(a) Homogeneous Linear Equations of Second Order

A second order differential equation is called **linear** if it can be written as

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=r(x) \text{ or } y''+p(x)y'+q(x)y=r(x)$$

and **nonlinear** if it cannot be written in this form.

The above equation is linear in the unknown function y and its derivatives, whereas p and q as well as r may be given function of x.

If $r(x)=0$, then

$$y''+p(x)y'+q(x)y=0 \text{ is called homogeneous.}$$

If $r(x)\neq 0$, then

$$y''+p(x)y'+q(x)y=r(x) \text{ is called nonhomogeneous.}$$

Fundamental Theorem for Homogeneous Equation

For a homogeneous linear differential equation any linear combination of two solutions is a again a solution.

If y_1 and y_2 are solutions of homogeneous linear differential equation, then

$$y=c_1y_1+c_2y_2 \text{ (Linear combination)}$$

is also a solution of the homogeneous linear differential equation.

Note: It does not hold for nonhomogeneous linear equations or nonlinear equations.

General Solution and Initial Value Problem

For second order homogeneous linear differential equations a general solutions is of the form

$$y=c_1y_1+c_2y_2 \quad (c_1,\ c_2 \text{ are arbitrary constants})$$

An initial value problem now consists of

$y''+p(x)y'+q(x)y=0$ and two initial conditions $y(x_0)=K_0$ and $y'(x_0)=K_1$.

Using these conditions we can find constants c_1 and c_2.

Basis

A basis of solutions is a pair y_1 and y_2 of linearly independent solutions.

Two functions $y_1(x)$ and $y_2(x)$ are linearly independent if

$$k_1y_1(x)+k_2y_2(x)=0 \Rightarrow k_1=0,\ k_2=0.$$

and we call them linearly dependent if $k_1y_1(x)+k_2y_2(x)=0$ for some constant $k_1,\ k_2$ not both zero.

If $k_1\neq 0,\ k_2\neq 0$, then $y_1=-\frac{k_2}{k_1}y_2$ or $y_2=-\frac{k_1}{k_2}y_1$.

How to Obtain Basis if One Solution is Known (Reduction of Order)

To get y_2, set $y_2=uy_1$. Then $y_2'=u'y_1+uy_1'$ and $y_2''=u''y_1+2u'y_1'+uy_1''$.

Substituting y_2, y_2' and y_2'' in $y''+p(x)y'+q(x)y=0$

$$\Rightarrow u''y_1+2u'y_1'+uy_1''+p(u'y_1+uy_1')y_2''+quy_1=0$$

Collecting terms in u'', u' and u, we have

$$u''y_1+u'(2y_1'+py_1)+u(y_1''+py_1'+qy_1)=0$$

Since y_1 is a solution then $y_1''+py_1'+qy_1=0$.

$$\Rightarrow u''y_1+u'(2y_1'+py_1)=0 \Rightarrow u''+u'\frac{2y_1'+py_1}{y_1}=0$$

Put $u'=U$ and $u''=U'$ then

$U'+\left(\frac{2y_1'}{y_1}+p\right)U=0$, this is the desired first-order equation.

$$\Rightarrow \frac{dU}{U}=-\left(\frac{2y_1'}{y_1}+p\right)dx \Rightarrow \ln|U|=-2\ln|y_1|-\int p\,dx \ \Rightarrow U=\frac{1}{y_1^2}e^{-\int p\,dx}$$

$$\because u'=U \Rightarrow u=\int U\,dx \Rightarrow y_2=y_1\int U\,dx$$

The quotient $\frac{y_2}{y_1}=u=\int U\,dx$ can not be constant (because $U\neq 0$), so that y_1 and y_2 form a basis.