4(f). The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = \oint_S \vec{A}\cdot d\vec{a}$$

This theorem has at least three special names: ***Gauss's theorem, Green's theorem***, or, simply, the ***divergence theorem***. Like the other "fundamental theorems," it says that the integral of a derivative (in this case the divergence) over a region (in this case a volume) is equal to the value of the function at the boundary (in this case the surface that bounds the volume). Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the "boundary" of a line is just two end points, but the boundary of a volume is a (closed) surface.

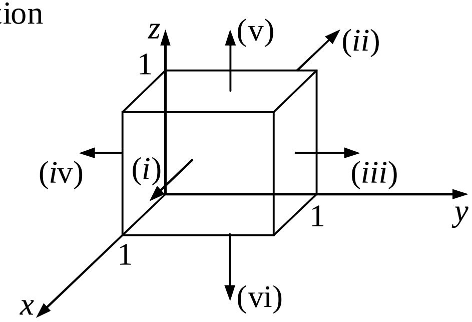
Geometrical Interpretation

If $\vec{A}$ represents the flow of an incompressible fluid, then "the flux of $\vec{A}$ (the right side of equation) is the total amount of fluid passing out through the surface, per unit time and the left side of equation shows an equal amount of liquid will be forced out through the boundaries of the region.

Example: Check the divergence theorem using the function

$$\vec{A} = y^2\hat{x} + \left(2xy + z^2\right)\hat{y} + \left(2yz\right)\hat{z}$$

and the unit cube situated at the origin.



Solution: In this case

$$\vec{\nabla}\cdot\vec{A} = 2(x+y),$$

and $$\int_V 2(x+y)d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\,dx\,dy\,dz,$$

$$\int_0^1 (x+y)dx = \frac{1}{2} + y,\ \int_0^1\left(\frac{1}{2}+y\right)dy = 1,\ \int_0^1 1dz = 1.$$

Evidently, $$\int_V \left(\vec{\nabla}\cdot\vec{A}\right)d\tau = 2$$

To evaluate the surface integral we must consider separately the six sides of the cube:

(i) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 y^2 dy\, dz = \frac{1}{3}$

(ii) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 y^2 dy\, dz = -\frac{1}{3}$

(iii) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 \left(2x + z^2\right) dx\, dz = \frac{4}{3}$

(iv) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 z^2 dx\, dz = -\frac{1}{3}$

(v) $\int \vec{A}\cdot d\vec{a} = \int_0^1\int_0^1 2y\, dx\, dy = 1$

(vi) $\int \vec{A}\cdot d\vec{a} = -\int_0^1\int_0^1 0\, dx\, dy = 0$

So the total flux is:

$$\oint_S \vec{A}\cdot d\vec{a} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2\,.$$