(e) Nonhomogeneous Linear Equations of Second Order

Consider nonhomogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = r(x) \qquad \text{.....(1)}$$

General Solution

A general solution of the nonhomogeneous equation (1) is of the form

$$y(x) = y_h(x) + y_p(x)$$

where $y_h(x) = c_1y_1(x) + c_2y_2(x)$ is a general solution of the homogeneous equation $y'' + p(x)y' + q(x)y = 0$ and $y_p(x)$ is any solution containing no arbitrary constants.

Methods of Finding $y_p(x)$:

(a) $\dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax}$

If $f(a) = 0$ then $\dfrac{1}{f(D)}e^{ax} = x \cdot \dfrac{1}{f'(a)} \cdot e^{ax}$

If $f'(a) = 0$ then $\dfrac{1}{f(D)}e^{ax} = x^2 \cdot \dfrac{1}{f''(a)} \cdot e^{ax}$.

Example: Find particular integral for $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 5e^{3x}$.

Solution: $y_p(x) = \dfrac{1}{D^2 + 6D + 9}.5e^{3x} = \dfrac{5e^{3x}}{3^2 + 6.3 + 9} = \dfrac{5e^{3x}}{36}$

Example: Find particular integral for $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 7e^{-2x} - \log 2$.

Solution: $y_p(x) = \dfrac{1}{D^2 - 6D + 9}.7e^{-2x} + \dfrac{1}{D^2 - 6D + 9}.(-\log 2)$

$$\Rightarrow y_p(x) = \frac{7e^{-2x}}{4 + 12 + 9} - \frac{\log 2 \times e^{0x}}{0 - 0 + 9} = \frac{7e^{-2x}}{25} - \frac{\log 2}{9}$$

Example: Find particular integral for $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = e^{3x}$.

Solution: $\because f(D) = D^2 - 6D + 9 \Rightarrow f(3) = 9 - 18 + 9 = 0$, $f'(3) = 6 - 6 = 0$

$\Rightarrow f''(D)=2 \Rightarrow y_p(x)=\dfrac{1}{\lambda^2-6\lambda+9}.e^{3x}=x^2\dfrac{1}{2}e^{3x}$

(b) $\dfrac{1}{f(D)}x^n=\left[f(D)\right]^{-1}x^n$. Expand $\left[f(D)\right]^{-1}$ and then operate.

Example: Find particular integral for $\left(D^2+5D+4\right)y=3-2x$.

Solution: $y_p(x)=\left[D^2+5D+4\right]^{-1}(3-2x)=\dfrac{1}{4}\left[1+\dfrac{5}{4}D+\dfrac{5}{4}D^2\right]^{-1}(3-2x)$

$\Rightarrow y_p(x)=\dfrac{1}{4}\left[1-\dfrac{5}{4}D-\dfrac{5}{4}D^2\right](3-2x)=\dfrac{1}{4}\left[3-2x-\dfrac{5}{4}\times-2\right]=\dfrac{1}{8}[11-4x]$

(c) $\dfrac{1}{f\left(D^2\right)}\sin ax=\dfrac{1}{f\left(-a^2\right)}\sin ax$ and $\dfrac{1}{f\left(D^2\right)}\cos ax=\dfrac{1}{f\left(-a^2\right)}\cos ax$

If $f\left(-a^2\right)=0$ then $\dfrac{1}{f\left(D^2\right)}\sin ax=x.\dfrac{1}{f'\left(-a^2\right)}\sin ax$.

Example: Find particular integral for $\left(D^2+4\right)y=\sin 3x$.

Solution: $y_p(x)=\dfrac{1}{D^2+4}\sin 3x=\dfrac{\sin 3x}{\left(-3^2\right)+4}=-\dfrac{1}{5}\sin 3x$

Example: Find particular integral for $\left(D^2+4\right)y=\cos 2x$.

Solution: $\because f\left(D^2\right)=\left(D^2+4\right)\Rightarrow f\left(-2^2\right)=\left(-4+4\right)=0,\ f'\left(-2^2\right)\neq 0$

$\Rightarrow y_p(x)=\dfrac{1}{D^2+4}\cos 2x=x\dfrac{1}{2D}\cos 2x=\dfrac{x}{2}\left(\dfrac{1}{2}\sin 2x\right)=\dfrac{x}{4}\sin 2x$.

Example: Find particular integral for $\left(D^2+D+1\right)y=\cos 2x$.

Solution: $y_p(x)=\dfrac{1}{D^2+D+1}\cos 2x=\dfrac{1}{-2^2+D+1}\cos 2x=\dfrac{1}{D-3}.\cos 2x$.

$\Rightarrow y_p(x)=\dfrac{D+3}{D^2-9}\cos 2x=\dfrac{D+3}{-2^2-9}\cos 2x=\dfrac{1}{13}\left(2\sin 2x-3\cos 2x\right)$

(d) $\dfrac{1}{f(D)}e^{ax}\phi(x)=e^{ax}\dfrac{1}{f(D+a)}\phi(x)$