3(b). The Divergence

The Operator $\vec{\nabla}$

The gradient has the formal appearance of a vector, $\vec{\nabla}$, "multiplying" a scalar V:

$$\vec{\nabla}V = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)V$$

The term in parentheses is called "del":

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

We should say that $\vec{\nabla}$ is a vector operator that acts upon V, not a vector that multiplies V.

There are three ways the operator $\vec{\nabla}$ can act:

1. on a scalar function V: $\vec{\nabla}V$ (the **gradient**);

2. on a vector function $\vec{A}$, via the dot product: $\vec{\nabla}\cdot\vec{A}$ (the **divergence**);

3. on a vector function $\vec{A}$, via the cross product: $\vec{\nabla}\times\vec{A}$ (the **curl**).

The Divergence

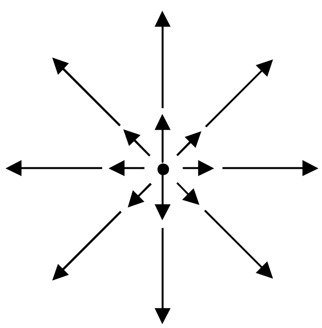
From the definition of $\vec{\nabla}$ we construct the divergence:

$$\vec{\nabla}\cdot\vec{A} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}\right) = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

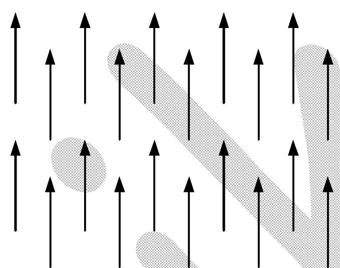
Observe that the divergence of a vector function $\vec{A}$ is itself a scalar $\vec{\nabla}\cdot\vec{A}$. (You can't have the divergence of a scalar: that's meaningless.)

Geometrical Interpretation

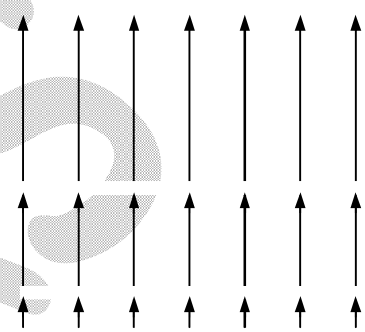
$\vec{\nabla}\cdot\vec{A}$ is a measure of how much the vector $\vec{A}$ spreads out (diverges) from the point in question. For example, the vector function in figure (a) has a large (positive) divergence (if the arrows pointed in, it would be a large negative divergence), the function in figure (b) has zero divergence, and the function in figure (c) again has a positive divergence. (Please understand that $\vec{A}$ here is a function-there's a different vector associated with every point in space.)



(a) $\vec{A}=x\hat{x}+y\hat{y}+z\hat{z}$



(b) $\vec{A}=\hat{z}$



(c) $\vec{A}=z\hat{z}$

Divergence in Spherical polar coordinates

$$\vec{\nabla}.\vec{A}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

Divergence in cylindrical coordinates

$$\vec{\nabla}.\vec{A}=\frac{1}{r}\frac{\partial}{\partial r}\left(rA_r\right)+\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}$$

Example: Suppose the function sketched in above figure are $\vec{A}=x\hat{x}+y\hat{y}+z\hat{z}$, $\vec{B}=\hat{z}$ and $\vec{C}=z\hat{z}$. Calculate their divergences.

Solution:

$$\vec{\nabla}\cdot\vec{A}=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3$$

$$\vec{\nabla}\cdot\vec{B}=\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(1)=0+0+0=0$$

$$\vec{\nabla}\cdot\vec{C}=\frac{\partial}{\partial x}(0)+\frac{\partial}{\partial y}(0)+\frac{\partial}{\partial z}(z)=0+0+1=1$$

Example: Given

(i) $\vec{A} = 2xy\hat{x} + z\hat{y} + yz^2\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$ at $(2,-1,3)$

(ii) $\vec{A} = 2r\cos^2\phi\hat{r} + 3r^2\sin z\hat{\phi} + 4z\sin^2\phi\hat{z}$, find $\vec{\nabla}\cdot\vec{A}$

(iii) $\vec{A} = 10\hat{r} + 5\sin\theta\hat{\theta}$, Find $\vec{\nabla}\cdot\vec{A}$

Solution: (i) In Cartesian coordinates $\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$A_x = 2xy, A_y = z, A_z = yz^2 \Rightarrow \vec{\nabla}\cdot\vec{A} = 2y + 0 + 2yz$, At $(2,-1,3)$, $\vec{\nabla}\cdot\vec{A} = -2-6 = -8$

(ii) In cylindrical coordinates $\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$

$$A_r = 2r\cos^2\phi, A_\phi = 3r^2\sin z, A_z = 4z\sin^2\phi$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r}4r\cos^2\phi + 0 + 4\sin^2\phi = 4\left(\cos^2\phi + \sin^2\phi\right) = 4$$

(iii) In spherical coordinates, $\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$

$$A_r = 10, A_\theta = 5\sin\theta, A_\phi = 0$$

$$\Rightarrow \vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}20r + \frac{1}{r\sin\theta}10\sin\theta\cos\theta = (2+\cos\theta)(10/r)$$