ [$\because A$ and B have share their eigen values, so $[AB] = 0$]

Q27. Consider a square well of depth $-V_0$ and width a with V_0 as fixed. Let $V_0 \to \infty$ and $a \to 0$. This potential well has

(a) No bound states (b) 1 bound state

(c) 2 bound states (d) Infinitely many bound states

Ans.: (b)

Solution: It forms delta potential, so it has only one bound state.

JEST-2015

Q28. Consider a harmonic oscillator in the state $|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^+}|0\rangle$, where $|0\rangle$ is the ground state, a^+ is the raising operator and α is a complex number. What is the probability that the harmonic oscillator is in the nth eigenstate $|n\rangle$?

(a) $e^{-|\alpha^2|}\frac{|\alpha|^{2n}}{n!}$ (b) $e^{-\frac{|\alpha|^2}{2}\frac{|\alpha|^n}{\sqrt{n!}}}$ (c) $e^{-|\alpha|^2}\frac{|\alpha|^n}{n!}$ (d) $e^{-\frac{|\alpha|^2}{2}}\frac{|\alpha|^{2n}}{n!}$

Ans.: (a)

Solution: $|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^+}|0\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{(\alpha a^+)^n}{\lfloor n}|0\rangle$ and $|n\rangle = \frac{(a^+)^n}{\sqrt{\lfloor n}}|0\rangle \Rightarrow (a^+)^n|0\rangle = \sqrt{\lfloor n}|n\rangle$

$$|\psi\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{(\alpha)^n\sqrt{\lfloor n}}{\lfloor n}|n\rangle \Rightarrow \langle\psi|\psi\rangle = e^{-|\alpha|^2}\sum\frac{(\alpha^*\alpha)^n\lfloor n}{(\lfloor n)^2}\langle n|n\rangle = e^{-|\alpha|^2}\sum_n\frac{|\alpha|^n}{\lfloor n} = e^{-|\alpha|^2}e^{|\alpha|^2} = 1$$

Probability that $|\psi\rangle$ is in $|n\rangle$ state is, $\frac{|\langle n|\psi\rangle|^2}{\langle\psi|\psi\rangle} = |\langle n|\psi\rangle|^2$

$$|\psi\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{(\alpha)^n\sqrt{\lfloor n}}{\lfloor n}|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \alpha^n\frac{1}{\sqrt{\lfloor n}}|n\rangle$$

$$\Rightarrow \langle n|\psi\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \alpha^n\frac{1}{\sqrt{\lfloor n}}\langle n|n\rangle = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{\lfloor n}}\alpha^n \Rightarrow |\langle n|\psi\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{\lfloor n}$$

Q29. A particle of mass m moves in $1-$dimensional potential $V(x)$, which vanishes at infinity. The exact ground state eigenfunction is $\psi(x) = A\sec h(\lambda x)$, where A and λ are constants. The ground state energy eigenvalue of this system is,

(a) $E = \frac{\hbar^2\lambda^2}{m}$ (b) $E = -\frac{\hbar^2\lambda^2}{m}$ (c) $E = -\frac{\hbar^2\lambda^2}{2m}$ (d) $E = \frac{\hbar^2\lambda^2}{2m}$

Ans.: (d)

Solution: $\because \psi(x) = A\sec h(\lambda x) \Rightarrow \frac{d\psi}{dx} = -A\lambda\sec h(\lambda x)\tanh(\lambda x)$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -A\lambda\left[-\sec h(\lambda x)\tan^2 h(\lambda x)\lambda + \lambda\sec h(\lambda x)\sec^2 h(\lambda x)\right]$$

$$= -A\lambda^2\left[\sec h(\lambda x)\left[-\tan^2 h(\lambda x) + \sec^2 h(\lambda x)\right]\right]$$

$$= -A\lambda^2\left[\sec h(\lambda x)\left[\sec^2 h(\lambda x) - \tan^2 h(\lambda x)\right]\right]$$

$$= -A\lambda^2\left[\sec h(\lambda x)\left[\sec^2 h(\lambda x) - \left[1 - \sec^2 h(\lambda x)\right]\right]\right]$$

$$\because \tan^2 h(\lambda x) = 1 - \sec^2 h(\lambda x)$$

$$= -A\lambda^2\left[\sec h(\lambda x)\left[\sec^2 h(\lambda x) - 1 + \sec^2 h(\lambda x)\right]\right]$$

$$\Rightarrow \frac{d^2\psi}{dx^2} = -A\lambda^2\left[2\sec^3 h(\lambda x) - \sec h(\lambda x)\right]$$

Now put the value $\frac{d^2\psi}{dx^2}$ in equation $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$

$$-\frac{\hbar^2}{2m}\lambda^2 A\left[2\sec^3 h(\lambda x) - \sec h(\lambda x)\right] + V(x)A\sec h(\lambda x) = EA\sec h(\lambda x)$$

$$\because V(x) \to 0 \quad as \quad x \to \infty$$

$$\Rightarrow +\frac{\hbar^2}{2m}\lambda^2 A\sec h(\lambda x) - \frac{\hbar^2\lambda^2}{2m}2A\sec^3 h(\lambda x) = EA\sec h(\lambda x)$$

Now we have to do approximation i.e. $\sec^3 h(\lambda x)$ dacays very fastly as $x \to \infty$ so second term

$\frac{\hbar^2\lambda^2}{2m}2A\sec^3 h(\lambda x) = 0$. Thus $\frac{\hbar^2\lambda^2}{2m}A\sec h(\lambda x) = EA\sec h(\lambda x) \Rightarrow E = \frac{\hbar^2\lambda^2}{2m}$

Q30. Consider a spin $-\frac{1}{2}$ particle characterized by the Hamiltonian $H = \omega S_z$. Under a perturbation $H' = gS_x$, the second order correction to the ground state energy is given by,

(a) $-\frac{g^2}{4\omega}$ (b) $\frac{g^2}{4\omega}$ (c) $-\frac{g^2}{2\omega}$ (d) $\frac{g^2}{2\omega}$

Ans.: (a)

Solution: $\because H = \omega s_z$ *and* $s_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow H = \frac{\omega\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $H' = gs_x = \frac{g\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

Ground state energy is $-\frac{\omega\hbar}{2}$ with eigenvector $|\phi_1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

and first excited state energy is $\frac{\omega\hbar}{2}$ with eigenvector $|\phi_2\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

Second order correction in ground state $E_1^2 = \sum_{m\neq 1}\frac{|\langle\phi_m|H'|\phi_1\rangle|^2}{E_1^0 - E_m^0} = \frac{|\langle\phi_m|H'|\phi_1\rangle|^2}{-\frac{\omega\hbar}{2} - \frac{\omega\hbar}{2}}$

$$\Rightarrow E_1^2 = \frac{g^2\hbar^2}{4}\frac{\left|\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right|^2}{-\frac{2\omega\hbar}{2}} = -\frac{g^2\hbar^2}{4\omega\hbar} = -\frac{g^2}{4\omega}\hbar$$

Q31. Given that ψ_1 and ψ_2 are eigenstates of a Hamiltonian with eigenvalues E_1 and E_2 respectively, what is the energy uncertainty in the state $(\psi_1 + \psi_2)$?

(a) $-\sqrt{E_1E_2}$

(b) $\frac{1}{2}|E_1 - E_2|$

(c) $\frac{1}{2}(E_1 + E_2)$

(d) $\frac{1}{\sqrt{2}}|E_2 - E_1|$

Ans.: (b)

Solution: $\langle E^2\rangle = \frac{1}{2}E_1^2 + \frac{1}{2}E_2^2 = \frac{(E_1^2 + E_2^2)}{2}$ and $\langle E\rangle = \frac{1}{2}E_1 + \frac{1}{2}E_2$

$$\because \Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2} = \sqrt{\frac{(E_1^2 + E_2^2)}{2} - \frac{1}{4}(E_1 + E_2)^2} = \sqrt{\frac{2E_1^2 + 2E_2^2 - E_1^2 - E_2^2 - 2E_1E_2}{4}}$$

$$\Rightarrow \Delta E = \sqrt{\frac{E_1^2 + E_2^2 - 2E_1E_2}{4}} = \frac{1}{2}|E_1 - E_2|$$

Q32. A particle moving under the influence of a potential $V(r)=\frac{kr^2}{2}$ has a wavefunction $\psi(r,t)$. If the wavefunction changes to $\psi(\alpha r,t)$, the ratio of the average final kinetic energy to the initial kinetic energy will be,

(a) $\frac{1}{\alpha^2}$ (b) α (c) $\frac{1}{\alpha}$ (d) α^2

Ans.: (c)

Solution: For $\psi(r,t)$ the average kinetic energy $\langle T\rangle=\int_0^\infty \psi^*(r,t)\left(-\frac{\hbar^2}{2m}\right)\left(\nabla^2\psi\right)r^2dr$, ∇^2 is written in spherical polar coordinate, which is dimension of $(\text{length})^{-2}$

For wave function $\psi(\alpha r,t)$

$$\langle T_\alpha\rangle=\int_0^\infty \psi^*(\alpha r,\, t)\left(-\frac{\hbar^2}{2m}\right)\left(\nabla^2\psi(\alpha r,\, t)\right)r^2dr$$

Put $\alpha r=r'$ or $r=\frac{r'}{\alpha}\Rightarrow dr=\frac{dr'}{\alpha}$ and $\nabla_r^2=\alpha^2\nabla_{r'}^2$

$$\langle T_\alpha\rangle=\frac{\alpha^2}{\alpha^3}\int_0^\infty \psi^*(r',t)\left(-\frac{\hbar^2}{2m}\right)\nabla^2\psi(r',t)r'^2dr'=\frac{1}{\alpha}\int_0^\infty \psi^*(r',t)\left(-\frac{\hbar^2}{2m}\right)\nabla^2\psi(r',t)r'^2dr'$$

$$\Rightarrow\langle T_\alpha\rangle=\frac{\langle T\rangle}{\alpha}\Rightarrow\frac{\langle T_\alpha\rangle}{\langle T\rangle}=\frac{1}{\alpha}$$

Q33. If a Hamiltonian H is given as $H=|0\rangle\langle 0|-|1\rangle\langle 1|+i(|0\rangle\langle 1|-|1\rangle\langle 0|)$, where $|0\rangle$ and $|1\rangle$ are orthonormal states, the eigenvalues of H are

(a) ± 1 (b) $\pm i$ (c) $\pm\sqrt{2}$ (d) $\pm i\sqrt{2}$

Ans: (c)

Solution: $H=|0\rangle\langle 0|-|1\rangle\langle 1|+i(|0\rangle\langle 1|-|1\rangle\langle 0|)$

$$H|0\rangle=|0\rangle-i|1\rangle \quad \text{and} \quad H|1\rangle=-|1\rangle+i|0\rangle$$

The matrix representation of H is $\begin{vmatrix}\langle 0|H|0\rangle & \langle 0|H|1\rangle\\ \langle 1|H|0\rangle & \langle 1|H|1\rangle\end{vmatrix}=\begin{pmatrix}1 & i\\ -i & -1\end{pmatrix}$

Eigenvalue of H $\begin{pmatrix}1-\lambda & i\\ -i & -1-\lambda\end{pmatrix}=0\Rightarrow-(1-\lambda^2)-1=-0\Rightarrow\lambda=\pm\sqrt{2}$

Q34. A particle of mass m is confined in a potential well given by $V(x)=0$ for $\frac{-L}{2}<x<\frac{L}{2}$ and $V(x)=\infty$ elsewhere. A perturbing potential $H'(x)=ax$ has been applied to the system. Let the first and second order corrections to the ground state be $E_0^{(1)}$ and $E_0^{(2)}$, respectively. Which one of the following statements is correct?

(a) $E_0^{(1)}<0$ and $E_0^{(2)}>0$ (b) $E_0^{(1)}=0$ and $E_0^{(2)}>0$

(c) $E_0^{(1)}>0$ and $E_0^{(2)}<0$ (d) $E_0^{(1)}=0$ and $E_0^{(2)}<0$

Ans.: (d)

Solution: $V(x)=\begin{cases}0 & -L/2<x<+L/2 \\ \infty & elsewhere\end{cases}$ and $H'(x)=\alpha x$

For ground state $|\phi_0\rangle=\sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}$

$$E_0^{(1)}=\frac{\langle\phi_0|H'|\phi_0\rangle}{\langle\phi_0|\phi_0\rangle}=\frac{2}{L}\alpha\int_{-L/2}^{L/2}x\cos^2\frac{\pi x}{L}=0$$

$$E_0^{(2)}=\sum_{m\neq 0}\frac{|\langle\phi_m|H'|\phi_0\rangle|^2}{E_0^0-E_m^0}\Rightarrow E_0^{(2)}<0 \quad \because E_0^0<E_m^0$$

JEST-2016

Q35. The wavefunction of a hydrogen atom is given by the following superposition of energy eigen functions $\psi_{nlm}(\vec{r})$ (n,l,m are the usual quantum numbers):

$$\psi(\vec{r}) = \frac{\sqrt{2}}{\sqrt{7}}\psi_{100}(\vec{r}) - \frac{3}{\sqrt{14}}\psi_{210}(\vec{r}) + \frac{1}{\sqrt{14}}\psi_{322}(\vec{r})$$

The ratio of expectation value of the energy to the ground state energy and the expectation value of L^2 are, respectively:

(a) $\frac{229}{504}$ and $\frac{12\hbar^2}{7}$ (b) $\frac{101}{504}$ and $\frac{12\hbar^2}{7}$

(c) $\frac{101}{504}$ and $\hbar^2$ (d) $\frac{229}{504}$ and $\hbar^2$

Ans.: (a)

Solution: $\langle E\rangle = \frac{2}{7}\times\frac{E_0}{1} + \frac{9}{14}\times\frac{E_0}{4} + \frac{1}{14}\times\frac{E_0}{9} = \frac{229}{504}E_0$

$$\langle L^2\rangle = \frac{2}{7}\times 0\hbar^2 + \frac{9}{14}\times 2\hbar^2 + \frac{1}{14}\times 6\hbar^2 = \frac{24}{14}\hbar^2 = \frac{12}{7}\hbar^2$$

Q36. A spin-$\frac{1}{2}$ particle in a uniform external magnetic field has energy eigenstates $|1\rangle$ and $|2\rangle$. The system is prepared in ket-state $\frac{(|1\rangle + |2\rangle)}{\sqrt{2}}$ at time $t = 0$. It evolves to the state described by the ket $\frac{(|1\rangle - |2\rangle)}{\sqrt{2}}$ in time T. The minimum energy difference between two levels is:

(a) $\frac{h}{6T}$ (b) $\frac{h}{4T}$ (c) $\frac{h}{2T}$ (d) $\frac{h}{T}$

Ans.: (c)

Solution: $|\psi(t=0)\rangle = \frac{(|1\rangle + |2\rangle)}{\sqrt{2}} \Rightarrow |\psi(t=t)\rangle = \frac{\left(|1\rangle\left(-i\frac{E_1 t}{\hbar}\right) + |2\rangle\exp\left(-i\frac{E_2 t}{\hbar}\right)\right)}{\sqrt{2}}$

$$|\psi(t=t)\rangle = \left(-i\frac{E_1 t}{\hbar}\right)\frac{\left(|1\rangle + |2\rangle\exp\left(-i\frac{(E_2 - E_1)t}{\hbar}\right)\right)}{\sqrt{2}}$$

$$\exp\left(-i\frac{(E_2-E_1)t}{\hbar}\right)=-1$$

$$\frac{(E_2-E_1)T}{\hbar}=\pi\Rightarrow(E_2-E_1)=\frac{\pi\hbar}{T}=\frac{h}{2T}$$

Q37. The energy of a particle is given by $E=|p|+|q|$ where p and q are the generalized momentum and coordinate, respectively. All the states with $E\le E_0$ are equally probable and states with $E>E_0$ are inaccessible. The probability density of finding the particle at coordinate q, with $q>0$ is:

(a) $\frac{(E_0+q)}{E_0^2}$ (b) $\frac{q}{E_0^2}$ (c) $\frac{(E_0-q)}{E_0^2}$ (d) $\frac{1}{E_0}$

Ans.: (c)

Solution: For condition, $E=|p|+|q|$ total number of accessible state upto energy E_0 for $q>0$ is area under the curve $=\frac{1}{2}\times 2\times E_0^2=E_0^2$

The probability density of finding the particle at coordinate q, with $q>0$

$$\frac{dpdq}{E_0^2}=\frac{pdq}{E_0^2}\Rightarrow\frac{(E_0-q)dq}{E_0^2}$$

For probability at point q, dq is insignificant so $p(q)=\frac{(E_0-q)}{E_0^2}$

Q38. Consider a quantum particle of mass m in one dimension in an infinite potential well, i.e., $V(x)=0$ for $\frac{-a}{2}<x<\frac{a}{2}$ and $V(x)=\infty$ for $|x|\ge\frac{a}{2}$. A small perturbation, $V'(x)=\frac{2\in|x|}{a}$ is added. The change in the ground state energy to $O(\in)$ is:

(a) $\frac{\in}{2\pi^2}(\pi^2-4)$ (b) $\frac{\in}{2\pi^2}(\pi^2+4)$

(c) $\frac{\in\pi^2}{2}(\pi^2+4)$ (d) $\frac{\in\pi^2}{2}(\pi^2-4)$

Ans.: (a)

Solution: $E_1^1 = \int_{-\frac{a}{2}}^{\frac{a}{2}} \phi_1^* V'(x)\phi_1 dx \Rightarrow \frac{2\in}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} |x|\frac{2}{a}\cos^2\frac{\pi x}{a}dx$

$$= \frac{2\in}{a}.2.\int_0^{\frac{a}{2}} x\,\frac{2}{a}\cos^2\frac{\pi x}{a}dx \Rightarrow \frac{4\in}{a^2}\int_0^{\frac{a}{2}} x\frac{2}{2}\left(\cos\frac{2\pi x}{a}+1\right)dx \Rightarrow \frac{4\in}{a^2}\int_0^{\frac{a}{2}} x\left(\cos\frac{2\pi x}{a}+1\right)dx$$

$$\Rightarrow \frac{4\in}{a^2}\int_0^{\frac{a}{2}} x\left(\cos\frac{2\pi x}{a}+1\right)dx = \frac{\in}{2\pi^2}\left(\pi^2-4\right)$$

Q39. If $Y_{xy} = \frac{1}{\sqrt{2}}\left(Y_{2,2} - Y_{2,-2}\right)$ where $Y_{l,m}$ are spherical harmonics then which of the following is true?

(a) Y_{xy} is an eigenfunction of both L^2 and L_z

(b) Y_{xy} is an eigenfunction of L^2 but not L_z

(c) Y_{xy} is an eigenfunction both of L_z but not L^2

(d) Y_{xy} is not an eigenfunction of either L^2 and L_z

Ans.: (b)

Solution: The $L^2 Y_{xy} = l(l+1)\hbar^2 Y_{xy}$, where $l = 2$ and $L_z Y_{xy} \neq m Y_{xy}$

So, Y_{xy} is an eigenfunction of L^2 but not L_z

Q40. A spin-1 particle is in a state $|\psi\rangle$ described by the column matrix $\frac{1}{\sqrt{10}}\begin{pmatrix} 2 \\ \sqrt{2} \\ 2i \end{pmatrix}$ in the S_z basis. What is the probability that a measurement of operator S_z will yield the result $\hbar$ for the state $S_x|\psi\rangle$?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

Ans.: (c)

Solution: $S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $|\psi\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 2 \\ \sqrt{2} \\ 2i \end{pmatrix}$

$$S_x|\psi\rangle = \frac{\sqrt{2}}{\sqrt{10}}\hbar\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix}$$

$$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigen state corresponding to eigen value $\hbar$ of S_z is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \quad P(\hbar) = \frac{\left|(1\ 0\ 0)\frac{\sqrt{2}}{\sqrt{10}}\hbar\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix}\right|^2}{\frac{2}{10}\hbar^2(1\ 1-i\ 1)\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix}} = \frac{1}{4}$$

Q41. The Hamiltonian of a quantum particle of mass m confined to a ring of unit radius is:

$$H = \frac{\hbar^2}{2m}\left(-i\frac{\partial}{\partial\theta} - \alpha\right)^2$$

where θ is the angular coordinate, α is a constant. The energy eigenvalues and eigenfunctions of the particle are (n is an integer):

(a) $\psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n-\alpha)^2$

(b) $\psi_n(\theta) = \frac{\sin(n\theta)}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n-\alpha)^2$

(c) $\psi_n(\theta) = \frac{\cos(n\theta)}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n-\alpha)^2$

(d) $\psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n+\alpha)^2$

Ans.: (a)

Solution: $H = \frac{\hbar^2}{2m}\left(-i\frac{\partial}{\partial\theta} - \alpha\right)^2 \Rightarrow \frac{\hbar^2}{2m}\left[-\frac{\partial^2\psi}{\partial\theta^2} + 2i\alpha\frac{\partial\psi}{\partial\theta} + \alpha^2\psi\right] = E\psi$

By inspection, $\left|\psi_n(\theta)\right\rangle = \frac{e^{in\theta}}{\sqrt{2\pi}}$, which will also satisfy boundary condition $\left|\psi_n(\theta+2\pi)\right\rangle = \left|\psi_n(\theta)\right\rangle$ and satisfies the eigen value equation with eigen value $E = \frac{\hbar^2(n-\alpha)^2}{2m}$

Q42. The adjoint of a differential operator $\frac{d}{dx}$ acting on a wavefunction $\psi(x)$ for a quantum mechanical system is:

(a) $\frac{d}{dx}$ (b) $-i\hbar\frac{d}{dx}$ (c) $-\frac{d}{dx}$ (d) $i\hbar\frac{d}{dx}$

Ans.: (c)

Q43. For a quantum mechanical harmonic oscillator with energies, $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$, where $n = 0,1,2\ldots$, the partition function is:

(a) $\frac{e^{\frac{\hbar\omega}{k_BT}}}{e^{\frac{\hbar\omega}{2k_BT}}-1}$ (b) $e^{\frac{\hbar\omega}{2k_BT}}-1$ (c) $e^{\frac{\hbar\omega}{2k_BT}}+1$ (d) $\frac{e^{\frac{\hbar\omega}{2k_BT}}}{e^{\frac{\hbar\omega}{k_BT}}-1}$

Ans.: (d)

Solution: $z = \exp\left(-\frac{\hbar\omega}{2kT}\right) + \exp\left(-\frac{3\hbar\omega}{2kT}\right) + \exp\left(-\frac{5\hbar\omega}{2kT}\right) + \exp\left(-\frac{7\hbar\omega}{2kT}\right)\ldots\ldots$

$$z = \exp\left(-\frac{\hbar\omega}{2kT}\right)\left(1 + \exp\left(-\frac{\hbar\omega}{kT}\right) + \exp\left(-\frac{2\hbar\omega}{kT}\right)\ldots\ldots\right)$$

$$z = \frac{\exp\left(-\frac{\hbar\omega}{2kT}\right)}{1-\exp\left(\frac{\hbar\omega}{kT}\right)} \Rightarrow \frac{1}{\exp\left(\frac{\hbar\omega}{2kT}\right) - \exp\left(-\frac{\hbar\omega}{2kT}\right)} \Rightarrow \frac{\exp\left(\frac{\hbar\omega}{2kT}\right)}{\exp\left(\frac{\hbar\omega}{kT}\right)-1}$$

Q44. In the ground state of hydrogen atom, the most probable distance of the electron from the nucleus, in units of Bohr radius a_0 is:

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) $\frac{3}{2}$

Ans.: (d)

Solution: $\psi_{100} = \frac{1}{\sqrt{\pi a_0^3}} e^{\frac{-r}{a_0}}$

$$P = \psi^*\psi = \frac{1}{\pi a_0^3} e^{\frac{-r}{a_0}} \Rightarrow r_p = \frac{dP}{dr} = 0 \Rightarrow r_p = a_0$$

Q45. For operators P and Q, the commutator $\left[P, Q^{-1}\right]$ is

(a) $Q^{-1}[P,Q]Q^{-1}$ (b) $-Q^{-1}[P,Q]Q^{-1}$ (c) $Q^{-1}[P,Q]Q$ (d) $-Q[P,Q]Q^{-1}$

Ans.: (b)

Solution: $\left[P, Q^{-1}\right] = PQ^{-1} - Q^{-1}P$

$$-Q^{-1}[P,Q]Q^{-1} = -Q^{-1}[PQ - QP]Q^{-1} = -Q^{-1}\left[PQQ^{-1} - QPQ^{-1}\right] = -Q^{-1}P + PQ^{-1} = \left[P, Q^{-1}\right]$$

Q46. A spin $\frac{1}{2}$ particle is in a state $\frac{(|\uparrow\rangle + |\downarrow\rangle)}{\sqrt{2}}$ where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of S_z operator. The expectation value of the spin angular momentum measured along x direction is:

(a) $\hbar$ (b) $-\hbar$ (c) 0 (d) $\frac{\hbar}{2}$

Ans.: (d)

Solution: $|\psi\rangle = \frac{(|\uparrow\rangle + |\downarrow\rangle)}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\langle S_x \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2}$$

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Q47. What is the dimension of $\frac{\hbar \partial \psi}{i \partial x}$, where ψ is a wavefunction in two dimensions?

(a) $kg\, m^{-1} s^{-2}$ (b) $kg\, s^{-2}$ (c) $kg\, m^{2} s^{-2}$ (d) $kg\, s^{-1}$

Ans. : (d)

Solution: Dimension of $\frac{\hbar \partial \psi}{i \partial x} = \frac{\dim of\ \hbar}{\dim\ of\ x} = \frac{kg \cdot m \cdot \sec^{-2} \cdot \sec}{m} = kg \sec^{-1}$

Q48. Suppose the spin degrees of freedom of a 2 - particle system can be described by a 21 - dimensional Hilbert subspace. Which among the following could be the spin of one of the particles?

(a) $\frac{1}{2}$ (b) 3 (c) $\frac{3}{2}$ (d) 2

Ans. : (b)

Solution: Dimension of Hilbert space $= (2s_1 + 1) \otimes (2s_2 + 1) = 7 \times 3 = 21$

So, $s_1 = 3, s_2 = 1$

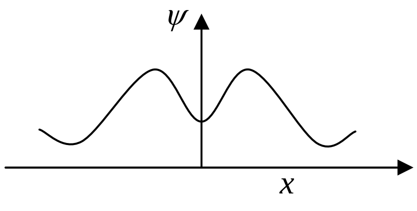
Q50. If the ground state wavefunction of a particle moving in a one dimensional potential is proportional to $\exp(-x^2/2)\cosh(\sqrt{2}x)$, then the potential in suitable units such that $\hbar = 1$, is proportional to

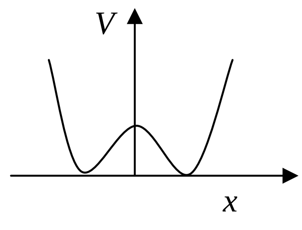
(a) x^2 (b) $x^2 - 2\sqrt{2}x \tanh(\sqrt{2}x)$

(c) $x^2 - 2\sqrt{2}x \tan(\sqrt{2}x)$ (d) $x^2 - 2\sqrt{2}x \coth(\sqrt{2}x)$

Ans. : (b)

Solution: From figure, we can conclude that option (b) is the correct answer.





Q51. A particle is described by the following Hamiltonian

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2+\lambda\hat{x}^4$$

where the quartic term can be treated perturbatively. If ΔE_0 and ΔE_1 denote the energy correction of $O(\lambda)$ to the ground state and the first excited state respectively, what is the fraction $\Delta E_1/\Delta E_0$?

Ans. : 5

Solution: $\hat{H}=\frac{\hat{P}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2+\lambda\hat{x}^4$

Now, energy correction of $O(\lambda)$ to ground state is

$$\Delta E_0=\langle 0|\hat{x}^4|0\rangle=\left(\frac{\hbar}{2m\omega}\right)^2\langle 0|6n^2+6n+3|0\rangle=\left(\frac{\hbar}{2m\omega}\right)^2\times 3$$

And energy correction of $O(\lambda)$ to first excited state is

$$\Delta E_1=\langle 1|\hat{x}^4|1\rangle=\left(\frac{\hbar}{2m\omega}\right)^2\langle 1|6n^2+6n+3|1\rangle$$

$$=\left(\frac{\hbar}{2m\omega}\right)^2\times[6+6+3]=15\left(\frac{\hbar}{2m\omega}\right)^2$$

Hence, $\frac{\Delta E_1}{\Delta E_0}=\frac{15}{3}=5$

Q52. If $\hat{x}(t)$ be the position operator at a time t in the Heisenberg picture for a particle described by the Hamiltonian, $\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2$ what is $e^{i\omega t}\langle 0|\hat{x}(t)\hat{x}(0)|0\rangle$ in units of $\frac{\hbar}{2m\omega}$ where $|0\rangle$ is the ground state?

Solution: Operator $\hat{X}(t)$ in Hisenburg picture is written as

$$\hat{X}(t)=e^{iHt/\hbar}\hat{X}(0)e^{iHt/\hbar}$$

Thus, $\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle=\langle 0|e^{iHt/\hbar}X(0)e^{-iHt/\hbar}X(0)|0\rangle$

Here, $\hat{X}(0)|0\rangle=\sqrt{\frac{\hbar}{2m\omega}}|1\rangle$

So, above equation reduces as,

$$\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 0|e^{iHt/\hbar}\hat{X}(0)e^{-iHt/\hbar}|1\rangle$$

In integral form,

$$\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\int \phi_0^*(t)\hat{X}(0)\phi_1(t)\,dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\int \phi_0^* e^{\frac{i\hbar\omega t}{2\hbar}}\hat{X}(0)\phi_1 e^{\frac{-i3\hbar\omega t}{2\hbar}}dx = \sqrt{\frac{\hbar}{2m\omega}}e^{-i\omega t}\int \phi_0^* x\phi_1\,dx$$

Therefore, $e^{i\omega t}\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle = \left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2\langle 0|a+a^\dagger|1\rangle$

$$e^{i\omega t}\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle = \frac{\hbar}{2m\omega}$$

Q53. Two classical particles are distributed among $N(>2)$ sites on a ring. Each site can accommodate only one particle. If two particles occupy two nearest neighbour sites, then the energy of the system is increased by $\in$. The average energy of the system at temperature T is

(a) $\dfrac{2\in e^{-\beta\in}}{(N-3)+2e^{-\beta\in}}$

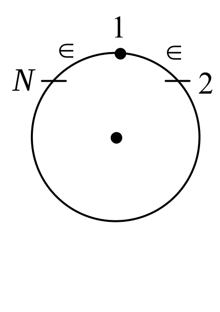
(b) $\dfrac{2N\in e^{-\beta\in}}{(N-3)+2e^{-\beta\in}}$

(c) $\dfrac{\in}{N}$

(d) $\dfrac{2\in e^{-\beta\in}}{(N-2)+2e^{-\beta\in}}$

Ans. : (a)

Solution: Since two particle two nearest neighbour sites, which energy of system increased by $\in$, and remaining $(N-3)$ particle has zero energy, then particle function is given



$$z = 2e^{-\beta\in} + (N-3)e^{-\beta.0} = (N-3)+2e^{-\beta\in}$$

then $\langle E\rangle = KT^2\dfrac{\partial}{\partial T}(\ln z)$

$$= \frac{KT^2}{z}.\left[0+2e^{-\beta\in}.\frac{\partial}{\partial T}\left(-\frac{\in}{KT}\right)\right] = \frac{KT^2}{z}.2e^{-\beta\in}\left(\frac{\in}{KT^2}\right)$$

$$\langle E\rangle = \frac{2\in e^{-\beta\in}}{\left[(N-3)+2e^{-\beta\in}\right]}$$

Q54. Consider a particle confined by a potential $V(x) = k|x|$, where k is a positive constant. The spectrum E_n of the system, within the WKB approximation is proportional to

(a) $\left(n+\frac{1}{2}\right)^{3/2}$ (b) $\left(n+\frac{1}{2}\right)^{2/3}$ (c) $\left(n+\frac{1}{2}\right)^{1/2}$ (d) $\left(n+\frac{1}{2}\right)^{4/3}$

Ans. : (b)

Solution: $V(x) = \begin{cases} kx & x > 0 \\ -kx & x < 0 \end{cases}$

$$\therefore \sqrt{2m}\int_0^b \sqrt{E - V(x)}dx = \left(n+\frac{1}{2}\right)\hbar\pi = 2\sqrt{2m}\int_0^{E/k} \sqrt{E-kx}dx = 2\sqrt{2m}\int_0^{E/k} \sqrt{E}\cdot\sqrt{1-\frac{k}{E}x}\,dx$$

$$= 2\sqrt{2mE}\int_0^1 \sqrt{1-t}\,\frac{E}{k}dt = \frac{2E}{k}\sqrt{2mE}\int_0^1 \sqrt{1-t}\,dt = 2E^{3/2}\frac{\sqrt{2m}}{k}\times\frac{2}{3}$$

$$= \left(n+\frac{1}{2}\right)\hbar\pi \Rightarrow E_n^{3/2} = \frac{3\hbar\pi k}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)$$

$$E_n = \left[\frac{3\hbar\pi k}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{2/3}$$

Q55. Consider the Hamiltonian

$$H(t) = \alpha\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \beta t\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

The time dependent function $\beta(t) = \alpha$ for $t \le 0$ and zero for $t > 0$. Find $\left|\langle\Psi(t<0)|\Psi(t>0)\rangle\right|^2$, where $|\Psi(t<0)\rangle$ is the normalised ground state of the system at a time $t<0$ and $|\Psi(t>0)\rangle$ is the state of the system at $t>0$.

(a) $\frac{1}{2}(1+\cos(2\alpha t))$ (b) $\frac{1}{2}(1+\cos(\alpha t))$

(c) $\frac{1}{2}(1+\sin(2\alpha t))$ (d) $\frac{1}{2}(1+\sin(\alpha t))$

Ans. : (a)

Solution: $H(t) = \alpha\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} + \beta(t)\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$

Time dependent function $\beta(t)=\begin{cases}\alpha & , t\le 0\\ 0 & , t>0\end{cases}$

When $t\le 0$

$$H(t)=\alpha\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$$

Eigen value are $0\ ,\ 2\alpha\ ,\ 2\alpha$.

For Eigen value zero, the ground state wave function is $\left|\psi(t\le 0)\right\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

And $\left|\psi(t\ge 0)\right\rangle=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\end{pmatrix}e^{\frac{-i\alpha t}{\hbar}}-\frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\end{pmatrix}e^{\frac{-i3\alpha t}{\hbar}}$

Now, $\left|\left\langle\psi(t<0)\middle|\psi(t>0)\right\rangle\right|^2=\frac{1}{4}\left|e^{\frac{-i\alpha t}{\hbar}}+e^{\frac{-i3\alpha t}{\hbar}}\right|^2$

$$=\frac{1}{4}\left[\left(\cos\frac{\alpha t}{\hbar}+\cos\frac{3\alpha t}{\hbar}\right)^2+\left(-\sin\frac{\alpha t}{\hbar}-\sin\frac{3\alpha t}{\hbar}\right)^2\right]$$

$$=\frac{1}{4}\left[1+1+2\left(\cos\frac{\alpha t}{\hbar}.\cos\frac{3\alpha t}{\hbar}+\sin\frac{\alpha t}{\hbar}\cos\frac{3\alpha}{\hbar}\right)\right]=\frac{1}{4}\left[2+2.\cos\frac{2\alpha t}{\hbar}\right]=\frac{1}{2}\left[1+\cos\frac{2\alpha t}{\hbar}\right]$$