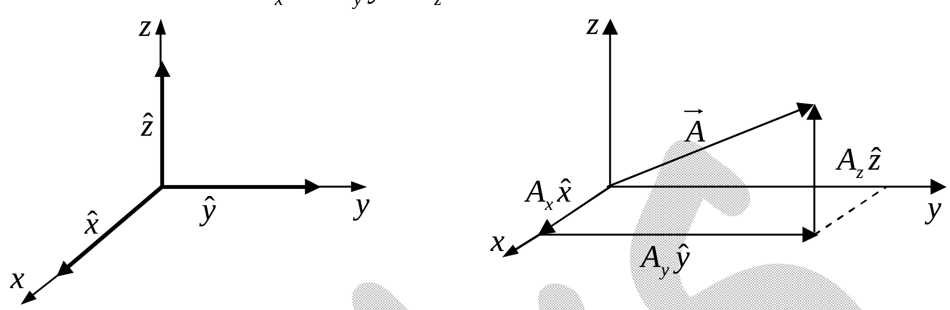
1(b). Vector Algebra: Component Form

Let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be unit vectors parallel to the x, y and z axis, respectively. An arbitrary vector $\vec{A}$ can be expanded in terms of these basis vectors

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$



The numbers A_x, A_y, and A_z are called component of $\vec{A}$; geometrically, they are the projections of $\vec{A}$ along the three coordinate axes.

(i) Rule: To add vectors, add like components.

$$\vec{A}+\vec{B} = \left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right)+\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right) = \left(A_x+B_x\right)\hat{x}+\left(A_y+B_y\right)\hat{y}+\left(A_z+B_z\right)\hat{z}$$

(ii) Rule: To multiply by a scalar, multiply each component.

$$\vec{A} = \left(aA_x\right)\hat{x}+\left(aA_y\right)\hat{y}+\left(aA_z\right)\hat{z}$$

Because $\hat{x}$, $\hat{y}$ and $\hat{z}$ are mutually perpendicular unit vectors

$$\hat{x}.\hat{x} = \hat{y}.\hat{y} = \hat{z}.\hat{z} = 1;\ \ \hat{x}.\hat{y} = \hat{x}.\hat{z} = \hat{y}.\hat{z} = 0$$

Accordingly, $\vec{A}.\vec{B} = \left(A_x\hat{x}+A_y\hat{y}+A_z\hat{z}\right).\left(B_x\hat{x}+B_y\hat{y}+B_z\hat{z}\right) = A_xB_x + A_yB_y + A_zB_z$

(iii) Rule: To calculate the dot product, multiply like components, and add.

In particular, $\vec{A}.\vec{A} = A_x^2 + A_y^2 + A_z^2 \Rightarrow A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

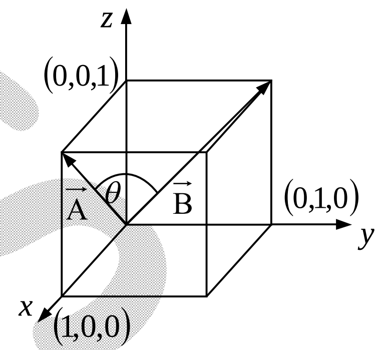
Similarly, $\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0,$

$$\hat{x}\times\hat{y} = -\hat{y}\times\hat{x} = \hat{z}$$
$$\hat{y}\times\hat{z} = -\hat{z}\times\hat{y} = \hat{x}$$
$$\hat{z}\times\hat{x} = -\hat{x}\times\hat{z} = \hat{y}$$

(iv) Rule: To calculate the cross product, form the determinant whose first row is $\hat{x}$, $\hat{y}$, $\hat{z}$, whose second row is $\vec{A}$ (in component form), and whose third row is $\vec{B}$.

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left(A_yB_z-A_zB_y\right)\hat{x}+\left(A_zB_x-A_xB_z\right)\hat{y}+\left(A_xB_y-A_yB_x\right)\hat{z}$$

Example: Find the angle between the face diagonals of a cube.



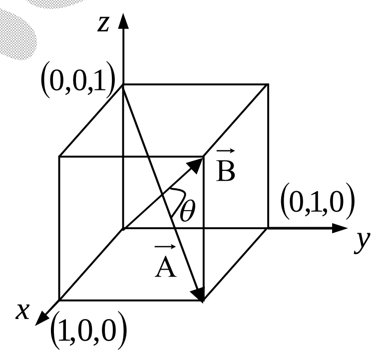
Solution: The face diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A}=1\hat{x}+0\hat{y}+1\hat{z};\quad \vec{B}=0\hat{x}+1\hat{y}+1\hat{z}$

So, $\Rightarrow \vec{A}.\vec{B}=1$

Also, $\Rightarrow \vec{A}.\vec{B}=AB\cos\theta=\sqrt{2}\sqrt{2}\cos\theta\Rightarrow\cos\theta=\frac{1}{2}\Rightarrow\theta=60^0$

Example: Find the angle between the body diagonals of a cube.



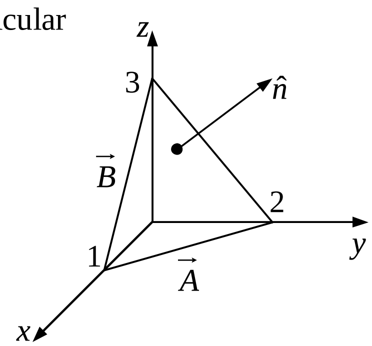
Solution: The body diagonals $\vec{A}$ and $\vec{B}$ are

$\vec{A}=\hat{x}+\hat{y}-\hat{z};\quad \vec{B}=\hat{x}+\hat{y}+\hat{z}$

So, $\Rightarrow \vec{A}.\vec{B}=1+1-1=1$

Also, $\Rightarrow \vec{A}.\vec{B}=AB\cos\theta=\sqrt{3}\sqrt{3}\cos\theta\Rightarrow\cos\theta=\frac{1}{3}\Rightarrow\theta=\cos^{-1}\left(\frac{1}{3}\right)$

Example: Find the components of the unit vector $\hat{n}$ perpendicular to the plane shown in the figure.



Solution: The vectors $\vec{A}$ and $\vec{B}$ can be defined as

$\vec{A}=-\hat{x}+2\hat{y};\quad \vec{B}=-\hat{x}+3\hat{z}\Rightarrow\hat{n}=\frac{\vec{A}\times\vec{B}}{\left|\vec{A}\times\vec{B}\right|}=\frac{6\hat{x}+3\hat{y}+2\hat{z}}{7}$