(c) Second Order Homogeneous Equation of the form $x^2y'' + axy' + by = 0$

Consider linear differential equations in y, xy' and x^2y'' that is

$$x^2y'' + axy' + by = 0$$

Substitute $x = e^z$,

then $dx = e^z dz \Rightarrow \frac{d}{dx} = \frac{1}{e^z}\frac{d}{dz} \Rightarrow e^z\frac{d}{dx} = \frac{d}{dz} \Rightarrow x\frac{d}{dx} = \frac{d}{dz} = D$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = \frac{1}{e^z}\frac{d}{dz}\left(\frac{1}{e^z}\frac{d}{dz}\right) = \frac{1}{e^z}\left(-\frac{1}{e^z}\frac{d}{dz} + \frac{1}{e^z}\frac{d^2}{dz^2}\right) = \frac{1}{e^{2z}}\left(-\frac{d}{dz} + \frac{d^2}{dz^2}\right) = \frac{1}{x^2}\left(D^2 - D\right)$$

$$\Rightarrow x^2\frac{d^2}{dx^2} = D(D-1)$$

Similarly $x^3\frac{d^3}{dx^3} = D(D-1)(D-2)$

Thus $x^2y'' + axy' + by = 0 \Rightarrow \left[D(D-1) + aD + b\right]y = 0 \Rightarrow \left[D^2 + (a-1)D + b\right]y = 0$

$$\Rightarrow y'' + (a-1)y' + by = 0 \text{ where } y' = \frac{dy}{dz}.$$

This is second order homogeneous equation with constant coefficients.

Example: Solve $x^2y'' - 2.5xy' - 2.0y = 0$.

Solution: Compare with $x^2y'' + axy' + by = 0$, then $a = -2.5$ and $b = -2.0$

Substitute $x = e^z \Rightarrow y'' + (a-1)y' + by = 0$ where $y' = \frac{dy}{dz}$.

$\Rightarrow y'' - 3.5y' - 2.0y = 0$

The **characteristic equation is**

$$\lambda^2 - 3.5\lambda - 2.0 = 0 \Rightarrow \lambda = -0.5,\ 4$$

Thus general solution is

$$y = c_1e^{-0.5z} + c_2e^{4z} = c_1e^{-\frac{z}{2}} + c_2e^{4z} = c_1\frac{1}{\sqrt{x}} + c_2x^4.$$