(d) Concept of Wronskian (Linear dependence and independence)

Consider homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

and two initial conditions $y(x_0) = K_0$ and $y'(x_0) = K_1$.

Its general solution is

$$y = c_1 y_1 + c_2 y_2$$

Using these conditions we can find constants c_1 and c_2.

If y_1 and y_2 are two solutions of homogeneous equation, then Wornskian is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

Note: The two solutions y_1 and y_2 are linearly dependent if and only if their Wornskian is zero. If Wronskian is not zero then y_1 and y_2 are linearly independent.

Example: Solution of $y'' + \omega^2 y = 0$ is $y_1 = \cos\omega x$ and $y_2 = \sin\omega x$. Check whether they are linearly dependent or independent.

Solution: Wronskian is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos\omega x & \sin\omega x \\ -\omega\sin\omega x & \omega\cos\omega x \end{vmatrix} = \omega\left(\cos^2\omega x + \sin^2\omega x\right) = \omega$$

If $\omega \neq 0$, they are linearly independent.

Example: Solution of $y'' - 2y' + y = 0$ is $y = (c_1 + c_2 x)e^x$. Check whether they are linearly dependent or independent.

Solution: Wronskian is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & (x+1)e^x \end{vmatrix} = (x+1)e^{2x} - xe^{2x} \neq 0$$

They are linearly independent.

Finding a Second Solution

Consider homogeneous equation

$$y'' + p(x)y' + q(x)y = 0$$

If y_1 and y_2 are two independent solutions, then Wronskian is defined by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

By differentiating Wronskian, we will get

$$W' = (y_1' y_2' + y_1 y_2'') - (y_1'' y_2 + y_1' y_2') = y_1 y_2'' - y_1'' y_2$$

$$\because y'' + p(x) y' + q(x) y = 0 \Rightarrow \frac{y''}{y} + p(x)\frac{y'}{y} = -q(x)$$

$$\Rightarrow \frac{y_1''}{y_1} + p(x)\frac{y_1'}{y_1} = -q(x) = \frac{y_2''}{y_2} + p(x)\frac{y_2'}{y_2}$$ (y_1 and y_2 are two independent solutions)

$$\Rightarrow y_1'' y_2 + p(x) y_1' y_2 = -q(x) y_1 y_2 \text{ and } y_2'' y_1 + p(x) y_2' y_1 = -q(x) y_1 y_2$$

$$\Rightarrow y_2'' y_1 - y_1'' y_2 + p(x)(y_2' y_1 - y_1' y_2) = 0$$

$$\Rightarrow y_2'' y_1 - y_1'' y_2 = -p(x)(y_2' y_1 - y_1' y_2)$$

$$\Rightarrow W' = -p(x) W \Rightarrow \frac{dW}{dx} = -p(x) W \Rightarrow \frac{dW}{W} = -p(x)\,dx$$

$$\Rightarrow W = e^{-\int p(x)dx}$$