(b) Second Order Homogeneous Equations with Constant Coefficients

Consider

$$y'' + ay' + by = 0 \qquad \text{.......(1)}$$

where coefficient a and b are constant.

Let us assume its solution is $y = e^{\lambda x}$

Substitute $y = e^{\lambda x}$, $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$ in equation (1), we will get

$$\left(\lambda^2 + a\lambda + b\right)e^{\lambda x} = 0$$

Hence $y = e^{\lambda x}$ is a solution of (1) if λ is a solution of quadratic equation

$$\boxed{\lambda^2 + a\lambda + b = 0} \qquad \text{.......(2)}$$

This equation is called the **characteristic equation** (or auxiliary equation) of (1). Its roots are

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) \text{ and } \lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right)$$

This shows that $y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are solution of (1).

Let us consider three different cases

Case I: two real roots if $a^2 - 4b > 0$

Case II: a real double root if $a^2 - 4b = 0$

Case III: complex conjugate roots if $a^2 - 4b < 0$

Case I: Two Distinct Real Roots λ_1 and λ_2

In this case,

$$y_1 = e^{\lambda_1 x} \text{ and } y_2 = e^{\lambda_2 x}$$

Constitute a basis of solutions of (1). The corresponding general solution is

$$\boxed{y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}}$$

Case II: Real Double Root $\lambda = -\frac{a}{2}$

If the discriminant $a^2 - 4b = 0$, then $\lambda = \lambda_1 = \lambda_2 = -\frac{a}{2}$. Hence only one solution

$$y_1 = e^{-(a/2)x}.$$

Let us obtain a second solution y_2 of $y'' + ay' + by = 0$,

$$\Rightarrow p = a \Rightarrow -\int p\,dx = -ax$$

$$\because U = \frac{1}{y_1^2} e^{-\int p\,dx} \Rightarrow U = \frac{1}{e^{-ax}} e^{-ax} = 1$$

$$\because y_2 = y_1 \int U\,dx \Rightarrow y_2 = e^{-(a/2)x} \int dx = x e^{-(a/2)x}$$

Thus a basis of solutions of $y'' + ay' + by = 0$ is

$$e^{-(a/2)x}, \; x e^{-(a/2)x}.$$

The corresponding general solution is

$$\boxed{y = (c_1 + c_2 x) e^{-(a/2)x}}$$

Case III: Complex Roots

If the discriminant $a^2 - 4b < 0$, then

$$\lambda_1 = \frac{1}{2}\left(-a + \sqrt{a^2 - 4b}\right) = -\frac{1}{2}a + \frac{1}{2} i\sqrt{4b - a^2} = -\frac{1}{2}a + i\sqrt{b - \frac{1}{4}a^2} = -\frac{1}{2}a + i\omega$$

and $\lambda_2 = \frac{1}{2}\left(-a - \sqrt{a^2 - 4b}\right) = -\frac{1}{2}a - i\omega$ where $\omega = \sqrt{b - \frac{1}{4}a^2}$

Thus

$$e^{\lambda_1 x} = e^{-\frac{1}{2}ax + i\omega x} = e^{-\frac{a}{2}x}\left(\cos\omega x + i\sin\omega x\right) \text{ and } e^{\lambda_2 x} = e^{-\frac{1}{2}ax - i\omega x} = e^{-\frac{a}{2}x}\left(\cos\omega x - i\sin\omega x\right).$$

$$\because \cos\omega x = \frac{1}{2}\left(e^{i\omega x} + e^{-i\omega x}\right) \text{ and } \sin\omega x = \frac{1}{2i}\left(e^{i\omega x} - e^{-i\omega x}\right)$$

Hence

$$y_1 = e^{-\frac{a}{2}x}\cos\omega x \text{ and } y_2 = e^{-\frac{a}{2}x}\sin\omega x.$$

The corresponding general solution is

$$y = e^{-\frac{a}{2}x}\left(A\cos\omega x + B\sin\omega x\right)$$

Summary of Cases I-III

$$y'' + ay' + by = 0$$

Case	Roots	Basis	General Solution
I	Distinct real $\lambda_1, \quad \lambda_2$	$e^{\lambda_1 x}, \ e^{\lambda_2 x}$	$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
II	Real Double root $\lambda = \lambda_1 = \lambda_2 = -\frac{1}{2}a$	$e^{-\frac{a}{2}x}, \quad xe^{-\frac{a}{2}x}$	$y = \left(c_1 + c_2 x\right)e^{-\frac{a}{2}x}$
III	Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$	$e^{-\frac{a}{2}x}\cos\omega x$ $e^{-\frac{a}{2}x}\sin\omega x$	$y = e^{-\frac{a}{2}x}\left(A\cos\omega x + B\sin\omega x\right)$