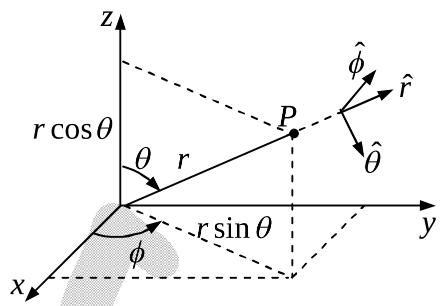
2(b). Spherical Polar Coordinate System

In spherical polar coordinate any general point P lies on the surface of a sphere. The spherical polar coordinates (r,θ,ϕ) of a point P are defined in figure shown below; r is the distance from the origin (the magnitude of the position vector), θ (the angle drawn from the z axis) is called the polar angle, and ϕ (the angle around from the x axis) is the azimuthal angle.



Their relation to cartesian coordinates (x,y,z) can be read from the figure:

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta$$

and $r = \sqrt{x^2+y^2+z^2}$, $\theta = \cos^{-1}\left(\frac{z}{r}\right)$, $\phi = \tan^{-1}\left(\frac{y}{x}\right)$

The range of r is $0 \to \infty$, θ goes from $0 \to \pi$, and ϕ goes from $0 \to 2\pi$.

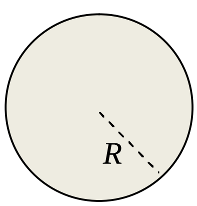
Figure shows three unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$, pointing in the direction of increase of the corresponding coordinates. They constitute an orthogonal (mutually perpendicular) basis set (just like $\hat{x}, \hat{y}, \hat{z}$), and any vector $\vec{A}$ can be expressed in terms of them in the usual way:

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$$

A_r, A_θ, and A_ϕ are the radial, polar and azimuthal components of $\vec{A}$.

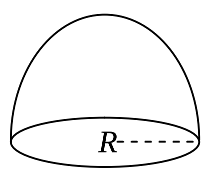
If we have sphere or any part of the sphere, then we can specify r,θ and ϕ. Lets consider some examples shown in figure below:

Sphere



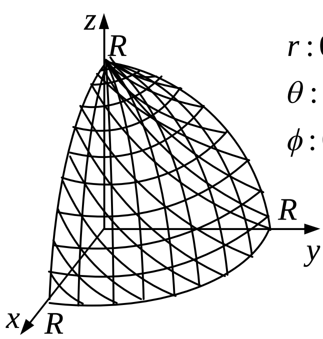
$r: 0 \to R$
$\theta: 0 \to \pi$
$\phi: 0 \to 2\pi$

Hemi-sphere



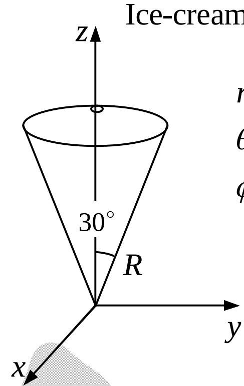
$r: 0 \to R$
$\theta: 0 \to \pi/2$
$\phi: 0 \to 2\pi$

Octant



$r : 0 \to R$
$\theta : 0 \to \pi/2$
$\phi : 0 \to \pi/2$

Ice-creame cone



$r : 0 \to R$
$\theta : 0 \to \pi/6$
$\phi : 0 \to 2\pi$

Infinitesimal Displacement Vector $\left(d\vec{l}\right)$

An infinitesimal displacement in the $\hat{r}$ direction is simply dr (figure *a*), just as an infinitesimal element of length in the x direction is dx:

$$dl_r = dr$$

On the other hand, an infinitesimal element of length in the $\hat{\theta}$ direction (figure *b*) is $r\,d\theta$

$$dl_\theta = r d\theta$$

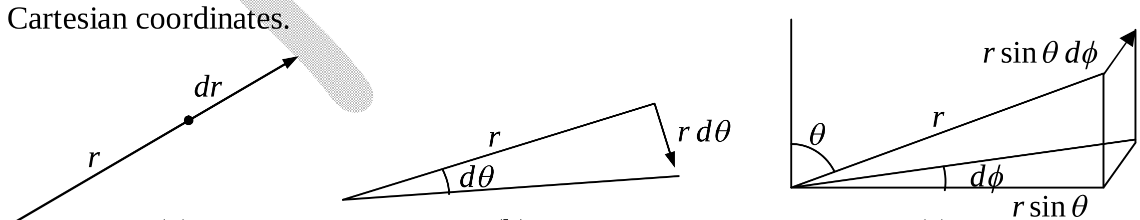
Similarly, an infinitesimal element of length in the $\hat{\phi}$ direction (figure *c*) is $r \sin\theta\, d\phi$

$$dl_\phi = r \sin\theta\, d\phi$$

Thus, the general infinitesimal displacement $d\vec{l}$ is

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\hat{\theta} + r\sin\theta\,d\phi\hat{\phi}$$

This plays the role (in line integrals, for example) that $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ played in Cartesian coordinates.



(a) (b) (c)

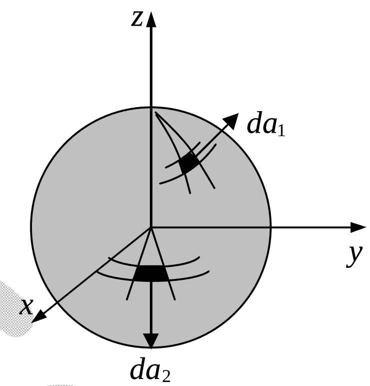
Area Element $\left(d\vec{a}\right)$

If we are integrating over the surface of a sphere, for instance, then r is constant, whereas θ and ϕ change, so

$$d\vec{a}_1 = dl_\theta dl_\phi \hat{r} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

on the other hand, if the surface lies in the xy plane, then θ is constant ($\theta = \pi/2$) while r and ϕ vary, then

$$d\vec{a}_2 = dl_r dl_\phi \hat{\theta} = r^2 \, dr \, d\phi \hat{\theta}$$



If we are integrating over the surface of an octant, for instance, then r is constant $(r = R)$, whereas θ and ϕ change,

so, $$d\vec{a}_1 = dl_\theta dl_\phi \hat{r} = R^2 \sin\theta d\theta d\phi \hat{r} .$$

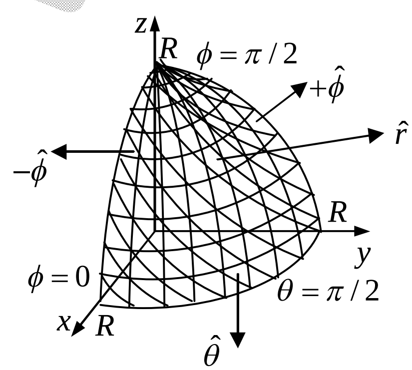
If the surface lies in the xy plane, then θ is constant ($\theta = \pi/2$) while r and ϕ vary, then

$$d\vec{a}_2 = dl_r dl_\phi \hat{\theta} = r^2 dr d\phi \hat{\theta}$$

If the surface lies in the yz plane, then ϕ is constant ($\phi = \pi/2$) while r and ϕ vary, then

$$d\vec{a}_3 = dl_r dl_\theta \hat{\phi} = r dr d\theta \hat{\phi}$$

Octant



If the surface lies in the xz plane, then ϕ is constant ($\phi = 0$) while r and ϕ vary, then

$$d\vec{a}_4 = -dl_r dl_\theta \hat{\phi} = -r dr d\theta \hat{\phi}$$

Volume Element $\left(d\tau\right)$

The infinitesimal volume element $d\tau$, in spherical coordinates, is the product of the three infinitesimal displacements:

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin\theta \, dr \, d\theta \, d\phi$$