(b) Energy Distribution of Ideal Gas in Three Dimensions

Entropy (S) is measurement of randomness of system. It is function of number of microstate (which is number of ways to achieve any energy E). Statistically it is seen so $S = k_B \ln W$. And Law of nature reveal that at equilibrium the entropy is maximum so $dS = 0 \Rightarrow \ln W = 0$

For distinguishable particle. Taking in for equation $W = N! \prod_{i=1}^{l} \left[\frac{g_i^{n_i}}{n_i!} \right]$

$$\ln W = \ln \lfloor N + \sum_{l=1}^{l} \left[n_i \ln g_i - \ln \lfloor n_i \right]$$

Using sterling approximation $\ln \lfloor n = n \ln n - n$

$$= N \ln N - N + \sum_{l=1}^{l} n_i \ln g_i - \sum_{l=1}^{l} n_i \ln n_i + \sum_{l=1}^{l} n_i$$

$$d \ln W = \sum_{L=1}^{l} dn_i \ln g_i - \sum_{i=1}^{l} \left[n_i \frac{1}{n_i} dn_i + \ln n_i dn_i \right] + \sum_{i=1}^{l} dn_i$$

$$dW = \sum_{l=1}^{l} dn_i \ln g_i - \sum_{i=1}^{l} \ln n_i \, dn_i = \sum_{i=1}^{l} \ln \frac{g_i}{n_i} dn_i = 0$$

$$\sum n_i = N \Rightarrow \sum dn_i = 0 \Rightarrow \alpha \sum dn_i = 0$$

$$\sum E_i n_i = U \Rightarrow \sum E_i dn_i = 0 \Rightarrow \beta \sum E_i dn_i = 0$$

where α and β are used as LaGrange's multiplier which is using to make equation dimensionless where dimension of β is inverse of energy can be related to equilibrium temperature T so $\beta = \frac{1}{k_B T}$. α can be identified later can be related to fugacity

$$\sum_{l=1}^{l} \ln \frac{g_i}{n_i} dn_i = \alpha \sum_{l=1}^{l} dn_i + \beta \sum_{l=1}^{l} E_i dn_i$$

Equating the coefficient of dn_i then $\frac{n_i}{g_i} = \exp(-\alpha)\exp(-\beta E)$

So, Maxwell distribution is given by $f(\varepsilon) = \frac{n_i}{g_i} = e^{-\alpha} e^{-\beta E}$

Calculation of α in three dimensional case

Total number of particles in ideal gas $N = \sum_{l=1}^{l} n_i \Rightarrow N = \sum g_i e^{-\alpha} e^{-p\in}$

where energy levels are continuous then $N = e^{-\alpha} \int_0^\infty g(E) e^{-\beta E} dE$

for three dimensional case $g(E) = \frac{2\pi V}{h^3} (2m)^{3/2} E^{1/2}$

$$N = e^{-\alpha} \int_0^\infty \frac{2\pi V}{h^3} (2m)^{3/2} E^{1/2} e^{-\beta E} dE$$

$$N = e^{-\alpha} \frac{2\pi V}{h^3} (2m)^{3/2} \int_0^\infty E^{1/2} e^{-\beta E} dE = e^{-\alpha} \frac{2\pi V}{h^3} (2m)^{3/2} \left(\frac{1}{\beta}\right)^{3/2} \left\lceil \frac{3}{2} \right.$$

where $\beta = \frac{1}{k_B T}, \left\lceil \frac{3}{2} \right. = \frac{1}{2}\sqrt{\pi}$

Derivation of Maxwell-Boltzmann Distribution in Three Dimension

The Maxwell-Boltzmann distribution law for the particles in the states is

$$n_i = g_i \exp(-\alpha - \beta E_i) \ , \ n_i = g_i (\exp - \alpha)(\exp - \beta E_i)$$

After using the values $e^{-\alpha} = \frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2}$, where $\beta = \frac{1}{k_B T}$

We get $\frac{n_i}{g_i} = f(E) = \frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} \exp\left(-\frac{E}{k_B T}\right)$

The number of particles $dN(E)$ having energies in the range from E to $E + dE$ is $dN(E) = f(E) g(E) dE$ where $f(E)$ is distribution function and $g(E)dE$ is number of level (quantum state) in the range of E to $E + dE$

The number of particles $dn(E)$ having energies in the range from E to $E + dE$ in three dimensional space

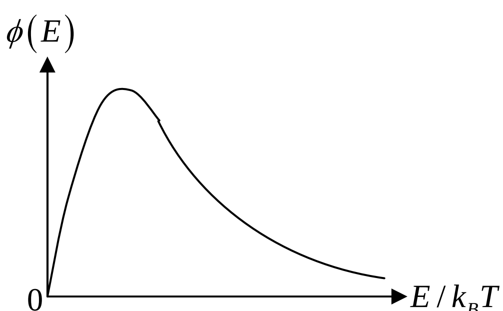

Figure 1

$$dN = \frac{N}{V}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} \exp\left(-\frac{E}{k_B T}\right) 2\pi V \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} E^{1/2} dE$$

$$dN = \frac{2\pi N}{\left(\pi k_B T\right)^{3/2}} E^{1/2} \exp\left(-\frac{E}{k_B T}\right) dE$$

$\frac{dN}{N}$ is fraction of particles that have energy between from E to $E + dE$ is $\frac{dN}{N} = \frac{2\pi}{\left(\pi k_B T\right)^{3/2}} E^{1/2} e^{-E/k_B T} dE$ which is popularly known as probability to find gas have energy between from E to $E + dE$ $\frac{dN}{N} = \phi(E) dE = \frac{2\pi}{\left(\pi k_B T\right)^{3/2}} E^{1/2} \exp\left(-\frac{E}{k_B T}\right) dE$,

where $\phi(E) = \frac{2\pi}{\left(\pi k_B T\right)^{3/2}} E^{1/2} \exp\left(-\frac{E}{k_B T}\right)$ is identified probability density

This is known as the Maxwell-Boltzmann energy distribution law for an ideal gas,

where $\lambda = \frac{h}{\sqrt{2\pi m k_B T}}$ is defined as the thermal wavelength.