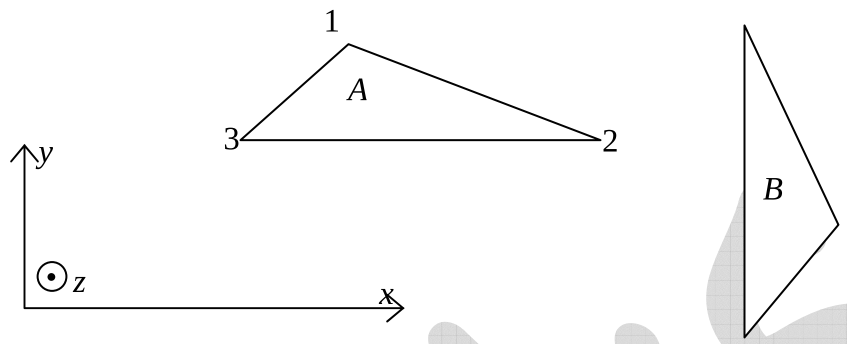
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

GS-2018-X (PHYSICS)

SECTION A

Q1.



Refer to the figure above. If the z-axis points out of the plane of the paper towards you, the triangle marked 'A' can be transformed (and suitably re-positioned) to the triangle marked 'B' by

(a) Rotation about z - direction by $\pi/2$, then reflection in the yz-plane

(b) Reflection in the xz - plane, then rotation by $-\pi/2$ about z-direction

(c) Reflection in the yz - plane, then rotation by $\pi/2$ about z-direction

(d) Rotation about x - direction by $\pi/2$, then rotation by $-\pi/2$ in the yz-plane

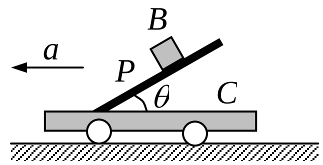
Q2. If a 2×2 matrix M is given by

$$M=\begin{pmatrix} 1 & (1-i)/\sqrt{2} \\ (1+i)/\sqrt{2} & 0 \end{pmatrix}$$

Then det exp $M =$

(a) $e^2\ e$ (b) e (c) $2i\sin\sqrt{2}$ (d) $\exp\left(-2\sqrt{2}\right)$

Q3. A small block B of mass m is placed on an inclined plane P, which makes an angle θ with a horizontal cart C, on which P is rigidly fixed (see figure). The coefficient of friction between the block B and the plane C is μ.



When the cart stays stationary the block slides down. If the cart C is moving in the horizontal direction with acceleration a, the minimum value of a for which the block remain static on the place is:

(a) $g\dfrac{\tan\theta-\mu}{\mu\tan\theta+1}$ (b) $g\left(\mu-\sin\theta\cos\theta\right)$

(c) $g\dfrac{1-\mu\tan\theta}{\mu+\tan\theta}$ (d) $g\left(\cos\theta-\mu\sin\theta\right)$

Q4. A particle of mass m moving in one-dimension x is subjected to the Lagrangian

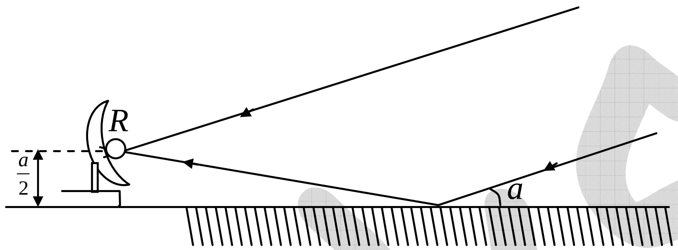
$$L=\frac{1}{2}m\left(x-\lambda x\right)^2$$

Where λ is a real constant. It is starts at the origin at $t=0$, its motion corresponds to the equation (a is a constant)

(a) $x=a\exp\lambda t$ (b) $x=a\sin\lambda t$

(c) $x=a\left\{1-\exp\left(-\lambda t\right)\right\}$ (d) $x=a\sinh\lambda t$

Q5. The sketch below shows a radio antenna located at the edge of a calm lake, which has a receiver R at the centre of the dish at a height $a/2$ above the ground. This is picking up a signal from a distant radio-emitting star which is just rising above the horizon. However, the receiver also picks up a reflected signal from the surface of the lake, which, at the relevant radio-wavelength, may be taken to be a plane.



If the instantaneous angle of the star above the horizon is denoted α, the receiver R will detect the first interference maximum when $\alpha =$

(a) $\arcsin\left(\frac{\lambda}{2a}\right)^{1/3}$ (b) $\arcsin\left(\frac{\lambda}{a}\right)^{1/3}$ (c) $\arcsin\frac{\lambda}{2a}$ (d) $\arcsin\frac{\lambda}{a}$

Q6. A particle is confined inside a one-dimensional box of length ℓ and left undisturbed for a long time. In the most general case, its wave-function MUST be

(a) The ground state of energy.

(b) Periodic, where ℓ equals an integer number of periods.

(c) Any one of the energy eigenfunctions.

(d) A linear superposition of the energy eigenfunctions.

Q7. A classical ideal gas of atoms with masses m is confined in a three-dimensional potential

$$V(x,y,z)=\frac{\lambda}{2}\left(x^2+y^2+z^2\right)$$

at a temperature T. If k_B is the Boltzmann constant, the root mean square (r.m.s.) distance of the atoms from the origin is

(a) $\left(\frac{3k_BT}{\lambda}\right)^{1/2}$ (b) $\left(\frac{2k_BT}{3\lambda}\right)^{1/2}$ (c) $\left(\frac{3k_BT}{2\lambda}\right)^{1/2}$ (d) $\left(\frac{k_BT}{\lambda}\right)^{1/2}$

Q8. The characteristic impedance of a co-axial cable is independent of the

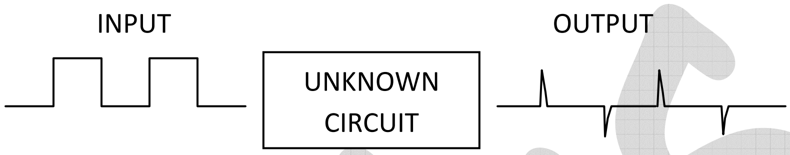
(a) Dielectric medium between the core and the outer mesh

(b) Outer diameter

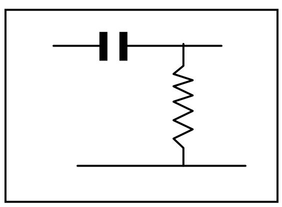
(c) Length of the cable

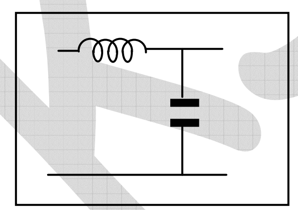
(d) Core diameter

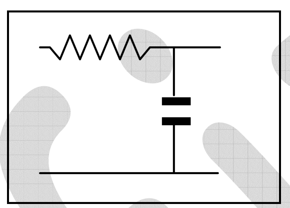
Q9. The figure below shows an unknown circuit, with an input and output voltage signal.

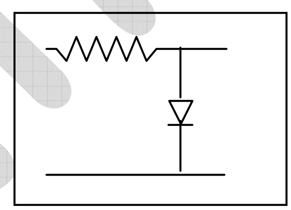


From the form of the input and output signals, one can infer that the circuit is likely to be

(a) 

(b) 

(c) 

(d) 

Q10. In Boolean terms, $(A+B)(A+C)$ is equal to

(a) ABC

(b) $A+BC$

(c) $A(B+C)$

(d) $(A+B+C)(A+B)$

Q11. Consider the two equations

$$\frac{x^2}{3}+\frac{y^2}{2}=1 \qquad\qquad x^3-y=1$$

How many simultaneous real solutions does this pair of equations have?

Q12. A heavy steel ball is struck by a small steel pellet moving horizontally with velocity $20ms^{-1}$. If the pellet bounces off the steel ball with no slippage, and then rises vertically to a height $10m$ above the point of contact, then what is 100 times the elastic coefficient of restitution (e) i.e. $100e$?

Q13. A particle is in the ground state of a cubical box of side ℓ. Suddenly one side of the box changes from ℓ to 4ℓ. If p is the probability of finding the particle in the ground state of the new box, what is $1000p$?

Q14. The wave-function ψ of a particle in a one-dimensional harmonic oscillator potential is given by

$$\psi = \left(\frac{1}{\pi\ell^2}\right)^{1/4}\left(1+\frac{\sqrt{2}x}{\ell}\right)\exp\left(-\frac{x^2}{2\ell^2}\right)$$

Where $\ell = 100\mu m$. Find the expectation value of the position x of this particle, in μm.

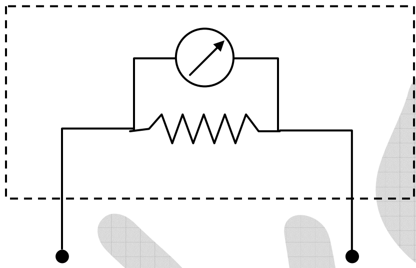
Q15. Consider a dipole antenna with length ℓ, charge q and frequency ω. The power emitted by the antenna at a large distance r is P. Now suppose the length ℓ is increased to $\sqrt{2}\ell$, the charge is increased to $\sqrt{3}q$ and the frequency is increased to $\sqrt{5}\omega$. By what factor is the radiated power increased?

Q16. Calculate the self-energy, in Joules, of a spherical conductor of radius $8.5cm$, which carries a charge $100\mu C$.

Q17. A heat engine is operated between two bodies that are kept at constant pressure. The constant-pressure heat capacity c_p of the reservoirs is independent of temperature. Initially the reservoirs are at temperatures $300K$ and $402K$. If, after some time, they come to a common final temperature T_f, the process remaining adiabatic, what is the value of T_f (in Kelvin)?

Q18. N Particles are distributed among three energy levels having energies: 0, $k_B T$ and $2k_B T$ respectively. If the total equilibrium energy of the system is approximately $42.5k_B T$ then find the value of N (to the closest integer).

Q19. A realistic voltmeter can be modeled as an ideal voltmeter with an input resistor in parallel a shown below:



Such a realistic voltmeter, with input resistance $1k\Omega$, gives a reading of $100mV$ when connected to a voltage source with source resistance 50Ω. .What will a similar voltmeter, with input resistance $1M\Omega$, read in mV, when connected to the same voltage source?

Q20. An electron enters a linear accelerator with a speed $v = 10ms^{-1}$. A vertical section of the accelerator tube is shown in the figure, where the lengths of the successive sections are designed such that the electron takes the same time $\tau = 20ms$ to traverse each section.

If the momentum of the electron increases by 2% every time it crosses the narrow gap between two sections, what is the length (in km) of the collider which will be required to accelerate it to $100kms^{-1}$?

SECTION B

Q21. If $y(x)$ satisfies the differential equation $y''-4y'+4y=0$ with boundary conditions

$$y(0)=1 \text{ and } y'(0)=0, \text{ then } y\left(-\frac{1}{2}\right)=$$

(a) $-\frac{2}{e}$ (b) $\frac{1}{2}\left(e+\frac{1}{e}\right)$ (c) $\frac{1}{e}$ (d) $\frac{e}{2}$

Q22. Given the following xy data

x	1.0	2.0	3.0	4.0	5.0
y	0.002	0.601	0.948	1.21	1.42

Which of the following would be the best curve, with constant positive parameters a and b, to fit this data?

(a) $y=ax-b$ (b) $y=a+\exp bx$ (c) $y=a\log_{10} bx$ (d) $y=a-\exp(-bx)$

Q23. The Hamiltonian of a dynamical system is equal to its total energy, provided that its Lagrangian

(a) Does not contain velocity-dependent terms.

(b) Has no explicit time dependence.

(c) Is separable in generalized coordinates and velocities.

(d) Does not have terms which explicitly depend on the coordinates.

Q24. A particle in a one-dimensional harmonic oscillator potential is described by a wave-function $\psi(x,t)$. If the wave function changes to $\psi(\lambda x,t)$ then the expectation value of kinetic energy T and the potential energy V will change, respectively, to

(a) $\lambda^2 T$ and $\lambda^2 V$ (b) T/λ^2 and $\lambda^2 V$

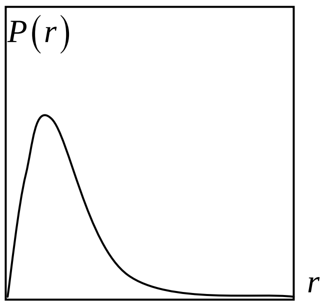
(c) T/λ^2 and V/λ^2 (d) $\lambda^2 T$ and V/λ^2

Q25. An electron is in the $2s$ level of the hydrogen atom, with the radial wave-function

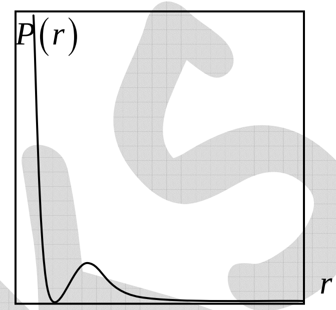
$$\psi(r) = \frac{1}{2\sqrt{2}a_0^{3/2}}\left(2 - \frac{r}{a_0}\right)\exp\left(-\frac{r}{2a_0}\right).$$

The probability $P(r)$ of finding this electron between distances r to $r+dr$ from the centre is best represented by the sketch

(a)



(b)



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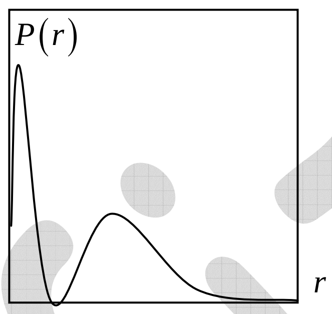
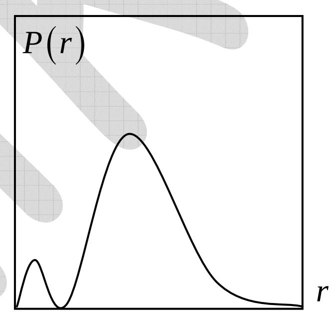


(d)



Q26. An atom of atomic number Z can be modeled as a point positive charge surrounded by a rigid uniformly negatively charged solid sphere of radius R. The electric Polaris ability a of this system is defined as

$$\alpha = \frac{p_E}{E}$$

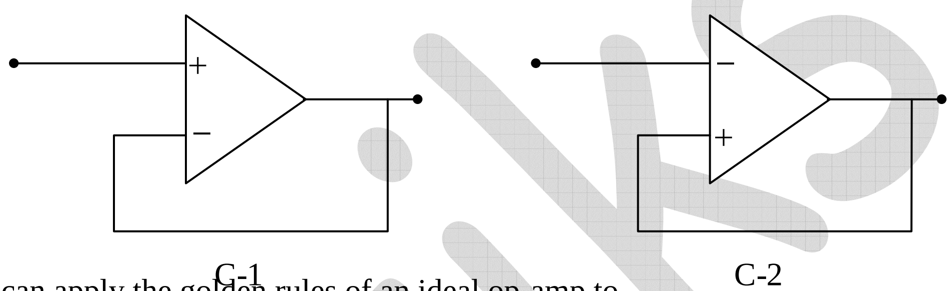
Where p_p is the dipole moment induced on application of electric field E which is small compared to the binding electric field inside the atom. It follows that $a =$

(a) $\frac{4\pi\varepsilon_0}{R^3}$ (b) $\frac{8\pi\varepsilon_0}{R^3}$ (c) $4\pi\varepsilon_0 R^3$ (d) $8\pi\varepsilon_0 R^3$

Q27. A many-body system undergoes a phase transition between two phases A and B at a temperature T_c The temperature-dependent specific heat at constant volume C_V of the two phases are given by $C_V^{(A)} = aT^3 + bT$ and $C_V^{(B)} = cT^3$. Assuming negligible volume change of the system, and no latent heat generated in the phase transition, T_c is

(a) $\sqrt{\frac{4b}{c-a}}$ (b) $\sqrt{\frac{3b}{c-a}}$ (c) $\sqrt{\frac{2b}{c}}$ (d) $\sqrt{\frac{b}{c-a}}$

Q28. Consider the following circuits C-1 and C-2.

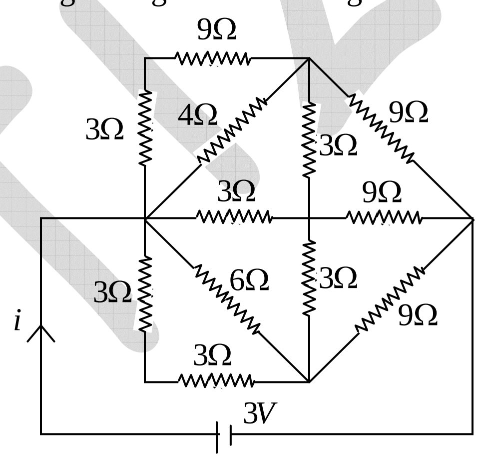


You can apply the golden rules of an ideal op-amp to

(a) Both C-1 and C-2 (b) Neither C-1 nor C

(c) Only C-1 (d) Only C-2

Q29. The current i flowing through the following circuit is



(a) $1.0A$ (b) $0.75A$ (c) $0.6A$ (d) $0.5A$

Q30. Two students A and B try to measure the time period T of a pendulum using the same stopwatch, but following two different methods. Student A measures the time taken for one oscillation, repeats this process $N(\gg 1)$ times and computes the average. On the over hand, Student B just once measures the time taken for N oscillations and divides that number by N.

The error in the measurement of the stopwatch is much smaller than the time period.

Assuming that accuracy of the stopwatch is sufficiently high which of the following statements is true about the errors in T as measured by A and by B?

(a) A and B will measure the time period with the same accuracy.

(b) It is not possible to determine if the measurement made by A or B has the larger error.

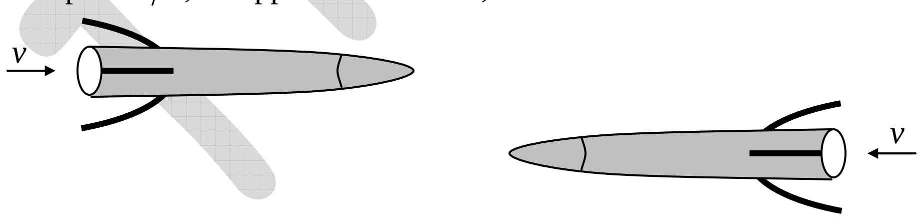
(c) The measurement made by A has a smaller error than that made by B.

(d) The measurement made by A has a larger error than that made by B.

Q31. Evaluate the integral

$$\int_{-\infty}^{+\infty} dx \exp\left(-x^2\right)\cos\left(\sqrt{2}x\right)$$

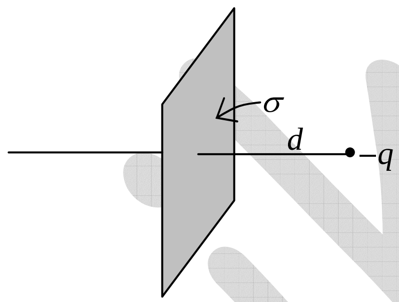
Q32. From an observational post E on the Earth, two ballistic missiles, each of rest length ℓ from nose-tip to tail-end, are observed to fly past each other, with the same uniform relativistic speed $c/2$, in opposite directions, as shown below.



What is the time taken for the tail-end of one of the missiles to cross the tail-end of the other missile, as measured from the post E?

Q33. A statistical system, kept at a temperature T, has n discrete energy levels with equal level-spacing ε, starting from energy 0. If, now, a single particle is placed in the system what will be the mean energy of the system in the limit as $n \to \infty$? [The answer should not be left as a summation]

Q34. Consider an infinite plane with a uniform positive charge density σ as shown below. A negative point charge $-q$ with mass m is held at rest at a distance d from the sheet and released. It will then undergo oscillatory motion. What is the time period of this oscillation?



[You may assume that the point charge can move freely though the charged plane without disturbing the charge density.].

Q35. Given a particle confined in a one-dimensional box between $x = -a$ and $x = +a$, a student attempts to find the ground state by assuming a wave-function

$$\psi(x) = \begin{cases} A\left(a^2 - x^2\right)^{3/2} & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$$

The ground state energy E_m is estimated by calculating the expectation value of energy with this trial wave-function. If E_0 is the true ground state energy, what is the ration E_m/E_0?

SECTION C

Q36. The Fourier series which reproduces, in the interval $0 \le x < 1$, the function

$$f(x) = \sum_{n=-\infty}^{+\infty} \delta(x-n)$$

Where n is an integer, is

(a) $1 + 2\cos 2\pi x + 2\cos 4\pi x + 2\cos 6\pi x + \cdots + (to\ \infty)$

(b) $1 + \cos \pi x + \cos 2\pi x + \cos 3\pi x + \cdots (to\ \infty)$

(c) $\cos \pi x + \cos 2\pi x + \cos 3\pi x + \cdots (to\ \infty)$

(d) $(\cos \pi x + \sin \pi x) + \frac{1}{2}(\cos 2\pi x + \sin 2\pi x) + \frac{1}{3}(\cos 3\pi x + \sin 3\pi x) + \cdots (to\ \infty)$

Q37. The value of the integral $\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}$ is.

(a) $1/2a$ (b) $1/2\pi a$ (c) $\exp(-a)/a$ (d) $\pi a \exp(-a)$

Q38. A dynamical system with one degree of freedom is described by canonical coordinate (p,q). The generator F of the canonical transformation $(p,q) \to (-q, p)$ is

(a) $F = -p\dot{q}$ (b) $F = pq$ (c) $F = p\dot{q}$ (d) $F = -\dot{p}q$

Q39. The electrostatic charge density $\rho(r)$ corresponding to the potential

$$\varphi(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\left(1 + \frac{\alpha r}{2}\right)\exp(-\alpha r)$$

is $\rho =$

(a) $q\delta(r) - 2q\alpha^3 \exp(-\alpha r)$ (b) $q\delta(r) - q\frac{\alpha^3}{4}\exp(-\alpha r)$

(c) $-q\delta(r) - 2q\alpha^3 \exp(-\alpha r)$ (d) $q\delta(r) - q\frac{a^3}{2}\exp(-ar)$

Q40. The Hamiltonian of a particle of charge q and mass m in an electromagnetic field is given by $H = \frac{1}{2m}\left|\vec{p} - q\vec{A}(\vec{x},t)\right|^2 + q\varphi(\vec{x},t)$.

Where $(\varphi, \vec{A})$ are the electromagnetic potentials? Clearly this Hamiltonian changes under a gauge transformation.

$$\varphi \to \varphi - \frac{\partial x}{\partial t} \qquad \vec{A} \to \vec{A} + \vec{\nabla}\chi$$

Where $\chi(\vec{x},t)$ is a gauge function? Nevertheless the motion of the particle is not affected because

(a) The action of the particle changes only by surface terms which do not vary.

(b) The Lorentz force is modified to balance the effect of the gauge transformation.

(c) The Lagrangian does not change under the gauge transformation.

(d) The motion of the particle is correctly described only in Lorenz gauge.

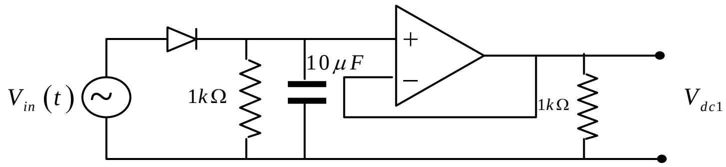
Q41. The electron of a free hydrogen atom is initially in a state with quantum numbers $n = 3$ and $\ell = 2$. It then makes an electric dipole transition to a lower energy state. Which one of the given states could it be in after the transition?

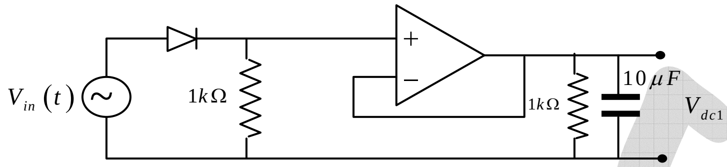
(a) $n = 3,\ \ell = 0$ (b) $n = 2,\ \ell = 1$ (c) $n = 3,\ \ell = 1$ (d) $n = 2,\ \ell = 2$

Q42. Consider a monatomic solid lattice at a low temperature $T \ll T_D$, where T_D is the characteristic Debye temperature of the solid $(T_D = \hbar\omega_m / k_B$ where ω_m is the maximum possible frequency of the lattice vibrations). The heat capacity of the solid is proportional to

(a) T/T_D (b) T_D/T (c) $(T/T_D)^3$ (d) $(T_D/T)^2$

Q43. A signal $V_{in}(t) = 5\sin(100\pi t)$ is sent to both the circuits sketched below.





If the DC output voltage of the top circuit has a value V_{dc1} and the bottom circuit has a value V_{dc2}, then which of the following statements about the relative value of V_{dc1} and V_{dc2} is correct?

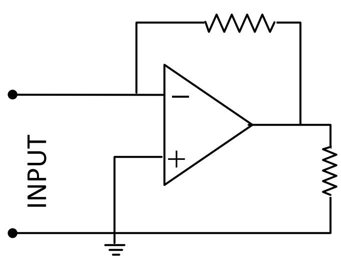
(a) It will depend on the slew rate of the op-amp.

(b) $V_{dc1} = V_{dc2}$

(c) $V_{dc1} < V_{dc2}$

(d) $V_{dc1} > V_{dc2}$

Q44. Consider the circuit shown on the right, which involves an op-amp and two resistors, with an input voltage marked INPUT.



Which of the following circuit components, when connected across the input terminals, is most likely to create a problem in the normal operation of the circuit?

(a) A voltage source with very high Thevenin resistance.

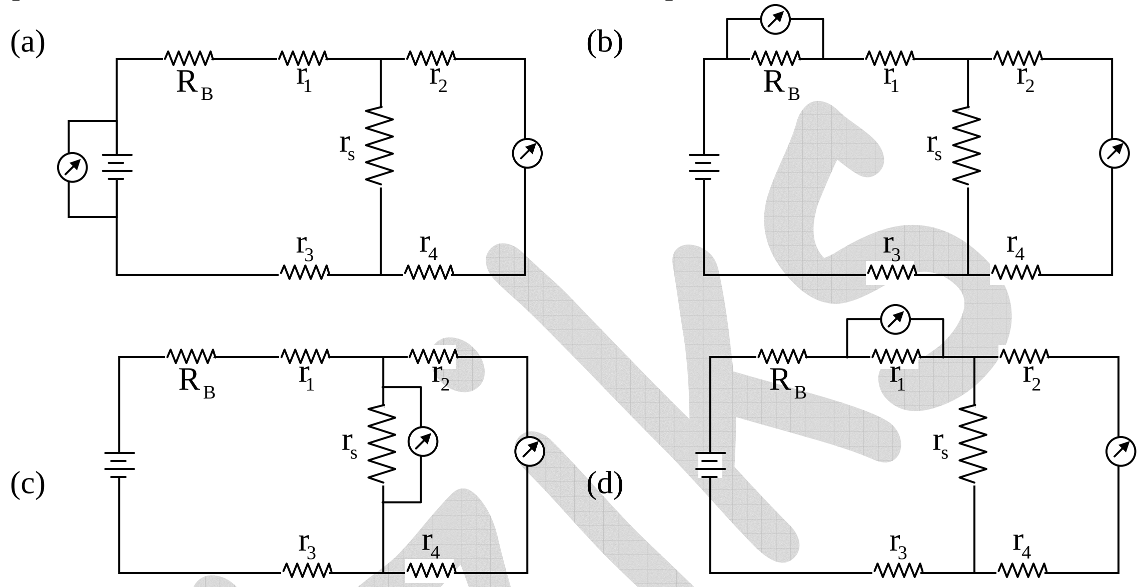
(b) A voltage source with a very low Thevenin resistance.

(c) A current source with a very high Norton resistance.

(d) A current source with a very low Norton resistance.

Q45. Which one of the following circuits, constructed only with resistors and voltmeters, will allow you to obtain the correct value of resistance r_s using the voltmeter readings? Note that the value of R_B is known while r_1, r_2, r_3, r_4 and r_s are all unknown.

[Assume that the voltmeters and resistors are ideal.]



Q46. A fourth rank Cartesian tensor T_{ijkt} satisfies the following identities

(i) $T_{ijk\ell} = T_{jik\ell}$ (ii) $T_{ijk\ell} = T_{ij\ell k}$ (iii) $T_{ijk\ell} = T_{k\ell ij}$

Assuming a space of three dimensions (i.e. $i, j, k = 1, 2, 3$), what is the number of independent components of $T_{ijk\ell}$?

Q47. If the velocity of the Earth in its orbit is v, find $\delta E/E$, where E is the translational (non-relativistic) kinetic energy of the Earth and δE is its relativistic correction to the lowest order in v/c.

Q48. A plane electromagnetic wave, which has an electric field

$$\vec{E}(\vec{x}, t) = \left(P\hat{i} + Q\hat{j}\right)\exp i\omega\left(t - \frac{z}{c}\right)$$

is passing through vacuum. Here P, Q and ω are all constants, while c is the speed of light in vacuo.

What is the average energy flux per unit time (in SI units) crossing a unit area placed normal to the direction of propagation of this wave, in terms of the above constants?

Q49. The state of a spin -1particle is given by

$$|\psi\rangle = \frac{1}{\sqrt{3}}\left(|1, -1\rangle + |1, 0\rangle \exp\frac{i\pi}{3} + |1, 1\rangle \exp\frac{2i\pi}{3}\right)$$

Where $|S, M_s\rangle$ denote the spin eigenstates with eigenvalues $h^2 S(S+1)$ and hM_s respectively. Find $\langle S_x \rangle$, i.e. the expectation value of the x component of the spin.

Q50. A particle of mass m moves in a two-dimensional space (x, y) under the influence of a Hamiltonian.

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{4}m\omega^2\left(5x^2 + 5y^2 + 6xy\right)$$

Find the ground state energy of this particle in a quantum-mechanical treatment.