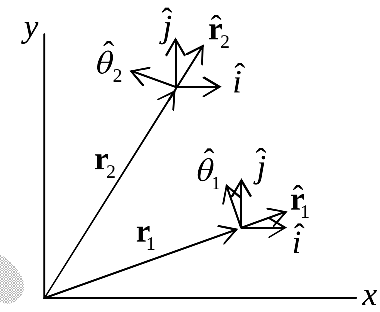
7. Central Force and Kepler's System

7.1 The Motion in Two Dimension in Polar Coordinate System

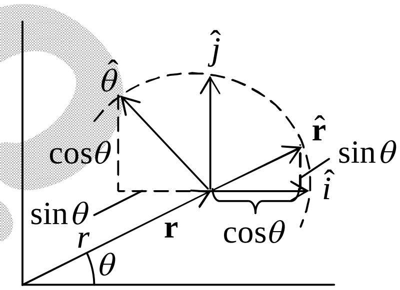
The drawing shows the unit vectors $\hat{i},\hat{j}$ and $\hat{r},\hat{\theta}$ at a point in the x-y plane. We see that the orthogonality of $\hat{\mathbf{r}}$ and $\hat{\theta}$ plus the fact that they are unit vectors,



$$|\hat{\mathbf{r}}|=1,\ |\hat{\theta}|=1,$$

$\hat{\mathbf{r}}=\hat{i}\cos\hat{\theta}+\hat{j}\sin\theta$ and $\hat{\theta}=-\hat{i}\sin\theta+\hat{j}\cos\theta$ which is shown.

The transformation can be shown by rotational Matrix



$$\begin{bmatrix}\hat{r}\\ \hat{\theta}\end{bmatrix}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\hat{i}\\ \hat{j}\end{bmatrix}$$

$$\frac{d\hat{r}}{dt}=-\hat{i}\sin\theta\dot{\theta}+\hat{j}\cos\theta\dot{\theta}\Rightarrow\frac{d\hat{r}}{dt}=\dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt}=-\hat{i}\cos\theta\dot{\theta}-\hat{j}\sin\theta\dot{\theta}\Rightarrow\frac{d\hat{\theta}}{dt}=-\hat{r}\dot{\theta}$$

7.1.1 The Position Vector in Polar Coordinate

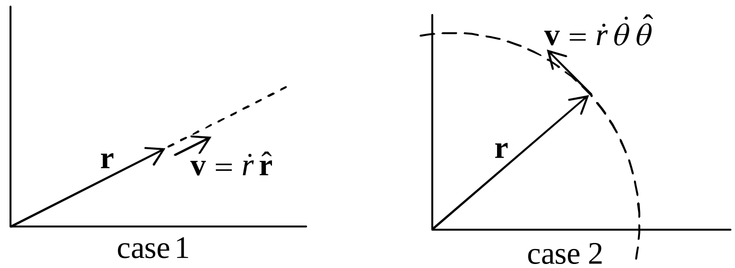
$$\vec{r}=x\hat{i}+y\hat{j},\ \vec{r}=|r|\left[\cos\theta\,\hat{i}+\sin\theta\,\hat{j}\right]$$

$$\vec{r}=|r|\cos\theta\,\hat{i}+|r|\sin\theta\,\hat{j}$$

$$\vec{r}=|r|\left[\cos\theta\,\hat{i}+\sin\theta\,\hat{j}\right]\Rightarrow\vec{r}=|r|\,\hat{r}$$

$\mathbf{r}=r\hat{\mathbf{r}}$ is sometimes confusing, because the equation as written seems to make no reference to the angle θ. We know that two parameters needed to specify a position in two dimensional space (in Cartesian coordinates they are x and y), but the equation $\mathbf{r}=r\hat{\mathbf{r}}$ seems to contain only the quantity r. The answer is that $\hat{\mathbf{r}}$ is not a fixed vector and we need to know the value of θ to tell how $\hat{\mathbf{r}}$ is origin. Although θ does not occur explicitly in $r\hat{\mathbf{r}}$, its value must be known to fix the direction of $\hat{\mathbf{r}}$. This would be apparent if we wrote $\mathbf{r}=r\hat{\mathbf{r}}(\theta)$ to emphasize the dependence of $\hat{\mathbf{r}}$ on θ. However, by common conversation $\hat{\mathbf{r}}$ is understood to stand for $\hat{\mathbf{r}}(\theta)$.

7.1.2 Velocity Vector in Polar Coordinate



$$\vec{v} = \frac{d}{dt}(r\hat{\mathbf{r}}) \Rightarrow \vec{v} = \frac{d(r\hat{r})}{dt} = \frac{dr}{dt}.\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} \Rightarrow \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

where $\dot{r}$ is radial velocity in $\hat{r}$ direction and $r\dot{\theta}$ is tangential velocity in $\hat{\theta}$ direction as shown in figure and the magnitude to velocity vector $|v| = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$

7.1.3 Acceleration Vector in Polar Coordinate

$$\frac{d\vec{v}}{dt} = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\dot{\theta}\hat{\theta} + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$\frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}(-\dot{\theta})\hat{r}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta} \Rightarrow \vec{a} = a_r\hat{r} + a_\theta\hat{\theta}$$

$a_r = \ddot{r} - r\dot{\theta}^2$ is radial acceleration and $a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$ is tangential acceleration .

So, Newton's law in polar coordinate can be written as

$f_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$, where f_r is force in radial direction .

$f_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$, where f_θ is force in tangential direction.

7.1.4 Circular Motion

For circular motion $r = r_0$, then $\ddot{r} = 0$. So, $f_r = ma_r = -mr_0\dot{\theta}^2$, where f_r is force in radial direction and $f_\theta = ma_\theta = mr_0\ddot{\theta}$, where f_θ is force in tangential direction.

If there is not any force in tangential direction i.e., $f_\theta = 0$ is condition, then motion is uniform circular motion i.e., $\dot{\theta} = \omega$ is constant known as angular speed and tangential speed is given by $v = r_0\omega$

For non-uniform circular motion radial acceleration is $a_r = -\frac{v^2}{r}$ and tangential acceleration is given by $a_t = \frac{dv}{dt}$

$$\Rightarrow a = \sqrt{a_r^2 + a_t^2}$$

7.2 Central Force

In classical mechanics, the **central-force problem** is to determine the motion of a particle under the influence of a single central force. A central force is a force that points from the particle directly towards (or directly away from) a fixed point in space, the center, and whose magnitude only depends on the distance of the object to the center.

In central force potential V is only function of r and only central force is always a conservative force; the magnitude F of a central force can always be expressed as the derivative of a time-independent potential energy

$$\vec{\nabla} \times \vec{F} = \frac{1}{r\sin\theta}\left(\frac{\partial F}{\partial \phi}\right)\hat{\theta} - \frac{1}{r}\left(\frac{\partial F}{\partial \theta}\right)\hat{\phi} = 0$$

And the force $\vec{F}$ is defined as $\vec{F} = -\frac{\partial V}{\partial r}\hat{r}$ (force is only in radial direction)

7.2.1 Angular Momentum and Areal Velocity

The equation of motion in polar coordinate is given by $m(\ddot{r} - r\dot{\theta}^2) = F_r$ and $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$ but for central force,

$$\vec{\tau} = \vec{r} \times \vec{F}_r \qquad \Rightarrow \vec{\tau} = r\hat{r} \times -\frac{\partial V}{\partial r}\hat{r} \ ,$$

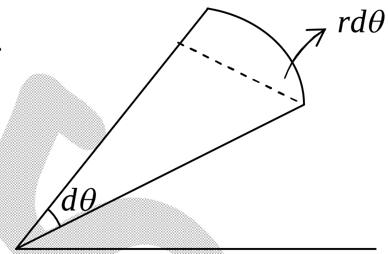
External torque $\vec{\tau} = 0$, so angular momentum is conserved

$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = F_\theta$ but for central force $F_\theta = 0$, so $m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \Rightarrow \frac{d(mr^2\dot{\theta})}{dt} = 0$ means angular momentum of $mr^2\dot{\theta} = J = |\vec{r} \times \vec{p}| \Rightarrow \dot{\theta} = \frac{J}{mr^2}$

$\vec{r}\cdot\vec{J}=\vec{r}\cdot(\vec{r}\times\vec{p})=0 \Rightarrow \vec{r}\perp\vec{J}$, hence position vector $\vec{r}$ is perpendicular to angular momentum vector $\vec{J}$ and hence $\vec{J}$ is conserved, its magnitude and direction both are fixed so direction of $\vec{r}$ is also fixed.

So, motion due to central force is confined into a plane and angular momentum $\vec{J}$ is perpendicular to that plane.

Central force problem. Prove that Areal velocity is constant.



For the central force problem $A=\frac{1}{2}r\cdot rd\theta$

Now, Areal velocity = $\frac{dA}{dt}=\frac{1}{2}r^2\frac{d\theta}{dt}=\frac{1}{2}r^2\dot{\theta}$

$\frac{dA}{dt}=\frac{1}{2}r^2\dot{\theta}$. It is given that $\dot{\theta}=\frac{J}{mr^2}$, so $\frac{dA}{dt}=\frac{J}{2m}$

which means equal area will swept in equal time.

7.2.2 Total Energy of the System

Total energy is not explicitly function of time t, so $\frac{\partial E}{\partial t}=0$. So, one can conclude that total energy in central potential is constant.

$$E=\frac{1}{2}mv^2+V(r) \text{ and Velocity } \vec{v}=\dot{r}\hat{r}+r\dot{\theta}\hat{\theta}$$

So, total energy, $E=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+V(r)$

$$=\frac{1}{2}m\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}^2+V(r), \text{ where } \dot{\theta}=\frac{J}{mr^2}$$

$$=\frac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}+V(r),\ r>0$$

$$\Rightarrow E=\frac{1}{2}m\dot{r}^2+V_{eff},$$

where $\frac{J^2}{2mr^2}+V(r)$ is identified as effective potential $V_{effective}$.

7.2.3 Condition for Circular Orbit

From equation of motion in radial part $m(\ddot{r}-r\dot{\theta}^2)=f(r)$

For circular orbit of radius r_0, $r=r_0$ and $\ddot{r}=0 \Rightarrow f(r)=m\left(0-r\dot{\theta}^2\right)=-\frac{J^2}{mr^3}$ at $r=r_0$

which can be also derived by $\left.\frac{\partial V_{effective}}{\partial r}\right|_{r=r_0}=0$ and $\dot{\theta}=\omega_0$ is identified as angular frequency of circular orbit.

Radius $r=r_0$ of circular orbit is also identified as stable equilibrium point so $\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0}\geq 0$. If somehow particle of mass m changes its orbit without changing its angular momentum and new orbit is bounded then new orbit is identified as elliptical orbit. The angular frequency in new elliptical orbit is

$$\omega=\sqrt{\frac{\left.\frac{\partial^2 V_{effective}}{\partial r^2}\right|_{r=r_0}}{m}}$$

7.2.4 Equation of Motion and Differential Equation of Orbit

So, Lagrangian can be reduced to

$$L=\frac{1}{2}\mu\left(\dot{r}^2+r^2\dot{\theta}^2\right)-V(r)$$

Since, $\frac{\partial L}{\partial \theta}=0$. So, θ is cyclic co-ordinate. Hence, angular momentum is conserved during the motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta}\right)=0 \Rightarrow mr^2\dot{\theta}=J \quad \text{or} \quad \dot{\theta}=\frac{J}{mr^2}$$

Equation of Motion

The Lagrangian equation of motion is given by $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)-\frac{\partial L}{\partial r}=0$

$$\Rightarrow m\ddot{r}-mr\dot{\theta}^2+\frac{\partial V}{\partial r}=0 \text{ or } m\ddot{r}-mr\dot{\theta}^2=-\frac{\partial V}{\partial r}$$

$$m\ddot{r} - mr\dot{\theta}^2 = f(r)$$

From equation of motion in radial part,

$$m(\ddot{r} - r\dot{\theta}^2) = f(r) \Rightarrow m\frac{d^2r}{dt^2} - \frac{J^2}{mr^3} = f(r) \quad \text{..........(1)}$$

where, $J = mr^2\dot{\theta} \Rightarrow d\theta = \frac{J}{mr^2}dt \Rightarrow \frac{d}{dt} = \frac{J}{mr^2}\frac{d}{d\theta}$

$$\frac{d^2}{dt^2} = \left(\frac{d}{dt}\right)\left(\frac{d}{dt}\right) = \left(\frac{J}{mr^2}\right)\frac{d}{d\theta}\left(\frac{J}{mr^2}\right)\frac{d}{d\theta}$$

Substituting in (1)

$$\frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{1}{r^2}\frac{dr}{d\theta}\right) - \frac{J^2}{mr^3} = f(r) \Rightarrow \frac{J^2}{m}\frac{1}{r^2}\frac{d}{d\theta}\left(\frac{d(-1/r)}{d\theta}\right) - \frac{J^2}{mr^3} = f(r)$$

$$-\left(\frac{J^2}{m}\frac{1}{r^2}\frac{d^2(1/r)}{d\theta^2} + \frac{J^2}{mr^3}\right) = f(r) \Rightarrow -\frac{J^2}{mr^2}\left(\frac{d^2(1/r)}{d\theta^2} + \frac{1}{r}\right) = f(r)$$

Let, $\frac{1}{r} = u \Rightarrow -\frac{J^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = f\left(\frac{1}{u}\right)$ (differential equation of an orbit)

Example: Consider the motion of a particle of mass m in the potential field $V(r) = \frac{kr^2}{2}$.

If l is angular momentum then

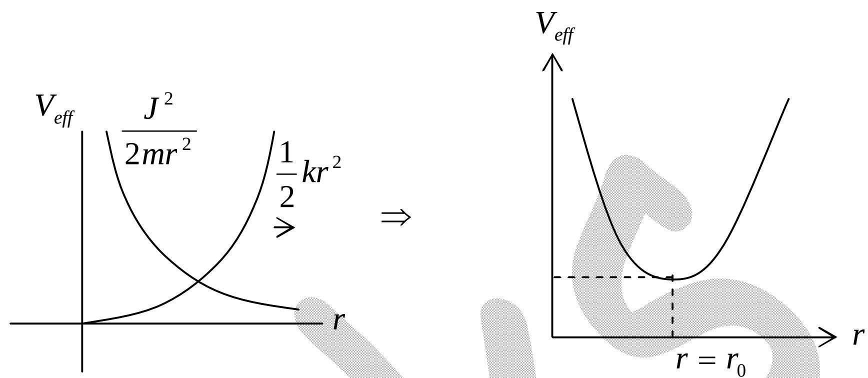
(a) What is effective potential (V_{eff}) of the system? Plot V_{eff} vs r

(b) Find value of energy such that motion is circular in nature.

(c) If particle is slightly disturbed from circular orbit such that its angular momentum remains constant. What will be the nature of new orbit? Find the angular frequency of new orbit in term of m, l, k.

Solution: (a) $V_{eff} = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2$

(b) $\frac{dV_{eff}}{dr} = -\frac{J^2}{mr^3} + kr = 0$ at $r = r_0$, so $r_0 = \left(\frac{J^2}{mk}\right)^{1/4}$ and $J = m\omega_0 r_0^2$



for circular motion $m\omega_0^2 r_0 = kr_0$, where r_0 is radius of circle $\omega_0 = \sqrt{\frac{k}{m}}$

total energy $E = \frac{J^2}{2mr^2} + \frac{1}{2}kr^2 = \frac{mkr_0^4}{2mr_0^2} + \frac{1}{2}kr_0^2 \Rightarrow E = kr_0^2$,

on putting $r_0 = \left(\frac{J^2}{mk}\right)^{1/4}$ gives $E = J\sqrt{\frac{k}{m}}$

(c) orbit is elliptical in nature

$$\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0} = \frac{3J^2}{mr^4} + k = \frac{3J^2}{m\left(\frac{J^2}{mk}\right)} + k = 4k$$

$$\omega = \sqrt{\frac{\left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0}}{m}} \Rightarrow \omega = \sqrt{\frac{4k}{m}} \Rightarrow \omega = 2\sqrt{\frac{k}{m}} \Rightarrow 2\omega_0$$

Example: A particle of mass m moves under the influence of an attractive central force $f(r)$.

(a) What is condition that orbit is circular in nature if J is the angular momentum of particle

(b) If force is in form of $f(r)=\frac{-k}{r^n}$ determine the maximum value of n for which the circular orbit can be stable.

Solution: (a) If $V_{eff}=\frac{J^2}{2mr^2}+V(r)$, for circular stable orbit $\frac{\partial V_{eff}}{\partial r}=0$, $\frac{\partial^2 V_{eff}}{\partial r^2}>0$

(b) $f(r)=\frac{-k}{r^n}$, for circular motion $\frac{\partial V_{eff}}{\partial r}=0\Rightarrow -\frac{J^2}{mr^3}+\frac{\partial V}{\partial r}=0$

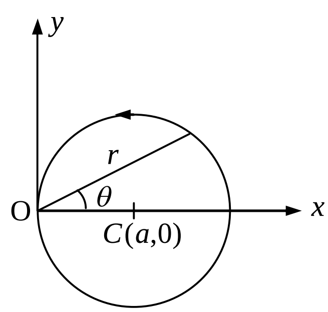
It is given $\frac{\partial V}{\partial r}=-f(r)$, if $f(r)=\frac{-k}{r^n}\ \Rightarrow\frac{\partial V}{\partial r}=\frac{k}{r^n}$

$$-\frac{J^2}{mr^3}+\frac{k}{r^n}=0\Rightarrow\frac{k}{r^n}=\frac{J^2}{mr^3}$$

$\frac{\partial^2 V_{eff}}{\partial r^2}>0\Rightarrow\frac{3J^2}{mr^4}-\frac{nk}{r^{n+1}}>0\Rightarrow\frac{3J^2}{mr^4}-\frac{n}{r}.\frac{J^2}{mr^3}>0$, so $n<3$

Example: A particle of mass m and angular momentum l is moving under the action of a central force $f(r)$ along a circular path of radius a as shown in the figure.

The force centre O lies on the orbit.

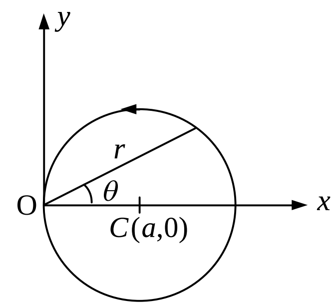


(a) Given the orbit equation in a central field motion.

$$\frac{d^2u}{d\theta^2}+u=-\frac{m}{l^2u^2}f\text{ , where } u=\frac{1}{r}.$$

Determine the form of force in terms of l, m, a and r.

(b) Calculate the total energy of the particle assuming that the potential energy $V(r)\to 0$ as $r\to\infty$.



Solution: (a) from the figure, $r = 2a\cos\theta \Rightarrow \frac{1}{r} = \frac{\sec\theta}{2a}$

$$-\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2}+u\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{4a^2}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta\right)+\frac{\sec\theta}{2a}\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^2\theta}{4a^2}\left[\frac{1}{2a}\left(\sec\theta\tan^2\theta+\sec^3\theta+\sec\theta\right)\right] = f\left(\frac{1}{u}\right)$$

$$-\frac{J^2\sec^3\theta}{8a^3m}\left[\tan^2\theta+\sec^2\theta+1\right] = f\left(\frac{1}{u}\right) \Rightarrow -\frac{2J^2\sec^5\theta}{8a^3m} = f\left(\frac{1}{u}\right) \Rightarrow f(r) \propto \frac{1}{r^5}$$

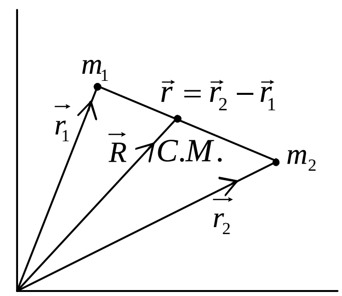
(b) $E = \frac{m\dot{r}^2}{2}+\frac{J^2}{2mr^2}+V(r)$, as $r\to\infty$, $V(r)\to 0 \Rightarrow \frac{J^2}{2mr^2}\to 0$

$E = \frac{m\dot{r}^2}{2}$ and $r = 2a\cos\theta$ and $\dot{r} = -2a\sin\theta\dot{\theta}$, $\dot{\theta} = \frac{J^2}{mr^2}$ as $r\to\infty$

hence, $\dot{\theta} = \frac{J^2}{mr^2}\to 0$, so $\dot{r}\to 0$, so $E = 0$

7.3 Two Body Problem

Reduction of two body central force problem to the equivalent one body problem: A system of two particles of mass m_1 and m_2 whose instantaneous position vectors of inertial frame with origin O are r_1 and r_2 respectively.



Position vector m_2 relative to m_1 is $\vec{r} = \vec{r}_2 - \vec{r}_1$

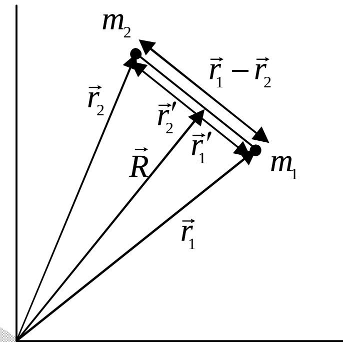
The potential energy V is only function of distance between the particles.

So, $V = V(|\vec{r}_2 - \vec{r}_1|)$

Total energy of the system is given in lab frame by

$$E = \frac{1}{2}m_1\dot{\vec{r}}_1^2 + \frac{1}{2}m_2\dot{\vec{r}}_2^2 + V\left(\left|\vec{r}_2 - \vec{r}_1\right|\right)$$

Let the position vectors of m_1 and m_2 be r_1 and r_2. The position vector of the center of mass, measured from the same origin, is



$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

The center of mass lies on the line joining m_1 and m_2. To show this, suppose first that the tip of $\vec{R}$ does not lie on the line, and consider the vectors $\vec{r}_1'$, $\vec{r}_2'$ from the tip of $\vec{R}$ to m_1 and m_2. From the sketch we see that

$$\vec{r}_1' = \vec{r}_1 - \vec{R}$$

$$\vec{r}_2' = \vec{r}_2 - \vec{R}$$

$$\vec{r}_1' = \vec{r}_1 - \frac{m_1\vec{r}_1}{m_1 + m_2} - \frac{m_2\vec{r}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}\left(\vec{r}_1 - \vec{r}_2\right)$$

$$\vec{r}_2' = \vec{r}_2 - \frac{m_1\vec{r}_1}{m_1 + m_2} - \frac{m_2\vec{r}_2}{m_1 + m_2} = -\left(\frac{m_1}{m_1 + m_2}\right)\left(\vec{r}_1 - \vec{r}_2\right)$$

$\vec{r}_1'$ and $\vec{r}_2'$ are proportional to $\vec{r}_1 - \vec{r}_2$, the vector from m_1 to m_2. Hence $\vec{r}_1'$ and $\vec{r}_2'$ lie along the line joining m_1 and m_2 as shown. Furthermore,

$$\vec{r}_1' = \frac{m_2}{m_1 + m_2}\left|\vec{r}_1 - \vec{r}_2\right| = \frac{m_2}{m_1 + m_2}\vec{r} \quad \text{and} \quad \vec{r}_2' = \frac{m_1}{m_1 + m_2}\left|\vec{r}_1 - \vec{r}_2\right| = \frac{m_1}{m_1 + m_2}\vec{r}$$

$E = \frac{1}{2}m_1\left|\dot{\vec{r}}_1'\right|^2 + \frac{1}{2}m_2\left|\dot{\vec{r}}_2'\right|^2 + V\left(\left|\vec{r}_2 - \vec{r}_1\right|\right)$. The total energy is transformed

$$E = \frac{1}{2}\left(m_1 + m_2\right)\left|\dot{\vec{R}}\right|^2 + \frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right)\left|\dot{\vec{r}}\right|^2 - V\left(r\right)$$

Centre of mass moving with constant momentum and equation of motion for three generalized co-ordinates or will not terms in R and $\dot{R}$. While discussing the motion of the system, one can ignore $\frac{1}{2}\left(m_1 + m_2\right)\dot{R}^2$.

So, energy in centre of mass reference frame is reduced to $E = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\left|\dot{\vec{r}}\right|^2 + V(r)$

where, $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is reduced to one body system in centre of mass reference frame.

7.3.1 Kepler's Problem

Kepler discuss the orbital motion of the sun and Earth system under the potential $V(r) = -\frac{k}{r}$ where, $k = Gm_s m_e$. Here m_s and m_e is mass of Sun and Earth respectively. Although Kepler discuss Sun and Earth system but method can be used for any system which is interacting with potential $V(r) = -\frac{k}{r}$

The reduced mass for Sun and Earth system is $\mu = \frac{m_e m_s}{m_e + m_s} \Rightarrow \frac{m_e}{1 + \frac{m_e}{m_s}} = m_e$, $m_s >> m_e$

Let us assume mass of Earth $m_e = m$

7.3.2 Kepler's First Law

Every planet (earth) moves in an elliptical orbit around the sun, the sun is being at one of the foci. Where sun and earth interact each other with potential $V(r) = -\frac{k}{r}$, we solve equation of motion in center of mass reference frame with reduced mass $\mu = m_e = m$

7.3.3 Equation of Motion

Lagrangian can be reduced to

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - V(r), \text{ put } \mu = m$$

$\frac{\partial L}{\partial \theta} = 0$, so θ is cyclic co-ordinate. Hence angular momentum is conserved during the motion.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 0 \Rightarrow mr^2\dot{\theta} = J \text{ or } \dot{\theta} = \frac{J}{mr^2}$$

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{k}{r^2}, \text{ put } \dot{\theta} = \frac{J}{mr^2}$$

$$m\ddot{r} - \frac{l^2}{mr^3} = -\frac{k}{r^2}$$

Equation of orbit is given by $\frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = -f\left(\frac{1}{u}\right)$

$$f(r) = -\frac{k}{r^2} \Rightarrow f\left(\frac{1}{u}\right) = -ku^2 \Rightarrow \frac{J^2u^2}{m}\left[\frac{d^2u}{d\theta^2} + u\right] = +ku^2 \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{ku^2m}{J^2u^2}$$

$\frac{d^2u}{d\theta^2} + \left(u - \frac{km}{J^2}\right) = 0$, put $u - \frac{km}{J^2} = y$, so $\frac{d^2u}{d\theta^2} = \frac{d^2y}{d\theta^2}$

The equation reduces to, $\frac{d^2y}{d\theta^2} + y = 0$

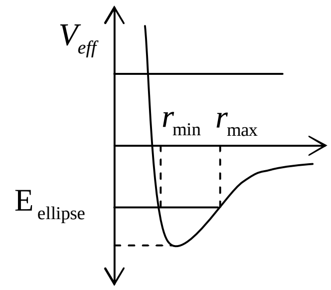
The solution of equation reduces to $y = A\cos\theta$ $u - \frac{km}{J^2} = A\cos\theta \Rightarrow u = \frac{km}{J^2} + A\cos\theta$

$$\frac{1}{r} = \frac{km}{J^2} + A\cos\theta \Rightarrow \frac{J^2/km}{r} = 1 + \left(\frac{AJ^2}{km}\right)\cos\theta$$

Put $\frac{J^2}{km} = l$ and $e = \frac{AJ^2}{km}$ the equation reduce to $\frac{l}{r} = 1 + e\cos\theta$ which is equation of conic, where l is latus rectum and e is eccentricity.

In a central force potential given by potential $V(r) = -\frac{k}{r}$, the trajectory can be any conic section depending on eccentricity e.

Now, we shall discuss the case specially of elliptical orbit as Kepler discuss for planetary motion.

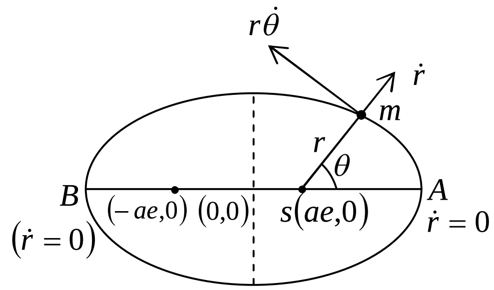


Total energy $E = \frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} - \frac{k}{r}$, where

$V_{effective} = \frac{J^2}{2mr^2} - \frac{k}{r}$ with constant angular momentum J.

If one will plot $V_{effective}$ vs r, it is clear that for negative energy the orbit is elliptical which is shown in figure.

Earth is orbiting in elliptical path with Sun at one of its foci as shown in figure.



Let equation of this ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $b=a\sqrt{1-e^2}$. Minimum value of r is $(a-ae)$ and maximum value of r is $(a+ae)$ and $r_{\max}+r_{\min}=2a$, from plot of effective potential it is identified that $r_{\max}$ and $r_{\min}$ is the turning point so at these points radial velocity is zero.

$E=\frac{J^2}{2mr^2}-\frac{k}{r}\Rightarrow 2mEr^2+2mkr-J^2=0$, given equation is quadratic in terms of r for their root at $r_{\max}$ and $r_{\min}$. Using theory of quadratic equation, sum of roots, $r_{\max}+r_{\min}=-\frac{2mk}{2mE}\Rightarrow E=-\frac{k}{2a}$, which is negative.

7.3.4 Relationship between Energy and Eccentricity

For central potential $V(r)=-\frac{k}{r}$ the solution of orbit is $\frac{l}{r}=1+e\cos\theta$ with $l=\frac{J^2}{km}$

The energy is given by $E=\frac{1}{2}m\dot{r}^2+\frac{J^2}{2mr^2}-\frac{k}{r}$

So, $\frac{-l}{r^2}\dot{r}=-e\sin\theta\dot{\theta}\Rightarrow \dot{r}=\frac{eJ\sin\theta}{ml}$, where $\dot{\theta}=\frac{J}{mr^2}$

after putting the value of $\frac{l}{r}=1+e\cos\theta$ and $\dot{r}=\frac{eJ\sin\theta}{ml}$ with $l=\frac{J^2}{km}$ in equation of energy,

one will get, $e=\sqrt{1+\frac{2EJ^2}{mk^2}}$.

The condition on energy for possible nature of orbit for potential

$E>0$; $e>1$ Hyperbola

$E=0$; $e=1$ Parabola

$E<0$; $e<1$ Ellipse

$E=-\frac{mk^2}{2J^2}$; $e=0$ circle

7.3.5 Kepler's Second Law

Equal Area will swept in equal time or Areal velocity is constant.

$\frac{dA}{dt}=\frac{J}{2m}$ (which is derived earlier)

7.3.6 Kepler's Third Law

The square of time period (T) of revolution in elliptical orbit is proportional to cube of semi major axis a ie $T^2 \propto a^3$

$$\frac{dA}{dt}=\frac{J}{2m} \Rightarrow \int dA=\frac{J}{2m}\int dt \Rightarrow \pi ab=\frac{J}{2m}T \quad (\pi ab \text{ is the area of ellipse})$$

$\pi a\cdot a\sqrt{1-e^2}=\frac{J}{2m}T$. It is given $e=\sqrt{1+\frac{2EJ^2}{mk^2}}$ and $E=-\frac{k}{2a}$

$$e^2=1-\frac{2kJ^2}{2amk^2} \Rightarrow 1-e^2=\frac{2kJ^2}{2amk^2}$$

$T^2=\frac{4m^2}{J^2}\pi^2a^2\cdot a^2(1-e^2)$ put value of $1-e^2=\frac{2kJ^2}{2amk^2}=\frac{J^2}{amk}$

$$T^2=\frac{4m^2}{J^2}\pi^2a^4(1-e^2) \Rightarrow T^2=\frac{4m^2}{J^2}\pi^2a^4.\frac{J^2}{mak}=\frac{4\pi^2ma^3}{k}$$

$\Rightarrow T^2=\frac{4\pi^2ma^3}{k}$ if $k=Gm_sm \Rightarrow T^2=\frac{4\pi^2a^3}{Gm_s}$, where m_s is mass of the sun.

Example: Given a classical model of tritium atom with nucleus of charge +1 and a single electron in a circular orbit of radius r_0. Suddenly the nucleus emits a negatron and changes to charge +2 (the emitted negatron escapes rapidly and we can forget about it) the orbit suddenly has a new situation.

(a) Find the ratio of the electron's energy after to before the emission of the negatron

(b) Describe qualitatively the new orbit

(c) Find the distance of closest and the farthest approach for the new orbits in units of r_0

Solution: (a) As the negatron leaves the system rapidly, we can assume that its leaving has no effect on the position and kinetic energy of the orbiting electron.

From the force relation for the electron,

$$\frac{mv_0^2}{r_0} = \frac{e^2}{4\pi\varepsilon_0 r_0^2}, \text{ and we find its kinetic energy } \frac{mv_0^2}{2} = \frac{e^2}{8\pi\varepsilon_0 r_0}$$

and its total mechanical energy

$$E_1 = \frac{mv_0^2}{2} - \frac{e^2}{4\pi\varepsilon_0 r_0} = -\frac{e^2}{8\pi\varepsilon_0 r_0}$$

before the emission of the negatron. After the emission the kinetic energy of the electron is still $\frac{e^2}{8\pi\varepsilon_0 r_0}$, while its potential energy suddenly changes to $\frac{-2e^2}{4\pi\varepsilon_0 r_0} = \frac{-e^2}{2\pi\varepsilon_0 r_0}$

Thus after the emission the total mechanical energy of the orbiting electron is

$$E_2 = \frac{mv_0^2}{2} - \frac{2e^2}{4\pi\varepsilon_0 r_0} = \frac{-3e^2}{8\pi\varepsilon_0 r_0}, \text{ giving } \frac{E_2}{E_1} = 3.$$

In other words, the total energy of the orbiting electron after the emission is three times as large as that before the emission.

(b) As $E_2 = \frac{-3e^2}{8\pi\varepsilon_0 r_0}$, the condition equation (i) for circular motion is no longer satisfied and the new orbit is an ellipse.

(c) Conservation of energy gives

$$\frac{-3e^2}{8\pi\varepsilon_0 r_0} = \frac{-e^2}{2\pi\varepsilon_0 r_0} + \frac{m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)}{2}$$

At positions where the orbiting electron is at the distance of closest or farthest approach to the atom, we have $\dot{r}=0$, for which

$$\frac{-3e^2}{8\pi\varepsilon_0 r_0}=\frac{mr^2\dot{\theta}^2}{2}-\frac{e^2}{2\pi\varepsilon_0 r}=\frac{J^2}{2mr^2}-\frac{e^2}{2\pi\varepsilon_0 r}$$

Then, with $J^2=m^2v_0^2r_0^2=\frac{me^2r_0}{4\pi\varepsilon_0}$

From above equations

$$3r^2-4r_0r+r_0^2=0$$

the solutions are $r=\frac{r_0}{3},\ r=r_0$

Hence, the distances of closest and farthest approach in the new orbit are respectively,

$$r_{\min}=\frac{1}{3},\ r_{\max}=1$$

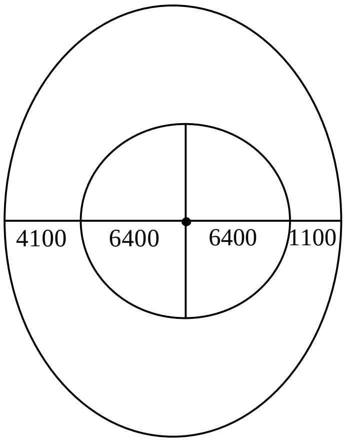
Example: A satellite of mass $m=2000$ kg is in elliptical orbit about earth. At perigee it has an altitude of 1,100 km and at apogee it has altitude 4,100 km. assume radius of the earth is $R_e=6,400$ km. it is given $GmM_e=8\times10^{17}\ J.m$

(a) What is major axis of the orbit?

(b) What is eccentricity of the orbit?

(c) What is angular momentum of the satellite?

(d) How much energy is needed to fix satellite in an orbit from surface of the earth?



Solution: $r_{\max}=4100+6400=10500\,km$

$r_{\min}=1100+6400=7500\,km$

(a) $r_{\max}+r_{\min}=2a\Rightarrow 18000=2a\Rightarrow a=9000\,km$

(b) $e=\frac{r_{\max}-r_{\min}}{r_{\max}+r_{\min}}\Rightarrow e=\frac{10500-7500}{10500+7500}=\frac{3000}{18000}=\frac{1}{6}\Rightarrow e=\frac{1}{6}$

(c) It is given, $k=8\times10^{17}\ J\cdot m$

$$E=-\frac{k}{2a}=-\frac{8\times10^{17}}{18000\times10^3},\ E_f=-\frac{8}{18}\times10^{11}\ J=-4.5\times10^{10}\ J$$

$$e=\sqrt{1+\frac{2EJ^2}{mk^2}} \Rightarrow \left(\frac{1}{6}\right)^2=1+\frac{2EJ^2}{mk^2}$$

$$\frac{1}{36}-1=\frac{2EJ^2}{mk^2} \Rightarrow 140\times10^{26}=J^2 \Rightarrow J=\sqrt{140}\times10^{13}=1.2\times10^{14}\,kgm/\sec^2$$

(d) When satellite is at surface of the earth, $R=6400\ Km$

$$E_i=-\frac{GMm}{R}=\frac{-8\times10^{17}}{6400\times10^3}=\frac{-10^{17}}{800\times10^3}=\frac{-10^{12}}{8}=-12.5\times10^{10}$$

$$E_f=-\frac{GMm}{2a}=-4.5\times10^{10}\,J \Rightarrow \Delta E=E_f-E_i=8\times10^{10}\,J$$

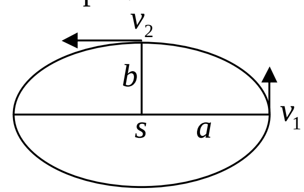
Example: For circular and parabolic orbits in an attractive $1/r$ potential having the same angular momentum, show that perihelion distance of the parabola is one-half the radius of the circle.

Solution: For Kepler's problem, $\frac{l}{r}=1+e\cos\theta$, for circular orbit $e=0 \Rightarrow \frac{l}{r_c}=1$ and for parabola $e=1$, $\frac{l}{r}=1+\cos\theta$, then r_p is minimum when $\cos\theta$ is maximum.

$$\frac{l}{r_p}=2 \text{ and } \frac{l}{l_c}=1 \Rightarrow \frac{r_p}{r_c}=\frac{1}{2}$$

Example: A planet of mass m moves in the inverse square central force field of the Sun of mass M. If the semi-major and semi-minor axes of the orbit are a and b respectively, then find total energy of the planet by assuming Sun is at the center of ellipse.

Solution: Assume Sun is at the centre of elliptical orbit.



Conservation of energy, $\frac{1}{2}mv_1^2-\frac{GMm}{a}=\frac{1}{2}mv_2^2-\frac{GMm}{b}$

Conservation of momentum, $L=mv_1a=mv_2b$

$$v_2=v_1\left(\frac{a}{b}\right)$$

$$\frac{1}{2}mv_1^2-\frac{1}{2}mv_2^2=\frac{GMm}{a}-\frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2-v_1^2\frac{a^2}{b^2}\right)=GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2-a^2}{b^2}\right)=GMm\left(\frac{b-a}{ab}\right)\Rightarrow\frac{1}{2}mv_1^2=GMm\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}$$

$$E=\frac{1}{2}mv_1^2-\frac{GMm}{a}=GMm\frac{b}{a}\frac{1}{(b+a)}-\frac{GMm}{a}$$

$$=\frac{GMm}{a}\left(\frac{b}{(b+a)}-1\right)=\frac{GMm}{a}\left(\frac{b-b-a}{(b+a)}\right)=-\frac{GMm}{(b+a)}$$