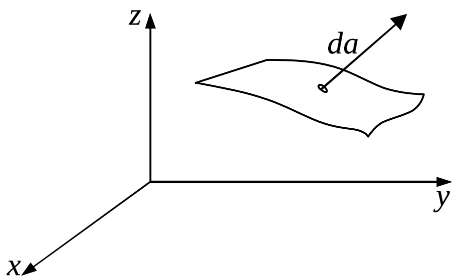
4(b). Surface Integrals



A surface integral is an expression of the form

$$\int_S \vec{A} \cdot d\vec{a}$$

where $\vec{A}$ is again some vector function, and $d\vec{a}$ is an infinitesimal patch of area, with direction perpendicular to the surface(as shown in figure). There are, of course, two directions perpendicular to any surface, so the sign of a surface integral is intrinsically ambiguous. If the surface is closed then “outward” is positive, but for open surfaces it’s arbitrary.

If $\vec{A}$ describes the flow of a fluid (mass per unit area per unit time), then $\int \vec{A} \cdot d\vec{a}$ represents the total mass per unit time passing through the surface-hence the alternative name, “flux.”

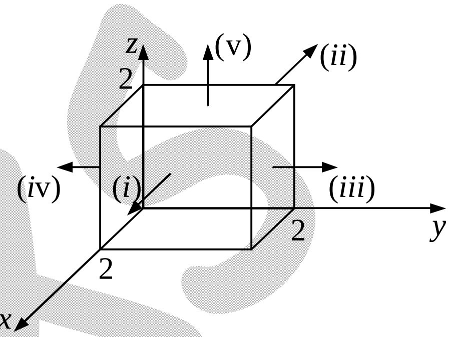
Ordinarily, the value of a surface integral depends on the particular surface chosen, but there is a special class of vector functions for which it is independent of the surface, and is determined entirely by the boundary line.

Example:

Calculate the surface integral of $\vec{A} = 2xz\hat{x} + (x+2)\hat{y} + y(z^2-3)\hat{z}$ over five sides (excluding the bottom) of the cubical box (side 2) as shown in figure. Let "upward and outward" be the positive direction, as indicated by the arrows.

Solution:

Taking the sides one at a time:



(i) $x = 2,\ d\vec{a} = dydz\hat{x},\ \vec{A}\cdot d\vec{a} = 2xzdydz = 4zdydz,$

so $\int \vec{A}\cdot d\vec{a} = 4\int_0^2 dy \int_0^2 zdz = 16$.

(ii) $x = 0,\ d\vec{a} = -dydz\hat{x},\ \vec{A}\cdot d\vec{a} = -2xzdydz = 0,$

so $\int \vec{A}\cdot d\vec{a} = 0$.

(iii) $y = 2,\ d\vec{a} = dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = (x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 (x+2)dx \int_0^2 dz = 12$.

(iv) $y = 0,\ d\vec{a} = -dx\,dz\,\hat{y},\ \vec{A}\cdot d\vec{a} = -(x+2)dx\,dz,$ so $\int \vec{A}\cdot d\vec{a} = -\int_0^2 (x+2)dx \int_0^2 dz = -12$.

(v) $z = 2,\ d\vec{a} = dx\,dy\,\hat{z},\ \vec{A}\cdot d\vec{a} = y(z^2-3)dx\,dy = y\,dx\,dy,$ so $\int \vec{A}\cdot d\vec{a} = \int_0^2 dx \int_0^2 ydy = 4$

Evidently the total flux is

$$\int_{\text{surface}} \vec{A}\cdot d\vec{a} = 16 + 0 + 12 - 12 + 4 = 20$$