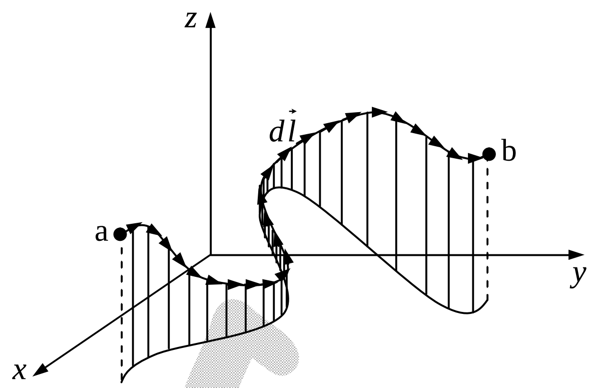
4(a). Line Integrals

A line integral is an expression of the form

$$\int_a^b \vec{A}\cdot d\vec{l},$$



where $\vec{A}$ is a vector function, $d\vec{l}$ is the infinitesimal displacement vector and the integral is to be carried out along a prescribed path *P* from point *a* to point *b*. If the path in question forms a closed loop (that is, if *b* = *a*), put a circle on the integral sign:

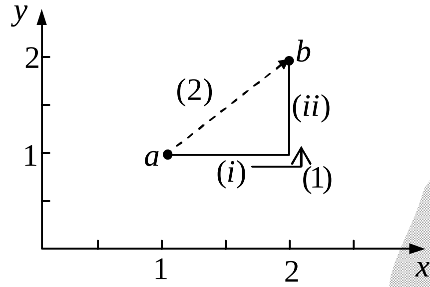
$$\oint \vec{A}\cdot d\vec{l}.$$

At each point on the path we take the dot product of $\vec{A}$ (evaluated at that point) with the displacement $d\vec{l}$ to the next point on the path. The most familiar example of a line integral is the work done by a force $\vec{F}$:

$$W = \int \vec{F}\cdot d\vec{l}$$

Ordinarily, the value of a line integral depends critically on the particular path taken from *a* to *b*, but there is an important special class of vector functions for which the line integral is independent of the path, and is determined entirely by the end points(A *force* that has this property is called ***conservative***.)

Example: Calculate the line integral of the function $\vec{A} = y^2\hat{x} + 2x(y+1)\hat{y}$ from the point a = (1, 1, 0) to the point b = (2, 2, 0), along the paths (1) and (2) as shown in figure. What is $\oint \vec{A}\cdot d\vec{l}$ for the loop that goes from a to b along (1) and returns to a along (2)?



Solution: Since $d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z}$. Path (1) consists of two parts. Along the "horizontal" segment $dy = dz = 0$, so

(i) $d\vec{l} = dx\hat{x}, \quad y = 1, \quad \vec{A}\cdot d\vec{l} = y^2 dx = dx$, so $\int \vec{A}\cdot d\vec{l} = \int_1^2 dx = 1$

On the "vertical" stretch $dx = dz = 0$, so

(ii) $d\vec{l} = dy\hat{y}, \quad x = 2, \quad \vec{A}\cdot d\vec{l} = 2x(y+1)dy = 4(y+1)dy$, so $\int \vec{A}\cdot d\vec{l} = 4\int_1^2 (y+1)dy = 10$.

By path (1), then,

$$\int_a^b \vec{A}\cdot d\vec{l} = 1 + 10 = 11$$

Meanwhile, on path (2) $x = y$, $dx = dy$, and $dz = 0$, so

$$d\vec{l} = dx\hat{x} + dx\hat{y}, \quad \vec{A}\cdot d\vec{l} = x^2 dx + 2x(x+1)dx = (3x^2 + 2x)dx$$

so

$$\int_a^b \vec{A}\cdot d\vec{l} = \int_1^2 (3x^2 + 2x)dx = \left(x^3 + x^2\right)\Big|_1^2 = 10$$

For the loop that goes out (1) and back (2), then,

$$\oint \vec{A}\cdot d\vec{l} = 11 - 10 = 1$$