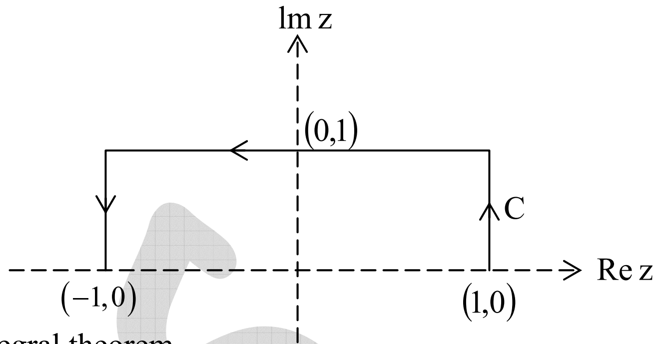
MATHEMATICAL PHYSICS SOLUTIONS

NET/JRF (JUNE-2011)

Q1. The value of the integral $\int_C dz\, z^2 e^z$, where C is an open contour in the complex z-plane as shown in the figure below, is:



(a) $\frac{5}{e}+e$ (b) $e-\frac{5}{e}$

(c) $\frac{5}{e}-e$ (d) $-\frac{5}{e}-e$

Ans. : (c)

Solution: If we complete the contour, then by Cauchy integral theorem

$$\int_{-1}^{1} dz z^2 e^z + \int_C dz z^2 e^z = 0 \Rightarrow \int_C dz z^2 e^z = -\int_{-1}^{1} dz z^2 e^z = -\left[z^2 e^z - 2ze^z + 2e^z\right]_{-1}^{1} = \frac{5}{e} - e$$

Q2. Which of the following matrices is an element of the group $SU(2)$?

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1+i}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1-i}{\sqrt{3}} \end{pmatrix}$

(c) $\begin{pmatrix} 2+i & i \\ 3 & 1+i \end{pmatrix}$ (d) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$

Ans. : (b)

Solution: $SU(2)$ is a group defined as following: $SU(2) = \left\{ \begin{pmatrix} \alpha & -\beta \\ \beta & \bar{\alpha} \end{pmatrix} : \alpha, \beta \in C; |\alpha|^2 + |\beta|^2 = 1 \right\}$

clearly (b) hold the property of $SU(2)$. $\alpha = \frac{1+i}{\sqrt{3}}, \beta = \frac{1}{\sqrt{3}}$ and $\bar{\alpha} = \frac{1-i}{\sqrt{3}}, \bar{\beta} = \frac{1}{\sqrt{3}}$.

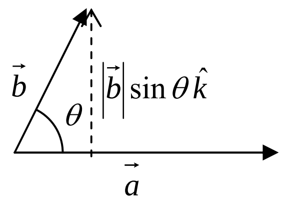
Note: $SU(2)$ has wide applications in electroweak interaction covered in standard model of particle physics.

Q3. Let $\vec{a}$ and $\vec{b}$ be two distinct three dimensional vectors. Then the component of $\vec{b}$ that is perpendicular to $\vec{a}$ is given by

(a) $\frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}$ (b) $\frac{\vec{b}\times(\vec{a}\times\vec{b})}{b^2}$ (c) $\frac{(\vec{a}\cdot\vec{b})\vec{b}}{b^2}$ (d) $\frac{(\vec{b}\cdot\vec{a})\vec{a}}{a^2}$

Ans. : (a)

Solution: $\vec{a}\times\vec{b} = ab\sin\theta\,\hat{n}$ where $\hat{n}$ is perpendicular to plane containing $\vec{a}$ and $\vec{b}$ and pointing upwards.



$$\vec{a}\times(\vec{a}\times\vec{b}) = ab\sin\theta(\vec{a}\times\hat{n}) = -a^2 b\sin\theta\,\hat{k}$$

$$b\sin\theta\,\hat{k} = \frac{-\vec{a}\times(\vec{a}\times\vec{b})}{a^2} \Rightarrow b\sin\theta\,\hat{k} = \frac{\vec{a}\times(\vec{b}\times\vec{a})}{a^2}.$$

Q4. Let $p_n(x)$ (where $n = 0,1,2,......$) be a polynomial of degree n with real coefficients, defined in the interval $2 \le n \le 4$. If $\int_2^4 p_n(x)p_m(x)dx = \delta_{nm}$, then

(a) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(-3-x)$ (b) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{3}(3+x)$

(c) $p_0(x) = \frac{1}{2}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$ (d) $p_0(x) = \frac{1}{\sqrt{2}}$ and $p_1(x) = \sqrt{\frac{3}{2}}(3-x)$

Ans. : (d)

Solution: For n not equal to m kroneker delta become zero. One positive and one negative term can make integral zero. So answer may be (c) or (d). Now take $n = m = 0$ so $p_0(x) = \frac{1}{\sqrt{2}}$ and then integrate. (d) is correct option because it satisfies the equation Check by integration and by orthogonal property of Legendre polynomial also.

Q5. Which of the following is an analytic function of the complex variable $z = x + iy$ in the domain $|z| < 2$?

(a) $(3 + x - iy)^7$ (b) $(1 + x + iy)^4(7 - x - iy)^3$

(c) $(1 - x - iy)^4(7 - x + iy)^3$ (d) $(x + iy - 1)^{1/2}$

Ans. : (b)

Solution: Put $z = x + iy$. If $\bar{z} = x - iy$ appears in any of the expressions then that expression is non-analytic. For option (d) we have a branch point singularity as the power is $\frac{1}{2}$ which is fractional. Hence only option (b) is analytic.

Q6. Consider the matrix $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

A. The eigenvalues of M are

(a) 0, 1, 2 (b) 0, 0, 3 (c) 1, 1, 1 (d) – 1, 1, 3

Ans. : (b)

Solution: For eigen values $\begin{bmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = 0$

$$(1-\lambda)\left((1-\lambda)^2 - 1\right) - (1-\lambda-1) + 1(1-(1-\lambda)) = 0$$

$$(1-\lambda)(1+\lambda^2 - 2\lambda - 1) + \lambda + \lambda = 0 \Rightarrow \lambda^2 - 2\lambda - \lambda^3 + 2\lambda^2 + 2\lambda = 0$$

$$\lambda^3 - 3\lambda^2 = 0 \Rightarrow \lambda^2(\lambda - 3) = 0 \Rightarrow \lambda = 0, 0, 3$$

For any $n \times n$ matrix having all elements unity eigenvalues are $0, 0, 0, \ldots, n$.

B. The exponential of M simplifies to (I is the 3 × 3 identity matrix)

(a) $e^M = I + \left(\frac{e^3 - 1}{3}\right)M$ (b) $e^M = I + M + \frac{M^2}{2!}$

(c) $e^M = I + 3^3 M$ (d) $e^M = (e-1)M$

Ans. : (a)

Solution: For e^M, let us try to diagonalize matrix M using similarity transformation.

For $\lambda = 3$, $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0,\ x_1 - 2x_2 + x_3 = 0,\ x_1 + x_2 - 2x_3 = 0$$

$$\Rightarrow -3x_2 + 3x_3 = 0 \text{ or } x_2 = x_3 \Rightarrow x_1 = x_2 = x_3 = k.$$

Eigen vector is $\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, where $k = 1$.

For $\lambda = 0$,

$$\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow x_1+x_2+x_3=0$$

Let $x_1=k_1,\ x_2=k_2$ and $x_3=(k_1+k_2)$. Eigen vector is $\begin{bmatrix}k_1\\k_2\\(k_1+k_2)\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$ where $k_1=k_2=1$.

Let $x_1=k_1, x_2=k_2$ and $x_3=-(k_1+k_2)$. Other Eigen vector $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix}$ where $k_1=1, k_2=-1$.

$$S=\begin{bmatrix}0&1&1\\-1&0&1\\1&-1&1\end{bmatrix}\Rightarrow S^{-1}=\begin{bmatrix}1&-2&1\\2&-1&-1\\1&-1&1\end{bmatrix}\Rightarrow D=S^{-1}MS, M=SDS^{-1}.$$

$$e^M=Se^DS^{-1}\Rightarrow e^D=\begin{bmatrix}1&0&0\\0&1&0\\0&0&e^3\end{bmatrix}\Rightarrow e^M=1+\frac{(e^3-1)M}{3}$$

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Q7. An unbiased dice is thrown three times successively. The probability that the numbers of dots on the uppermost surface add up to 16 is

(a) $\frac{1}{16}$ (b) $\frac{1}{36}$ (c) $\frac{1}{108}$ (d) $\frac{1}{216}$

Ans. : (b)

Solution: We can get sum of dice as 16 in total six ways i.e. three ways (6, 5, 5) and three ways (6, 6, 4).

Total number of ways for 3 dice having six faces $=6\times6\times6$

$$=\frac{6}{6\times6\times6}=\frac{1}{36}$$

Q8. The generating function $F(x,t)=\sum_{n=0}^{\infty}P_n(x)t^n$ for the Legendre polynomials $P_n(x)$ is $F(x,t)=(1-2xt+t^2)^{-1/2}$. The value of $P_3(-1)$ is

(a) $5/2$ (b) $3/2$ (c) $+1$ (d) -1

Ans. : (d)

Solution: $P_3 = \frac{1}{2}\left(5x^3 - 3x\right) \Rightarrow P_3(-1) = \frac{1}{2}\left(5(-1)^3 - 3(-1)\right) = \frac{1}{2}[-5+3] = -1$

Q9. The equation of the plane that is tangent to the surface $xyz = 8$ at the point $(1,2,4)$ is

(a) $x + 2y + 4z = 12$ (b) $4x + 2y + z = 12$

(c) $x + 4y + 2 = 0$ (d) $x + y + z = 7$

Ans. : (b)

Solution: To get a normal at the surface, lets take the gradient

$$\vec{\nabla}(xyx) = yz\hat{i} + zx\hat{j} + \hat{k}xy = 8\hat{i} + 4\hat{j} + 2\hat{k}$$

We want a plane perpendicular to this so: $\left(\vec{r} - \vec{r_0}\right) \cdot \frac{\left(8\hat{i} + 4\hat{j} + 2\hat{k}\right)}{\sqrt{64+16+4}} = 0$.

$\left[(x-1)\hat{i} + (y-2)\hat{j} + (z-4)\hat{k}\right] \cdot \left[8\hat{i} + 4\hat{j} + 2\hat{k}\right] = 0 \Rightarrow 4x + 2y + z = 12$.

Q10. A 3×3 matrix M has $Tr[M] = 6$, $Tr\left[M^2\right] = 26$ and $Tr\left[M^3\right] = 90$. Which of the following can be a possible set of eigenvalues of M ?

(a) $\{1,1,4\}$ (b) $\{-1,0,7\}$ (c) $\{-1,3,4\}$ (d) $\{2,2,2\}$

Ans. : (c)

Solution: $Tr\left[M^2\right] = (-1)^2 + (3)^2 + (4)^2$ also $Tr\left[M^3\right] = (-1)^3 + (3)^3 + (4)^3 = 90$.

Q11. Let $x_1(t)$ and $x_2(t)$ be two linearly independent solutions of the differential equation $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + f(t)x = 0$ and let $w(t) = x_1(t)\frac{dx_2(t)}{dt} - x_2(t)\frac{dx_1(t)}{dt}$. If $w(0) = 1$, then $w(1)$ is given by

(a) 1 (b) e^2 (c) $1/e$ (d) $1/e^2$

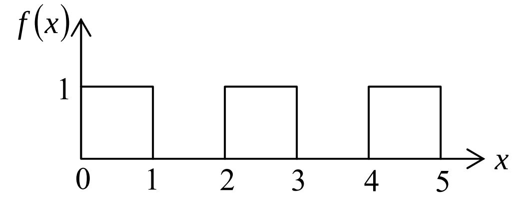
Ans. : (d)

Solution: $W(t)$ is Wronskian of D.E.

$$W = e^{-\int Pdt} = e^{-2t} \Rightarrow W(1) = e^{-2} \text{ since } P = 2.$$

Q12. The graph of the function $f(x)=\begin{cases}1 & \text{for } 2n \le x \le 2n+1 \\ 0 & \text{for } 2n+1 \le x \le 2n+2\end{cases}$

where $n=(0,1,2,......)$ is shown below. Its Laplace transform $\tilde{f}(s)$ is



(a) $\dfrac{1+e^{-s}}{s}$ (b) $\dfrac{1-e^{-s}}{s}$

(c) $\dfrac{1}{s\left(1+e^{-s}\right)}$ (d) $\dfrac{1}{s\left(1-e^{-s}\right)}$

Ans. : (c)

Solution: $L(f(x))=\int_0^\infty e^{-sx} f(x)dx=\int_0^1 e^{-sx}\cdot 1dx+\int_1^2 e^{-sx}\cdot 0dx+\int_2^3 e^{-sx}\cdot 1dx+......$

$$=\left[\frac{e^{-sx}}{-s}\right]_0^1+0+\left[\frac{e^{-sx}}{-s}\right]_2^3+......=\frac{1}{-s}\left[e^{-s}-1\right]+\frac{1}{-s}\left[e^{-3s}-e^{-2s}\right]+......$$

$$=\frac{1}{-s}\left[-1+e^{-s}-e^{-2s}+e^{-3s}+........\right]=\frac{1}{s}\left[1-e^{-s}+e^{-2s}-e^{-3s}+....\right]$$

Since $S_\infty=\dfrac{a}{1-r}$ where $r=-e^{-s}$ and $a=1$ $\Rightarrow S_\infty=\dfrac{1}{s}\left[\dfrac{1}{\left(1+e^{-s}\right)}\right]$.

Q13. The first few terms in the Taylor series expansion of the function $f(x)=\sin x$ around $x=\dfrac{\pi}{4}$ are:

(a) $\dfrac{1}{\sqrt{2}}\left[1+\left(x-\dfrac{\pi}{4}\right)+\dfrac{1}{2!}\left(x-\dfrac{\pi}{4}\right)^2+\dfrac{1}{3!}\left(x-\dfrac{\pi}{4}\right)^3.....\right]$

(b) $\dfrac{1}{\sqrt{2}}\left[1+\left(x-\dfrac{\pi}{4}\right)-\dfrac{1}{2!}\left(x-\dfrac{\pi}{4}\right)^2-\dfrac{1}{3!}\left(x-\dfrac{\pi}{4}\right)^3.....\right]$

(c) $\left[\left(x-\dfrac{\pi}{4}\right)-\dfrac{1}{3!}\left(x-\dfrac{\pi}{4}\right)^3.....\right]$

(d) $\dfrac{1}{\sqrt{2}}\left[1-x+\dfrac{x^2}{2!}-\dfrac{x^3}{3!}.....\right]$

Ans. : (b)

Solution: $f(x)=\sin x$

$$f\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}},\quad f'\left(\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}},\quad f''\left(\frac{\pi}{4}\right)=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$$

So Taylor's series is given by $\frac{1}{\sqrt{2}}\left[1+\left(x-\frac{\pi}{4}\right)-\frac{1}{2!}\left(x-\frac{\pi}{4}\right)^2-\frac{1}{3!}\left(x-\frac{\pi}{4}\right)^3.....\right]$

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Q14. A vector perpendicular to any vector that lies on the plane defined by $x+y+z=5$, is

(a) $\hat{i}+\hat{j}$ (b) $\hat{j}+\hat{k}$ (c) $\hat{i}+\hat{j}+\hat{k}$ (d) $2\hat{i}+3\hat{j}+5\hat{k}$

Ans. : (c)

Solution: Let $\phi=x+y+z-5 \Rightarrow \vec{\nabla}\phi=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z-5)=\hat{i}+\hat{j}+\hat{k}$.

Q15. The eigen values of the matrix $A=\begin{pmatrix}1&2&3\\2&4&6\\3&6&9\end{pmatrix}$ are

(a) $(1,4,9)$ (b) $(0,7,7)$ (c) $(0,1,13)$ (d) $(0,0,14)$

Ans. : (d)

Solution: For eigenvalues $|A-\lambda I|=0 \Rightarrow \begin{bmatrix}1-\lambda&2&3\\2&4-\lambda&6\\3&6&9-\lambda\end{bmatrix}=0$

$$(1-\lambda)[(4-\lambda)(9-\lambda)-36]-2[2(9-\lambda)-18]+3[12-3(4-\lambda)]=0$$

$$(1-\lambda)(4-\lambda)(9-\lambda)-36(1-\lambda)-4(9-\lambda)+36+9\lambda=0$$

$$\lambda^3-14\lambda^2=0 \Rightarrow \lambda^2(\lambda-14)=0 \Rightarrow \lambda=0,\ 0,\ 14\ .$$

Q16. The first few terms in the Laurent series for $\frac{1}{(z-1)(z-2)}$ in the region $1\le|z|\le 2$ and around $z=1$ is

(a) $\frac{1}{2}\left[1+z+z^2+....\right]\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+....\right]$ (b) $\frac{1}{1-z}-z-(1-z)^2+(1-z)^3+....$

(c) $\frac{1}{z^2}\left[1+\frac{1}{z}+\frac{1}{z^2}+....\right]\left[1+\frac{2}{z}+\frac{4}{z^2}+....\right]$ (d) $2(z-1)+5(z-1)^2+7(z-1)^3+....$

Ans. : (b)

Solution: $\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}=\frac{1}{1-z}+\frac{1}{(z-1)-1}=\frac{1}{1-z}-(1+(1-z))^{-1}$

$$=\frac{1}{1-z}-\left[1+(1-z)+\frac{(-1)(-2)}{2!}(1-z)^2+\frac{(-1)(-2)(-3)}{3!}(1-z)^3\ldots\right]$$

$$=\frac{1}{1-z}-\left[z+(1-z)^2-(1-z)^3+\ldots.\right]$$

Q17. Let $u(x,y)=x+\frac{1}{2}(x^2-y^2)$ be the real part of analytic function $f(z)$ of the complex variable $z=x+iy$. The imaginary part of $f(z)$ is

(a) $y+xy$ (b) xy (c) y (d) y^2-x^2

Ans. : (a)

Solution: $u(x,y)=x+\frac{1}{2}(x^2-y^2)$, $v(x,y)=?$

Check $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$.

$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial y}=1+x$, $v=y+xy+f(x)$

$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\Rightarrow\frac{\partial v}{\partial x}=+y$, $v=yx+f(y)$

$y+xy+f(x)=yx+f(y)$

If $f(x)=0$, $f(y)=y$

$v=xy+y$

Q18. Let $y(x)$ be a continuous real function in the range 0 and 2π, satisfying the inhomogeneous differential equation: $\sin x\frac{d^2y}{dx^2}+\cos x\frac{dy}{dx}=\delta\left(x-\frac{\pi}{2}\right)$

The value of $dyldx$ at the point $x=\pi/2$

(a) is continuous (b) has a discontinuity of 3

(c) has a discontinuity of 1/3 (d) has a discontinuity of 1

Ans. : (d)

Solution: After dividing by $\sin x$, $\frac{d^2y}{dx^2}+\cot x\frac{dy}{dx}=\operatorname{cosec} x\cdot\delta\left(x-\frac{\pi}{2}\right)$

Integrating both sides, $\frac{dy}{dx}+\int\cot x\left(\frac{dy}{dx}\right)dx=\int\operatorname{cosec} x\delta\left(x-\frac{\pi}{2}\right)dx$

$\frac{dy}{dx}+\cot x\cdot y-\int\operatorname{cosec}^2 x\cdot ydx=1$

Using Dirac delta property: $\int f(x)\delta(x-x_0)=f(x_0)$ (it lies with the limit).

$\frac{dy}{dx}+y\cdot\frac{\cos x}{\sin x}-\int y\operatorname{cosec}^2 xdx=1$, at $x=\pi;\sin x=0$. So this is point of discontinuity.

Q19. A ball is picked at random from one of two boxes that contain 2 black and 3 white and 3 black and 4white balls respectively. What is the probability that it is white?

(a) $34/70$ (b) $41/70$ (c) $36/70$ (d) $29/70$

Ans. : (b)

Solution: Probability of picking white ball

$2B\ \ 3W$	$3B\ \ 4W$
I	II

From box $I=\frac{3}{5}$ and from box $II=\frac{4}{7}$

Probability of picking a white ball from either of the two boxes is $=\frac{1}{2}\left[\frac{3}{5}+\frac{4}{7}\right]=\frac{41}{70}$

Q20. The eigenvalues of the antisymmetric matrix,

$$A=\begin{pmatrix}0 & -n_3 & n_2\\ n_3 & 0 & -n_1\\ -n_2 & n_1 & 0\end{pmatrix}$$

where n_1,n_2 and n_3 are the components of a unit vector, are

(a) $0,i,-i$ (b) $0,1,-1$ (c) $0,1+i,-1,-i$ (d) $0,0,0$

Ans. : (a)

Solution: $A=\begin{bmatrix}0 & -n_3 & n_2\\ n_3 & 0 & -n_1\\ -n_2 & n_1 & 0\end{bmatrix}\Rightarrow -A^T=\begin{bmatrix}0 & -n_3 & n_2\\ n_3 & 0 & -n_1\\ -n_2 & n_1 & 0\end{bmatrix}$

$$(A-\lambda I)=0,\ \begin{bmatrix}0-\lambda & -n_3 & n_2\\ n_3 & 0-\lambda & -n_1\\ -n_2 & n_1 & 0-\lambda\end{bmatrix}=0$$

$\Rightarrow \lambda_1 = 0 \Rightarrow \lambda_2 = -\sqrt{-n_1^2 - n_2^2 - n_3^2} \Rightarrow \lambda_3 = \sqrt{-n_1^2 - n_2^2 - n_3^2}$

but $\sqrt{n_1^2 + n_2^2 + n_3^2} = 1$

so, $\lambda_1 = 0,\ \lambda_2 = i,\ \lambda_3 = -i$

$A = -A^T$ (Antisymmetric). Eigenvalues are either zero or purely imaginary.

Q21. Which of the following limits exists?

(a) $\lim_{N\to\infty}\left(\sum_{m=1}^{N}\frac{1}{m} + \ln N\right)$ (b) $\lim_{N\to\infty}\left(\sum_{m=1}^{N}\frac{1}{m} - \ln N\right)$

(c) $\lim_{N\to\infty}\left(\sum_{m=1}^{N}\frac{1}{\sqrt{m}} - \ln N\right)$ (d) $\lim_{N\to\infty}\sum_{m=1}^{N}\frac{1}{m}$

Ans. : (b)

Q22. A bag contains many balls, each with a number painted on it. There are exactly n balls which have the number n (namely one ball with 1, two balls with 2, and so on until N on them). An experiment consists of choosing a ball at random, noting the number on it and returning it to the bag. If the experiment is repeated a large number of times, the average value the number will tend to

(a) $\frac{2N+1}{3}$ (b) $\frac{N}{2}$ (c) $\frac{N+1}{2}$ (d) $\frac{N(N+1)}{2}$

Ans. : (a)

Solution: Total number of balls $1+2+3+4+.....+N = \frac{N(N+1)}{2}$

The probability for choosing a k^{th} ball at random $= \dfrac{k}{\frac{N(N+1)}{2}}$

Average of it is given by $\langle k\rangle = \Sigma k\cdot P = \frac{2\Sigma k^2}{N(N+1)} = \frac{2}{N(N+1)}\cdot\frac{N(N+1)(2N+1)}{6}$

$= \frac{2N+1}{3}$ where $\Sigma k^2 = \frac{N(N+1)(2N+1)}{6}$.

Q23. Consider a sinusoidal waveform of amplitude $1V$ and frequency f_0. Starting from an arbitrary initial time, the waveform is sampled at intervals of $\frac{1}{2f_0}$. If the corresponding Fourier spectrum peaks at a frequency $\bar{f}$ and an amplitude $\bar{A}$, them

(a) $\bar{f} = 2f_0$ and $\bar{A} = 1V$ (b) $\bar{f} = 2f_0$ and $0 \le \bar{A} \le 1V$

(c) $\bar{f} = 0$ and $\bar{A} = 1V$ (d) $\bar{f} = \frac{f_0}{2}$ and $\bar{A} = \frac{1}{\sqrt{2}}V$

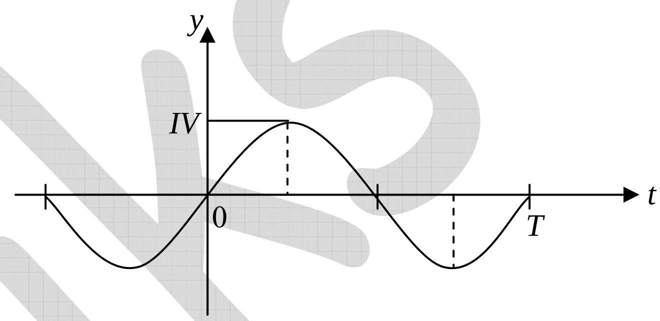
Ans. : (b)

Solution: $y = 1\sin(2\pi f_0 t)$.

The Fourier transform is:

$$F(y) = \frac{1}{2}[\delta(f + f_0)] - \delta[f - f_0]$$

In Fourier domain $\bar{f} = f_0, \bar{A} = \frac{1}{2}$.



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Q24. The unit normal vector of the point $\left[\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right]$ on the surface of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

(a) $\frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{a^2 + b^2 + c^2}}$ (b) $\frac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2 + b^2 + c^2}}$

(c) $\frac{b\hat{i} + c\hat{j} + a\hat{k}}{\sqrt{a^2 + b^2 + c^2}}$ (d) $\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

Ans. : All the options given are incorrect.

Solution: Here $\phi = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Unit normal vector is $\frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}$.

So, $\vec{\nabla}\phi = \left(i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right) = \frac{2x\hat{i}}{a^2} + \frac{2y\hat{j}}{b^2} + \frac{2z\hat{k}}{c^2}$

$$\vec{\nabla}\phi \Big|_{\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)} = \frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}$$

$$\left|\vec{\nabla}\phi\right| = \sqrt{\frac{4}{3a^2} + \frac{4}{3b^2} + \frac{4}{3c^2}} = \frac{2}{\sqrt{3}}\sqrt{\frac{b^2c^2 + a^2c^2 + a^2c^2}{a^2b^2c^2}}$$

$$\frac{\vec{\nabla}\phi}{\left|\vec{\nabla}\phi\right|}\Bigg|_{\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)} = \frac{\frac{2}{a\sqrt{3}}\hat{i} + \frac{2}{b\sqrt{3}}\hat{j} + \frac{2}{c\sqrt{3}}\hat{k}}{\frac{2}{\sqrt{3}}\frac{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}{abc}} = \frac{bc\hat{i} + ca\hat{j} + ab\hat{k}}{\sqrt{b^2c^2 + c^2a^2 + a^2b^2}}$$

Q25. Given a 2×2 unitary matrix U satisfying $U^\dagger U = UU^\dagger = 1$ with $\det U = e^{i\varphi}$, one can construct a unitary matrix $V\left(V^\dagger V = VV^\dagger = 1\right)$ with $\det V = 1$ from it by

(a) multiplying U by $e^{-i\varphi/2}$

(b) multiplying any single element of U by $e^{-i\varphi}$

(c) multiplying any row or column of U by $e^{-i\varphi/2}$

(d) multiplying U by $e^{-i\varphi}$

Ans. : (a)

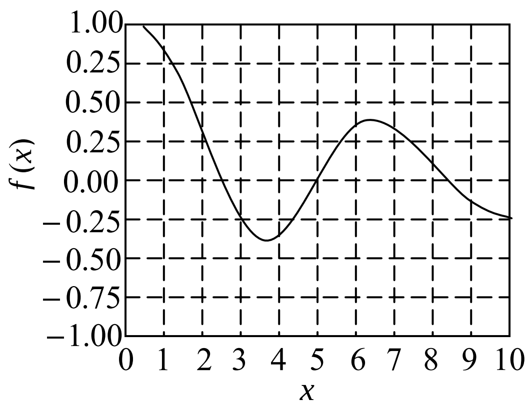
Q26. The graph of the function $f(x)$ shown below is best described by

(a) The Bessel function $J_0(x)$

(b) $\cos x$

(c) $e^{-x}\cos x$

(d) $\frac{1}{x}\cos x$



Ans. : (a)

Q27. In a series of five Cricket matches, one of the captains calls "Heads" every time when the toss is taken. The probability that he will win 3 times and lose 2 times is

(a) $1/8$ (b) $5/8$ (c) $3/16$ (d) $5/16$

Ans. : (d)

Solution: $P = \left(\frac{1}{2}\right)^3 \left(1-\frac{1}{2}\right)^{5-3} \frac{5!}{3!(5-3)!} = \frac{1}{8}\times\left(\frac{1}{2}\right)^2 \cdot \frac{5!}{3!(5-3)!}$

$$= \frac{1}{32}\cdot\frac{5\times4\times3!}{3!\ \times2!} = \frac{20}{32\times2} = \frac{5}{8\times2} = \frac{5}{16}$$

The probability of getting exactly k successes in n trials is given by probability mass function $= \frac{n!}{k!(n-k)!} p^k \cdot (1-p)^{n-k}$, $k =$ successes, $n =$ trials.

Q28. The Taylor expansion of the function $\ln(\cosh x)$, where x is real, about the point $x = 0$ starts with the following terms:

(a) $-\frac{1}{2}x^2 + \frac{1}{12}x^4 + ...$ (b) $\frac{1}{2}x^2 - \frac{1}{12}x^4 + ...$

(c) $-\frac{1}{2}x^2 + \frac{1}{6}x^4 + ...$ (d) $\frac{1}{2}x^2 + \frac{1}{6}x^4 + ...$

Ans. : (b)

Solution: $\cosh x = \frac{e^x + e^{-x}}{2}$ .Tailor's series expansion of $f(x)$ about $x = a$

$$f(x) = f(a) + \frac{f'(a)}{1\ !}(x-a) + \frac{f''(a)}{2\ !}(x-a)^2 + \frac{f'''}{3\ !}(x-a)^3 + ... \text{ Here } a = 0.$$

$$f(x) = \log\left[\frac{e^x + e^{-x}}{2}\right]\Bigg|_{x=0} = 0,\ f'(x)\big|_{x=0} = \frac{1}{\frac{e^x + e^{-x}}{2}} \cdot \frac{e^x - e^{-x}}{2} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x = 0$$

$$f''(x) = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2} = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2} = 1 - \tanh^2 x$$

At $x = 0,\quad f''(x) = 1, f'''(x) = 0, f''''(x) = -2, ...$

$$\Rightarrow f(x) = \frac{1}{2}x^2 - \frac{1}{12}x^4 + ...$$

Q29. The value of the integral $\int_C \frac{z^3 dz}{(z^2 - 5z + 6)}$, where C is a closed contour defined by the equation $2|z| - 5 = 0$, traversed in the anti-clockwise direction, is

(a) $-16\pi i$ (b) $16\pi i$ (c) $8\pi i$ (d) $2\pi i$

Ans. : (a)

Solution: $z^2-5z+6=0 \Rightarrow z^2-2z-3z+6=0 \Rightarrow z(z-2)-3(z-2)=0 \Rightarrow z=3,2$

$2|z|=5 \Rightarrow |z|=2.5$, only 2 will be inside.

$$\text{Residue} = (z-2)\frac{z^3}{(z-3)(z-2)}\Big|_{z=2} = \frac{8}{2-3} = -8 \Rightarrow \int_c \frac{z^3 dz}{z^2-5z+6} = 2\pi i(-8) = -16\pi i$$

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Q30. Given that $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx}$

the value of $H_4(0)$ is

(a) 12 (b) 6 (c) 24 (d) – 6

Ans. : (a)

Solution: $\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{-t^2+2tx} \Rightarrow \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!} = e^{-t^2} = 1-t^2+\frac{t^4}{2!}-\frac{t^6}{3!}$

$$\Rightarrow \frac{H_4(0)}{4!}t^4 = \frac{t^4}{2!} \Rightarrow H_4(0) = \frac{4!}{2!} = 12.$$

Q31. A unit vector $\hat{n}$ on the xy-plane is at an angle of 120° with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60° if

(a) $b = \sqrt{3}a/2$ (b) $b = 2a/\sqrt{3}$ (c) $b = a/2$ (d) $b = a$

Ans. : (c)

Solution: $\vec{u} = a\hat{i} + b\hat{n}$, $\vec{v} = a\hat{n} + b\hat{i}$

$$\Rightarrow \vec{u}\cdot\vec{v} = (a\hat{i}+b\hat{n})\cdot(a\hat{n}+b\hat{i}) \Rightarrow |\vec{u}||\vec{v}|\cos 60 = a^2\hat{i}\cdot\hat{n} + ab + ba + b^2\hat{n}.\hat{i}$$

$$\left(\sqrt{a^2+b^2+2ab\cos 120}\right)^2 \cdot \cos 60 = a^2\cos 120 + 2ab + b^2\cos 120$$

$$\left(a^2+b^2-2ab\times\frac{1}{2}\right)\cdot\cos 60 = -\frac{1}{2}(a^2+b^2)+2ab \Rightarrow \frac{1}{2}(a^2+b^2) - \frac{ab}{2} = -\frac{1}{2}(a^2+b^2)+2ab$$

$$\Rightarrow a^2+b^2 = \frac{5ab}{2} \Rightarrow b = \frac{a}{2}.$$

Q32. With $z = x + iy$, which of the following functions $f(x, y)$ is NOT a (complex) analytic function of z ?

(a) $f(x,y) = (x + iy - 8)^3 (4 + x^2 - y^2 + 2ixy)^7$

(b) $f(x,y) = (x + iy)^7 (1 - x - iy)^3$

(c) $f(x,y) = (x^2 - y^2 + 2ixy - 3)^5$

(d) $f(x,y) = (1 - x + iy)^4 (2 + x + iy)^6$

Ans. : (d)

Solution: $f(x,y) = (1 - x + iy)^4 (2 + x + iy)^6 = \{1 - (x - iy)\}^4 (2 + x + iy)^6$

Due to present of $\bar{z} = (x - iy)$

Q33. The solution of the partial differential equation

$$\frac{\partial^2}{\partial t^2} u(x,t) - \frac{\partial^2}{\partial x^2} u(x,t) = 0$$

satisfying the boundary conditions $u(0,t) = 0 = u(L,t)$ and initial conditions $u(x,0) = \sin(\pi x / L)$ and $\frac{\partial}{\partial t} u(x,t)\big|_{t=0} = \sin(2\pi x / L)$ is

(a) $\sin(\pi x / L)\cos(\pi t / L) + \frac{L}{2\pi}\sin(2\pi x / L)\cos(2\pi t / L)$

(b) $2\sin(\pi x / L)\cos(\pi t / L) - \sin(\pi x / L)\cos(2\pi t / L)$

(c) $\sin(\pi x / L)\cos(2\pi t / L) + \frac{L}{\pi}\sin(2\pi x / L)\sin(\pi t / L)$

(d) $\sin(\pi x / L)\cos(\pi t / L) + \frac{L}{2\pi}\sin(2\pi x / L)\sin(2\pi t / L)$

Ans. : (d)

Solution: $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$, $u(x,0) = \sin\frac{\pi x}{L}$ and $\frac{\partial u}{\partial t}\Big|_{t=0} = \sin\frac{2\pi x}{L}$

This is a wave equation

So solution is given by $u(x,t) = \sum_n \left(A_n \cos\frac{an\pi\ \ t}{L} + B_n \sin\frac{an\ \ \pi\ \ t}{L} \right) \sin\left(\frac{n\pi x}{L}\right)$

with $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx, \quad B_n = \frac{2}{an\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$

Comparing $a^2\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$, We have $a = 1$ and $f(x) = \sin\frac{\pi x}{L}$, $g(x) = \sin\frac{2\pi \ x}{L}$,

$$A_n = \frac{2}{L}\int_0^L \sin\frac{\pi x}{L}\sin\frac{n\pi \ x}{L}dx \Rightarrow \frac{2}{L}\int_0^L \sin^2\frac{\pi x}{L}dx = \frac{2}{L}\int_0^L\left(\frac{1-\cos\frac{2\pi x}{L}}{2}\right)dx = \frac{2}{L}\cdot\frac{L}{2} = 1 \text{ (let } n = 1\text{)}$$

Putting $n = 2$, $B_n = \frac{2}{an\pi}\int_0^L \sin\frac{2\pi x}{L}\cdot\sin\frac{n\pi x}{L}dx$

$$\Rightarrow \frac{2}{2\pi}\int_0^L \sin^2\frac{2\pi x}{L}dx = \frac{2}{2\pi}\int_0^L\left(\frac{1-\cos\frac{4\pi x}{L}}{2}\right)dx = \frac{2}{2\pi}\cdot\frac{L}{2} = \frac{L}{2\pi}$$

Q34. The solution of the differential equation

$$\frac{dx}{dt} = x^2$$

with the initial condition $x(0) = 1$ will blow up as t tends to

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) ∞

Ans. : (a)

Solution: $\frac{dx}{dt} = x^2 \Rightarrow \int\frac{dx}{x^2} = \int dt \Rightarrow \frac{x^{-2+1}}{-2+1} = t + C \Rightarrow \frac{-1}{x} = t + C$

$\Rightarrow x(0) = 1 \Rightarrow \frac{-1}{1} = 0 + C \Rightarrow C = -1 \Rightarrow \frac{-1}{x} = t - 1 \Rightarrow x = \frac{1}{1-t}$ as $t \to 1$, x blows up

Q35. The inverse Laplace transforms of $\frac{1}{s^2(s+1)}$ is

(a) $\frac{1}{2}t^2e^{-t}$ (b) $\frac{1}{2}t^2 + 1 - e^{-t}$

(c) $t - 1 + e^{-t}$ (d) $\frac{1}{2}t^2(1 - e^{-t})$

Ans. : (c)

Solution: $f(s)=\frac{1}{s+1}\Rightarrow f(t)=e^{-t}\ \Rightarrow L^{-1}\left[\frac{1}{s(s+1)}\right]=\int_0^t e^{-t}dt=\left(-e^{-t}\right)_0^t=\left(-e^{-t}+1\right)$

$$\Rightarrow L^{-1}\left[\frac{1}{s^2(s+1)}\right]=\int_0^t\left(-e^{-t}+1\right)\ dt=\left(e^{-t}+t\right)_0^t=e^{-t}+t-1.$$

Q36. The approximation $\cos\theta\approx 1$ is valid up to 3 decimal places as long as $|\theta|$ is less than: (take $180^\circ/\pi\approx 57.29^\circ$)

(a) 1.28^0 (b) 1.81^0 (c) 3.28^0 (d) 4.01^0

Ans. : (b)

Solution: $\cos\theta=1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}-.......\approx 1-\frac{\theta^2}{2!}$

$$\cos\theta\approx 1 \text{ when } \theta=1.81^\circ\approx\frac{\pi}{100}=0.0314$$

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Q37. If $\vec{A}=\hat{i}yz+\hat{j}xz+\hat{k}xy$, then the integral $\oint_C\vec{A}\cdot d\vec{l}$ (where C is along the perimeter of a rectangular area bounded by $x=0, x=a$ and $y=0,\ y=b$) is

(a) $\frac{1}{2}\left(a^3+b^3\right)$ (b) $\pi\left(ab^2+a^2b\right)$ (c) $\pi\left(a^3+b^3\right)$ (d) 0

Ans. : (d)

$$\oint_C\vec{A}\cdot d\vec{l}=\int_S\left(\vec{\nabla}\times\vec{A}\right)d\vec{a}=0 \text{ since } \vec{\nabla}\times\vec{A}=0.$$

Q38. If A, B and C are non-zero Hermitian operators, which of the following relations must be false?

(a) $[A,\ B]=C$ (b) $AB+BA=C$ (c) $ABA=C$ (d) $A+B=C$

Ans. : (a)

Solution: $[A,\ \ B]=C\Rightarrow AB-BA=C\ \ \Rightarrow(AB-BA)^\dagger=C^\dagger$

$$((AB)^\dagger-(BA)^\dagger)=C^\dagger\quad\Rightarrow(B^\dagger A^\dagger)-(A^\dagger B^\dagger)=C^\dagger$$

Hence A,B and C are hermitian then

$$BA-AB=C\neq[A,\ B]=C$$

Q39. Which of the following functions cannot be the real part of a complex analytic function of $z = x + iy$?

(a) $x^2 y$ (b) $x^2 - y^2$ (c) $x^3 - 3xy^2$ (d) $3x^2 y - y - y^3$

Ans. : (a)

Solution: Let $x^2 y$ be real part of a complex function. Use Milne Thomson's method to write analytic complex function. The real part of that function should be (1) but that is not the case. So this cannot be real part of an analytic function. Also,

$z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$, Real part option (2)

$z^3 = (x+iy)^3 = x^3 - iy^3 + 3ixy(x+iy)$

$= x^3 - iy^3 + 3ix^2 y - 3xy^2$, Real part option (3)

Q40. The expression

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right)\frac{1}{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)}$$

is proportional to

(a) $\delta(x_1 + x_2 + x_3 + x_4)$ (b) $\delta(x_1)\delta(x_2)\delta(x_3)\delta(x_4)$

(c) $\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{-3/2}$ (d) $\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^{-2}$

Ans. : (b)

Solution: $\left[\frac{\partial}{\partial x_1}\left(\frac{1}{x_1^2 + x_2^2 + x_3^2 + x_4^2}\right)\right] = \frac{-2x_1}{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^2}$

$$\frac{\partial^2}{\partial x_1^2} \to -2\left[\frac{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^2 \cdot 1 - 2\cdot 2x_1 \cdot x_1\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)}{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^4}\right]$$

$$= -2\left[\frac{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^2 - 4x_1^2}{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^3}\right] = \frac{8x_1^2 - 2\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)}{\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right)^3}$$

Now similarly solving all and add up then we get

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}\right)\left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2}\right)$$

$$=\frac{8\left(x_1^2+x_2^2+x_3^2+x_4^2\right)-8\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)^3}=0$$

also if all x_1, x_2, x_3, x_4 becomes zero it should be infinity.

So $\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}+\frac{\partial^2}{\partial x_4^2}\right)\frac{1}{\left(x_1^2+x_2^2+x_3^2+x_4^2\right)}=\delta(x_1)\cdot\delta(x_2)\cdot\delta(x_3)\cdot\delta(x_4)$

Q41. Given that the integral $\int_0^\infty \frac{dx}{y^2+x^2}=\frac{\pi}{2y}$, the value of $\int_0^\infty \frac{dx}{\left(y^2+x^2\right)^2}$ is

(a) $\frac{\pi}{y^3}$ (b) $\frac{\pi}{4y^3}$ (c) $\frac{\pi}{8y^3}$ (d) $\frac{\pi}{2y^3}$

Ans. : (b)

Solution: $\int_0^\infty \frac{dx}{\left(y^2+x^2\right)^2}=\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{\left(y^2+x^2\right)^2}$, pole is of 2nd order at $x=iy$, residue $=1/\left(4iy^3\right)$

$$Integral=\left(\frac{1}{2}\right)(2\pi i)\frac{1}{4iy^3}=\frac{\pi}{\left(4y^3\right)}$$

Q42. The Fourier transform of the derivative of the Dirac δ - function, namely $\delta'(x)$, is proportional to

(a) 0 (b) 1 (c) $\sin k$ (d) ik

Ans. : (d)

Solution: Fourier transform of $\delta'(x)$

$$H(K)=\int_{-\infty}^{\infty}\delta'(x)e^{ikx}dx=ike^{(k\cdot 0)}=ik$$

Q43. Consider an $n\times n(n>1)$ matrix A, in which A_{ij} is the product of the indices i and j (namely $A_{ij}=ij$). The matrix A

(a) has one degenerate eigevalue with degeneracy $(n-1)$

(b) has two degenerate eigenvalues with degeneracies 2 and $(n-2)$

(c) has one degenerate eigenvalue with degeneracy n

(d) does not have any degenerate eigenvalue

Ans. : (a)

Solution: If matrix is 2×2 let $\begin{pmatrix}1 & 2\\ 2 & 4\end{pmatrix}$ then eigen value is given by

$$\begin{pmatrix}1-\lambda & 2\\ 2 & 4-\lambda\end{pmatrix}=0 \Rightarrow (1-\lambda)(4-\lambda)-4=0 \Rightarrow \lambda=0,5$$

If If matrix is 3×3 let $\begin{pmatrix}1 & 2 & 3\\ 2 & 4 & 6\\ 3 & 6 & 9\end{pmatrix}$ then eigen value is given by

$$\begin{pmatrix}1-\lambda & 2 & 3\\ 2 & 4-\lambda & 6\\ 3 & 6 & 9-\lambda\end{pmatrix}=0$$

$$(1-\lambda)\left[(4-\lambda)(9-\lambda)-36\right]+2\left[18-2(9-\lambda)\right]+3\left[12-3(4-\lambda)\right]=0$$

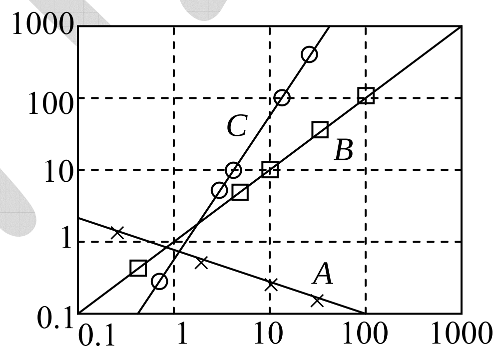
$$(1-\lambda)\left[\lambda^2-13\lambda+36-36\right]+2\left[18-18+2\lambda\right]+3\left[12-12+3\lambda\right]=0$$

$$\lambda^2-13\lambda-\lambda^3+13\lambda^2+13\lambda=0 \Rightarrow \lambda^3-14\lambda^2=0 \Rightarrow \lambda=0,\ \ 0,\lambda=14$$

i.e. has one degenerate eigenvalue with degeneracy 2.

Thus one can generalized that for n dimensional matrix has one degenerate eigevalue with degeneracy $(n-1)$.

Q44. Three sets of data A, B and C from an experiment, represented by $\times$, $\square$ and O, are plotted on a log-log scale. Each of these are fitted with straight lines as shown in the figure.



The functional dependence $y(x)$ for the sets A, B and C are respectively

(a) $\sqrt{x}$, x and x^2 (b) $-\frac{x}{2}$, x and $2x$ (c) $\frac{1}{x^2}$, x and x^2 (d) $\frac{1}{\sqrt{x}}$, x and x^2

Ans. : (d)

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Q45. Consider the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0$$

with the initial conditions $x(0)=0$ and $\dot{x}(0)=1$. The solution $x(t)$ attains its maximum value when t is

(a) 1/2 (b) 1 (c) 2 (d) ∞

Ans. : (b)

Solution: $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+x=0 \Rightarrow m^2+2m+1=0 \Rightarrow (m+1)^2=0 \Rightarrow m=-1,\,-1$

$\Rightarrow x=(c_1+c_2t)e^{-t}$, since $x(0)=0 \Rightarrow 0=c_1 \Rightarrow x=c_2te^{-t}$

$\Rightarrow \dot{x}=c_2\left[-te^{-t}+e^{-t}\right]$

Since $\dot{x}(0)=1 \Rightarrow 1=c_2 \Rightarrow x=te^{-t}$

For maxima or minima $\dot{x}=0 \Rightarrow \dot{x}=-te^{-t}+e^{-t}=0 \Rightarrow \dot{x}=e^{-t}(1-t)$

$\Rightarrow e^{-t}=0,\, 1-t=0 \Rightarrow t=\infty,\; t=1$

$\ddot{x}=e^{-t}(-1)+(1-t)e^{-t}(-1)=-e^{-t}+(t-1)e^{-t} \Rightarrow \ddot{x}(1)=-e^{-1}+0e^{-t}<0$

Q46. Consider the matrix

$$M=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$$

The eigenvalues of M are

(a) $-5,\,-2,\,7$ (b) $-7,\,0,\,7$ (c) $-4i,\,2i,\,2i$ (d) $2,\,3,\,6$

Ans. : (b)

Solution: $M=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$, $M^{+}=\begin{pmatrix} 0 & 2i & 3i \\ -2i & 0 & 6i \\ -3i & -6i & 0 \end{pmatrix}$

$M^{+}=M$

Matrix is Hermitian so roots are real and trace = 0.

$\lambda_1+\lambda_2+\lambda_3=0,\; \lambda_1\cdot\lambda_2\cdot\lambda_3=0 \Rightarrow \lambda=-7,0,7$

Q47. If C is the contour defined by $|z| = \frac{1}{2}$, the value of the integral

$$\oint_C \frac{dz}{\sin^2 z}$$

is

(a) ∞ (b) $2\pi i$ (c) 0 (d) πi

Ans. : (c)

Solution: $f(z) = \frac{1}{\sin^2 z} \quad \left(|z| = \frac{1}{2}\right)$

$$\sin z = z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} .... \Rightarrow \frac{1}{\sin^2 z} = \frac{1}{\left(z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} ....\right)^2}$$

$$\Rightarrow \frac{1}{\sin^2 z} = \frac{1}{z^2}\left[1 - \frac{z^2}{\underline{|3}} + \frac{z^4}{\underline{|5}} ....\right]^{-2} \Rightarrow \oint_C \frac{dz}{\sin^2 z} = 0$$

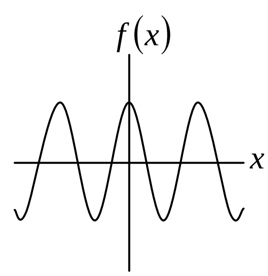
Q48. Given $\sum_{n=0}^{\infty} P_n(x) t^n = (1 - 2xt + t^2)^{-1/2}$, for $|t| < 1$, the value of $P_5(-1)$ is

(a) 0.26 (b) 1 (c) 0.5 (d) -1

Ans. : (d)

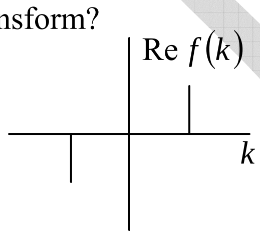
Solution: $P_n(-1) = -1$ if n is odd $\Rightarrow P_5(-1) = -1$

Q49. The graph of a real periodic function $f(x)$ for the range $[-\infty, \infty]$ is shown in the figure.

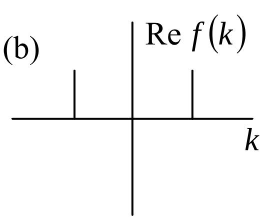


Which of the following graphs represents the real part of its Fourier transform?

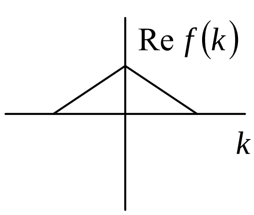
(a)



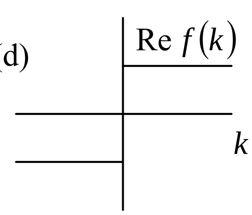
(b)



(c)



(d)



Ans. : (b)

Solution: This is cosine function

$$f(x) = A\cos x \Rightarrow F(k) = \frac{A}{2}\left[\delta(k-k_0)+\delta(k+k_0)\right]$$

NET/JRF (DEC-2014)

Q50. Let $\vec{r}$ denote the position vector of any point in three-dimensional space, and $r = |\vec{r}|$. Then

(a) $\vec{\nabla}\cdot\vec{r} = 0$ and $\vec{\nabla}\times\vec{r} = \vec{r}/r$

(b) $\vec{\nabla}\cdot\vec{r} = 0$ and $\nabla^2 r = 0$

(c) $\vec{\nabla}\cdot\vec{r} = 3$ and $\nabla^2\vec{r} = \vec{r}/r^2$

(d) $\vec{\nabla}\cdot\vec{r} = 3$ and $\vec{\nabla}\times\vec{r} = 0$

Ans. : (d)

Solution: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{\nabla}\cdot\vec{r} = = \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z}{\partial z} = 1+1+1 = 3$$

$$\vec{\nabla}\times\vec{r} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = \hat{x}\left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)+\hat{y}\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right)+\hat{z}\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right) = 0$$

Q51. The column vector $\begin{pmatrix} a \\ b \\ a \end{pmatrix}$ is a simultaneous eigenvector of $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ if

(a) $b = 0$ or $a = 0$

(b) $b = a$ or $b = -2a$

(c) $b = 2a$ or $b = -a$

(d) $b = a/2$ or $b = -a/2$

Ans. : (b)

Solution: *Let* $b = a$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix} = \begin{pmatrix} a \\ a \\ a \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ a \\ a \end{pmatrix} = 2\begin{pmatrix} a \\ a \\ a \end{pmatrix}$$

$Let\ \ b=-2a$

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}a\\-2a\\a\end{pmatrix}=\begin{pmatrix}a\\-2a\\a\end{pmatrix}\quad\text{and}\quad\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}a\\-2a\\a\end{pmatrix}=\begin{pmatrix}-a\\2a\\-a\end{pmatrix}=-1\begin{pmatrix}a\\-2a\\a\end{pmatrix}$$

For other combination above relation is not possible.

Q52. The principal value of the integral $\int_{-\infty}^{\infty}\frac{\sin(2x)}{x^3}dx$ is

(a) -2π (b) $-\pi$ (c) π (d) 2π

Ans. : (a)

Solution: Let $f(z)=\frac{e^{i2z}}{z^3}$

$$\lim_{z\to0}(z-0)^3 f(z)=\lim_{z\to0}(z-0)^3\frac{e^{i2z}}{z^3}=1(\,finite\ and\neq0)\Rightarrow z=0\ \text{ is pole of order 3.}$$

$$\text{Residue }\ R=\frac{1}{2!}\lim_{z\to0}\frac{d^2}{dz^2}\left[(z-0)^3\frac{e^{i2z}}{z^3}\right]=-2$$

$$\Rightarrow\int_{-\infty}^{\infty}f(x)dx=\pi i\Sigma R=\pi i(-2)=-2\pi i\Rightarrow \text{Im}.Part=-2\pi\Rightarrow\int_{-\infty}^{\infty}f(x)dx=-2\pi$$

Q53. The Laurent series expansion of the function $f(z)=e^z+e^{1/z}$ about $z=0$ is given by

(a) $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ for all $|z|<\infty$

(b) $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ only if $0<|z|<1$

(c) $\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!}$ for all $0<|z|<\infty$

(d) $\sum_{n=-\infty}^{\infty}\frac{z^n}{n!}$ only if $|z|<1$

Ans. : (c)

Solution: $e^z=\left(1+z+\frac{z^2}{2!}+....\right)=\sum_{n=0}^{\infty}\frac{z^n}{n!}$ and $e^{1/z}=1+\frac{1}{z}+\frac{1}{2!}\frac{1}{z^2}+....=\sum_{n=0}^{\infty}\frac{1}{z^n n!}$

$$\Rightarrow f(z)=\left(e^z+e^{1/z}\right)=\sum_{n=0}^{\infty}\left(z^n+\frac{1}{z^n}\right)\frac{1}{n!},\ \text{for all }0<|z|<\infty$$

Q54. Two independent random variables m and n, which can take the integer values $0, 1, 2, ..., \infty$, follow the Poisson distribution, with distinct mean values μ and ν respectively. Then

(a) the probability distribution of the random variable $l = m + n$ is a binomial distribution.

(b) the probability distribution of the random variable $r = m - n$ is also a Poisson distribution.

(c) the variance of the random variable $l = m + n$ is equal to $\mu + \nu$

(d) the mean value of the random variable $r = m - n$ is equal to 0.

Ans. : (c)

Solution: $\sigma_l^2 = \sigma_m^2 + \sigma_n^2 = \mu + \nu$

Q55. Consider the function $f(z) = \frac{1}{z}\ln(1-z)$ of a complex variable $z = re^{i\theta}(r \geq 0, \quad -\infty < \theta < \infty)$. The singularities of $f(z)$ are as follows:

(a) branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ only for $0 \leq \theta < 2\pi$

(b) branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ for all θ other than $0 \leq \theta < 2\pi$

(c) branch points at $z = 1$ and $z = \infty$; and a pole at $z = 0$ for all θ

(d) branch points at $z = 0$, $z = 1$ and $z = \infty$.

Ans. : None of the above is correct

Solution: For $f(z) = \frac{1}{z}\ln(1-z) = \frac{1}{z}\left(-z - \frac{z^2}{2} - \frac{z^3}{3} - .....\right) = -1 - \frac{z}{2} - \frac{z^2}{3} - .....$

There is no principal part and when $z \to 0$, $f(z) = -1$. So there is removable singularity at $z = 0$. Also $z = 1$ and $z = \infty$ is Branch point.

Q56. The function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$, satisfies the differential equation

(a) $x^2\frac{d^2 f}{dx^2} + x\frac{df}{dx} + (x^2 + 1)f = 0$

(b) $x^2\frac{d^2 f}{dx^2} + 2x\frac{df}{dx} + (x^2 - 1)f = 0$

(c) $x^2\frac{d^2 f}{dx^2} + x\frac{df}{dx} + (x^2 - 1)f = 0$

(d) $x^2\frac{d^2 f}{dx^2} - x\frac{df}{dx} + (x^2 - 1)f = 0$

Ans. : (c)

Solution: $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}\left(\frac{x}{2}\right)^{2n+1}$ is generating function (Bessel Function of first kind)

which satisfies the differential equation $x^2\frac{d^2 f}{dx^2}+x\frac{df}{dx}+\left(x^2-n^2\right)f=0$, put $n=1$.

Q57. Let α and β be complex numbers. Which of the following sets of matrices forms a group under matrix multiplication?

(a) $\begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix}$

(b) $\begin{pmatrix}1 & \alpha\\ \beta & 1\end{pmatrix}$, where $\alpha\beta\neq 1$

(c) $\begin{pmatrix}\alpha & \alpha^*\\ \beta & \beta^*\end{pmatrix}$, where $\alpha\beta^*$ is real

(d) $\begin{pmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{pmatrix}$, where $|\alpha|^2+|\beta|^2=1$

Ans. : (d)

Solution: $\because \begin{vmatrix}\alpha & \beta\\ -\beta^* & \alpha^*\end{vmatrix}=|\alpha|^2+|\beta|^2=1$

Q58. The expression $\sum_{i,j,k=1}^{3}\in_{ijk}\left\{x_i,\left\{p_j,L_k\right\}\right\}$ (where $\in_{ijk}$ is the Levi-Civita symbol, $\vec{x},\vec{p},\vec{L}$ are the position, momentum and angular momentum respectively, and $\{A,B\}$ represents the Poisson Bracket of A and B) simplifies to

(a) 0 (b) 6 (c) $\vec{x},\left(\vec{p}\times\vec{L}\right)$ (d) $\vec{x}\times\vec{p}$

Ans. : (b)

NET/JRF (JUNE-2015)

Q59. The value of integral $\int_{-\infty}^{\infty}\frac{dx}{1+x^4}$

(a) $\frac{\pi}{\sqrt{2}}$ (b) $\frac{\pi}{2}$ (c) $\sqrt{2}\pi$ (d) 2π

Ans. (a)

Solution: $\int_{-\infty}^{\infty}\frac{dz}{1+z^4}\quad \because |z|=R$

Now, pole $z=e^{(2n+1)\frac{\pi}{4}}$

$n=0,\quad \Rightarrow z_0=e^{\frac{i\pi}{4}}=\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}, n=2\Rightarrow z_2=\frac{-1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}$

$n=1 \Rightarrow z_1 = e^{\frac{i3\pi}{4}} = \frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}, n=3 \Rightarrow z_3 = +\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

only z_0 and z_1 lies in contour

i.e., residue at $\left(z = e^{\frac{i\pi}{4}}\right) = \frac{1}{4}\left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

residue at $\left(z = e^{\frac{i3\pi}{4}}\right) = \frac{1}{4}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

now $\int_{-\infty}^{\infty} \frac{dx}{x^4+1} = 2\pi i \Sigma \operatorname{Re} S = \frac{\pi}{\sqrt{2}}$

Q60. Consider the differential equation $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$. If $x=0$ at $t=0$ and $x=1$ at $t=1$, the value of x at $t=2$ is

(a) e^2+1 (b) e^2+e (c) $e+2$ (d) $2e$

Ans. (b)

Solution: $D^2 - 3D + 2 = 0$

$(D-1)(D-2) = 0 \Rightarrow D = 1,2 \Rightarrow x = c_1 e^{2t} + c_2 e^t$

using boundary condition $x=0, t=0 \Rightarrow c_1 = -c_2$

again using boundary condition $x=1, t=1$

$c_2 = \frac{1}{e-e^2}, c_1 = \frac{1}{e^2-e} \Rightarrow x = \frac{e^{2t}}{e^2-e} + \frac{1}{e-e^2}e^t$

again using $t=2$ then $x = e^2 + e$

Q61. The Laplace transform of $6t^3 + 3\sin 4t$ is

(a) $\frac{36}{s^4} + \frac{12}{s^2+16}$ (b) $\frac{36}{s^4} + \frac{12}{s^2-16}$

(c) $\frac{18}{s^4} + \frac{12}{s^2-16}$ (d) $\frac{36}{s^3} + \frac{12}{s^2+16}$

Ans. (a)

Solution: $L\left[6t^3 + 3\sin 4t\right] \quad \because L\left[t^n\right] = \frac{\lceil n+1}{s^{n+1}}$

$\because L[\sin at] = \frac{a}{(s^2 + a^2)}$

$L\left[6t^3 + 3\sin 4t\right] = \frac{6\times\lfloor 4}{s^4} + \frac{3\times 4}{s^2+16} = \frac{36}{s^4} + \frac{12}{s^2+16}$

Q62. Let $f(x,t)$ be a solution of the wave equation $\frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x^2}$ in 1-dimension. If at $t=0,\ f(x,0) = e^{-x^2}$ and $\frac{\partial f}{\partial t}(x,0) = 0$ for all x, then $f(x,t)$ for all future times $t>0$ is described by

(a) $e^{-\left(x^2-v^2t^2\right)}$

(b) $e^{-(x-vt)^2}$

(c) $\frac{1}{4}e^{-(x-vt)^2} + \frac{3}{4}e^{-(x+vt)^2}$

(d) $\frac{1}{2}\left[e^{-(x-vt)^2} + e^{-(x+vt)^2}\right]$

Ans. (d)

Solution: For $\frac{\partial^2 f}{\partial t^2} = v^2\frac{\partial^2 f}{\partial x^2}$

$\frac{\partial f}{\partial t}(x,0) = 0$ and $f(x,0) = e^{-x^2}$

$f = \frac{1}{2}\left[f(x+vt) + f(x-vt)\right]$. Therefore, solution is $f = \frac{1}{2}\left(e^{-(x-vt)}\right)^2 + e^{-(x+vt)^2}$

NET/JRF (DEC-2015)

Q63. In the scattering of some elementary particles, the scattering cross-section σ is found to depend on the total energy E and the fundamental constants h (Planck's constant) and c (the speed of light in vacuum). Using dimensional analysis, the dependence of σ on these quantities is given by

(a) $\sqrt{\frac{hc}{E}}$ (b) $\frac{hc}{E^{3/2}}$ (c) $\left(\frac{hc}{E}\right)^2$ (d) $\frac{hc}{E}$

Ans. : (c)

Solution: The dimension of σ is dimension of "Area"

$h = Joul - \sec$

$c = m/\sec$

$E = Joul$

$\left(\frac{hc}{E}\right)^2 = m^2$ dimension of area

Q64. If $y = \frac{1}{\tanh(x)}$, then x is

(a) $\ln\left(\frac{y+1}{y-1}\right)$ (b) $\ln\left(\frac{y-1}{y+1}\right)$ (c) $\ln\sqrt{\frac{y-1}{y+1}}$ (d) $\ln\sqrt{\frac{y+1}{y-1}}$

Ans. : (d)

Solution: $y = \frac{1}{\tanh x}$

$$y = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x}+1}{e^{2x}-1}$$

$$ye^{2x} - y = e^{2x} + 1 \Rightarrow ye^{2x} - e^{2x} = 1 + y \Rightarrow e^{2x}(y-1) = (1+y)$$

$$2x = \ln\left(\frac{y+1}{y-1}\right) \Rightarrow x = \frac{1}{2}\ln\left(\frac{y+1}{y-1}\right) = \ln\left(\frac{y+1}{y-1}\right)^{\frac{1}{2}}$$

Q65. The function $\frac{z}{\sin \pi z^2}$ of a complex variable z has

(a) a simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ for $n = 1, 2, 3...$

(b) a simple pole at 0 and poles of order 2 at $\pm\sqrt{n}$ and $\pm i\sqrt{n}$ for $n = 1, 2, 3...$

(c) poles of order 2 at $\pm\sqrt{n}$, $n = 0, 1, 2, 3...$

(d) poles of order 2 at $\pm n$, $n = 0, 1, 2, 3...$

Ans. : (b)

Solution: $f(z) = \frac{z}{\sin \pi z^2} = \frac{z}{\pi z^2 \frac{\sin \pi z^2}{\pi z^2}}$

at $z = 0$, it is a simple pole since, $\lim_{z\to 0}\frac{\sin \pi z^2}{\pi z^2} = 1$

Also, $\sin \pi z^2 = \sin n\pi \Rightarrow \pi z^2 = \pm n\pi$, $z = \pm\sqrt{n}, \pm i\sqrt{n}$

$\lim_{z\to\sqrt{n}}\left(z - \sqrt{n}\right)^2 . \frac{z}{\sin \pi z^2}$, exists. So its pole of order 2.

Q66. The Fourier transform of $f(x)$ is $\tilde{f}(k)=\int_{-\infty}^{+\infty} dx e^{ikx} f(x)$.

If $f(x)=\alpha\delta(x)+\beta\delta'(x)+\gamma\delta''(x)$, where $\delta(x)$ is the Dirac delta-function (and prime denotes derivative), what is $\tilde{f}(k)$?

(a) $\alpha+i\beta k+i\gamma k^2$
(b) $\alpha+\beta k-\gamma k^2$
(c) $\alpha-i\beta k-\gamma k^2$
(d) $i\alpha+\beta k-i\gamma k^2$

Ans. : (c)

Solution: $\tilde{f}(k)=\int_{-\infty}^{\infty} dx\, e^{ikx}\left(\alpha\delta(x)+\beta\delta'(x)+\gamma\delta''(x)\right)$

$$\int_{-\infty}^{\infty}\alpha\delta(x)e^{ikx}dx=\alpha$$

$$\int_{-\infty}^{\infty}\beta\delta'(x)e^{ikx}dx=\beta\left[e^{ikx}\delta(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}ike^{ikx}\delta(x)dx\right]=-i\beta k$$

$$\int_{-\infty}^{\infty}\gamma\delta''(x)e^{ikx}dx=-\gamma k^2$$

Q67. The solution of the differential equation $\frac{dx}{dt}=2\sqrt{1-x^2}$, with initial condition $x=0$ at $t=0$ is

(a) $x=\begin{cases}\sin 2t, & 0\le t<\frac{\pi}{4}\\ \sinh 2t, & t\ge\frac{\pi}{4}\end{cases}$

(b) $x=\begin{cases}\sin 2t, & 0\le t<\frac{\pi}{2}\\ 1, & t\ge\frac{\pi}{2}\end{cases}$

(c) $x=\begin{cases}\sin 2t, & 0\le t<\frac{\pi}{4}\\ 1, & t\ge\frac{\pi}{4}\end{cases}$

(d) $x=1-\cos 2t,\quad t\ge 0$

Ans. : (c)

Solution: $\frac{dx}{dt}=2\sqrt{1-x^2}$, $\frac{dx}{\sqrt{1-x^2}}=2dt$, $\sin^{-1}x=2t+c$, $x=0,\ t=0$ so, $c=0\Rightarrow x=\sin 2t$

x should not be greater than 1 at $x=1$

$1=\sin 2t,\ \ \sin\frac{\pi}{2}=\sin 2t,\ t=\frac{\pi}{4}.$

So, $$x=\begin{cases}\sin 2t, & 0\le t<\frac{\pi}{4}\\ 1\quad , & t\ge\frac{\pi}{4}\end{cases}$$

Q68. The Hermite polynomial $H_n(x)$, satisfies the differential equation

$$\frac{d^2H_n}{dx^2}-2x\frac{dH_n}{dx}+2nH_n(x)=0$$

The corresponding generating function

$G(t,x)=\sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)t^n$, satisfies the equation

(a) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2t\frac{\partial G}{\partial t}=0$ (b) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}-2t^2\frac{\partial G}{\partial t}=0$

(c) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2\frac{\partial G}{\partial t}=0$ (d) $\frac{\partial^2 G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2\frac{\partial^2 G}{\partial x\partial t}=0$

Ans. : (a)

Solution: $G=\frac{1}{n!}H_n(x)t^n$, $G'=\frac{1}{n!}H_n'(x)t^n$, $G''=\frac{1}{n!}H_n''(x)t^n$

$$\frac{\partial G}{\partial t}=\frac{1}{n!}H_n(x)nt^{n-1}$$

Let's check the options one by one

$$\frac{\partial G}{\partial x^2}-2x\frac{\partial G}{\partial x}+2t\frac{\partial G}{\partial t}=0$$

$$\frac{1}{n!}H_n''(x)t^n-2x\frac{1}{n!}H_n'(x)t^n+2t\frac{1}{n!}H_n(x)nt^{n-1}$$

$H_n''(x)-2xH_n'(x)+2xH_n(x)=0$, which is Hermite Differential Equation.

Q69. The value of the integral $\int_0^8\frac{1}{x^2+5}dx$, valuated using Simpson's $\frac{1}{3}$ rule with $h=2$ is

(a) 0.565 (b) 0.620 (c) 0.698 (d) 0.736

Ans. : (a)

Solution: $I = \frac{2}{3}\left[y_0 + 4\left(y_1 + y_2\right) + 2y_3 + y_4\right]$

$= \frac{2}{3}\left[\frac{1}{5} + 4\left(\frac{1}{9} + \frac{1}{4}\right) + 2\times\frac{1}{21} + \frac{1}{69}\right]$

$= \frac{2}{3}\left[\frac{1}{5} + 0.5734 + 0.09523 + 0.0145\right]$

$= \frac{2}{3}\left[0.2 + 0.5734 + 0.09523 + 0.0145\right]$

$= \frac{2}{3}\times 0.8831 = 0.5887$

x	$y = \frac{1}{x^2+5}$
0	$\frac{1}{5}$
2	$\frac{1}{9}$
4	$\frac{1}{21}$
6	$\frac{1}{31}$
8	$\frac{1}{69}$

Q70. Consider a random walker on a square lattice. At each step the walker moves to a nearest neighbour site with equal probability for each of the four sites. The walker starts at the origin and takes 3 steps. The probability that during this walk no site is visited more than one is

(a) $12/27$ (b) $27/64$ (c) $3/8$ (d) $9/16$

Ans. : (d)

Solution: Total number of ways $= 4\times 4\times 4$

Number of preferred outcome $= 4\times 3\times 3$

($\because$ Any four option in step-1 and only 3 option in step $2\,\&\,3$ because he can not go to previous position)

probability $= \frac{4\times 3\times 3}{4\times 4\times 4} = \frac{9}{16}$

NET/JRF (JUNE-2016)

Q71. The radius of convergence of the Taylor series expansion of the function $\frac{1}{\cosh(x)}$ around $x=0$, is

(a) ∞ (b) π (c) $\frac{\pi}{2}$ (d) 1

Ans. : (c)

Q72. The value of the contour integral $\frac{1}{2\pi i}\oint_C \frac{e^{4z}-1}{\cosh(z)-2\sinh(z)}dz$ around the unit circle C traversed in the anti-clockwise direction, is

(a) 0 (b) 2 (c) $\frac{-8}{\sqrt{3}}$ (d) $-\tanh\left(\frac{1}{2}\right)$

Ans. : (c)

Solution: $f(z)=\frac{e^{4z}-1}{\cosh z-2\sinh z}=\frac{e^{4z}-1}{\frac{e^z+e^{-z}}{2}-(e^z-e^{-z})}=\frac{e^{4z}-1}{-\frac{e^z}{2}+\frac{3}{2}e^{-z}}$

$$\Rightarrow f(z)=\frac{2e^z(e^{4z}-1)}{(3-e^{2z})}=\frac{2(e^{5z}-e^z)}{(3-e^{2z})}$$

For pole at $z=z_0, 3-e^{2z_0}=0\Rightarrow e^{2z_0}=3\Rightarrow z_0=\frac{\ln 3}{2}$

It has simple pole at z_0

$$\text{Re}(z_0)=\lim_{z\to z_0}(z-z_0)f(z)=\lim_{z\to z_0}(z-z_0)\frac{2(e^{5z}-e^z)}{3-e^{2z}}$$

$$=\lim_{z\to z_0}\frac{(z-z_0)\times 2(5e^{5z}-e^z)+2(e^{5z}-e^z)\times 1}{-2e^{2z}}=-\left(\frac{e^{5z_0}-e^{z_0}}{e^{2z_0}}\right)$$

$$=-\left(\frac{(\sqrt{3})^5-\sqrt{3}}{3}\right)=-\left(\frac{9\sqrt{3}-\sqrt{3}}{3}\right)=-\frac{8}{\sqrt{3}}$$

$$\frac{1}{2\pi i}\oint f(z)dz=\frac{1}{2\pi i}\times 2\pi i\sum \text{Residue}=-\frac{8}{\sqrt{3}}$$

Q73. The Gauss hypergeometric function $F(a,b,c,z)$, defined by the Taylor series expansion around $z=0$ as $F(a,b,c,z)=$

$$\sum_{n=0}^{\infty}\frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{c(c+1)...(c+n-1)n!}z^n,$$

satisfies the recursion relation

(a) $\frac{d}{dz}F(a,b,c;z)=\frac{c}{ab}F(a-1,b-1,c-1;z)$

(b) $\frac{d}{dz}F(a,b,c;z)=\frac{c}{ab}F(a+1,b+1,c+1;z)$

(c) $\frac{d}{dz}F(a,b,c;z)=\frac{ab}{c}F(a-1,b-1,c-1;z)$

(d) $\frac{d}{dz}F(a,b,c;z)=\frac{ab}{c}F(a+1,b+1,c+1;z)$

Ans. : (d)

Solution: $\frac{dF}{dz}=\sum_{n=0}^{\infty}\frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{c(c+1)...(c+n-1)n!}nz^{n-1}$

$$=\sum_{n=0}^{\infty}\frac{a(a+1)...(a+n-1)b(b+1)...(b+n-1)}{c(c+1)...(c+n-1)}\frac{z^{n-1}}{\underline{|n-1}}$$

$$=\frac{ab}{c}\sum\frac{(a+1)...(a+n-1)(b+1)...(b+n-1)}{(c+1)...(c+n-1)}\frac{z^{n-1}}{\underline{|n-1}}$$

$$=\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)...[a+(n-1)-1](b+1)...[b+(n-1)-1]}{(c+1)...[c+(n-1)-1]}\frac{z^{n-1}}{\underline{|n-1}}$$

$$\frac{dF}{dz}=\frac{ab}{c}F(a+1,b+1,c+1,z)$$

Q74. Let X and Y be two independent random variables, each of which follow a normal distribution with the same standard deviation σ, but with means $+\mu$ and $-\mu$, respectively. Then the sum $X+Y$ follows a

(a) distribution with two peaks at $\pm\mu$ and mean 0 and standard deviation $\sigma\sqrt{2}$

(b) normal distribution with mean 0 and standard deviation 2σ

(c) distribution with two peaks at $\pm\mu$ and mean 0 and standard deviation 2σ

(d) normal distribution with mean 0 and standard deviation $\sigma\sqrt{2}$

Ans. : (d)

Solution: $\mu' = \mu_x + \mu_y = \mu - \mu = 0$

$$\sigma'^2 = \sigma_x^2 + \sigma_y^2 = \sigma^2 + \sigma^2$$

$$\sigma' = \sqrt{2}\sigma$$

Q75. Using dimensional analysis, Planck defined a characteristic temperature T_P from powers of the gravitational constant G, Planck's constant h, Boltzmann constant k_B and the speed of light c in vacuum. The expression for T_P is proportional to

(a) $\sqrt{\frac{hc^5}{k_B^2 G}}$ (b) $\sqrt{\frac{hc^3}{k_B^2 G}}$ (c) $\sqrt{\frac{G}{hc^4 k_B^2}}$ (d) $\sqrt{\frac{hk_B^2}{Gc^3}}$

Ans. : (a)

Solution: $E = h\nu \Rightarrow h = \frac{E}{\nu} = \frac{ML^2T^{-2}}{T^{-1}} = ML^2T^{-1}$

$$E = k_B T \Rightarrow k_B = \frac{E}{T} = \frac{ML^2T^{-2}}{T_P} = ML^2T^{-2}T_P^{-1}$$

$$F = G\frac{m_1 m_2}{r^2} \Rightarrow G = \frac{MLT^{-2} \times L^2}{M^2} \Rightarrow G = M^{-1}L^3T^{-2}$$

$$\sqrt{\frac{hc^5}{k_B^2 G}} = \sqrt{\frac{ML^2T^{-1} \times \left(LT^{-1}\right)^5}{M^2L^4T^{-4}T_P^{-2} \times M^{-1}L^3T^{-2}}} = \sqrt{\frac{ML^7T^{-6}}{ML^7T^{-6}T_P^{-2}}} = \sqrt{T_P^2} = T_P$$

Q76. What is the Fourier transform $\int dx e^{ikx} f(x)$ of

$$f(x) = \delta(x) + \sum_{n=1}^{\infty} \frac{d^n}{dx^n}\delta(x)$$

where $\delta(x)$ is the Dirac delta-function?

(a) $\frac{1}{1-ik}$ (b) $\frac{1}{1+ik}$ (c) $\frac{1}{k+i}$ (d) $\frac{1}{k-i}$

Ans. : (b)

Solution: $f(x) = \delta(x) + \sum_{n=1}^{\infty} \frac{d^n}{dx^n}\delta(x) = \sum_{n=0}^{\infty} \frac{d^n}{dx^n}\delta(x) = \sum_{n=0}^{\infty} \delta^{(n)}(x)$

$$\because F\left[\delta(x)\right] = 1 \Rightarrow F\left[\delta^{(n)}(x)\right] = (-ik)^n F\left[\delta(x)\right] = (-ik)^n$$

$$\because f(x)=\sum_{n=0}^{\infty}\delta^{(n)}(x)$$

$$\Rightarrow F\left[f(x)\right]=\sum_{n=0}^{\infty}(-ik)^n=1-ik+(ik)^2-(ik)^3+....=\frac{1}{1-(-ik)}=\frac{1}{1+ik}$$

Q77. In finding the roots of the polynomial $f(x)=3x^3-4x-5$ using the iterative Newton-Raphson method, the initial guess is taken to be $x=2$. In the next iteration its value is nearest to

(a) 1.671 (b) 1.656 (c) 1.559 (d) 1.551

Ans. : (b)

Solution: $f(x)=3x^3-4x-5$; $f'(x)=9x^2-4$

$$x_{n+1}=x_n-\frac{3x_n^3-4x_n-5}{9x_n^2-4}\Rightarrow x_1=x_0-\frac{3x_0^3-4x_0-5}{9x_0^2-4}$$

Let $x_0=2\Rightarrow x_1=2-\frac{3\times 8-4\times 2-5}{9\times 4-4}=2-\frac{11}{32}\Rightarrow x_1=1.656$

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Q78. The matrix $M=\begin{pmatrix}1 & 3 & 2\\ 3 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$ satisfies the equation

(a) $M^3-M^2-10M+12I=0$ (b) $M^3+M^2-12M+10I=0$

(c) $M^3-M^2-10M+10I=0$ (d) $M^3+M^2-10M+10I=0$

Ans. : (c)

Solution: The characteristic equation is

$$\begin{vmatrix}(1-\lambda) & 3 & 2\\ 3 & (-1-\lambda) & 0\\ 0 & 0 & (1-\lambda)\end{vmatrix}=0$$

$$\Rightarrow \quad (1-\lambda)(-1-\lambda)(1-\lambda)-(3)\times 3(1-\lambda)=0$$

$$\Rightarrow \quad -(\lambda^2-1)(\lambda-1)-9(1-\lambda)=0\Rightarrow \lambda^3-10\lambda-\lambda^2+10=0$$

Thus the matrix M satisfies the equation

$M^3-M^2-10M+10I=0$ then the correct option is (c)

Q79. The Laplace transform of

$$f(t)=\begin{cases}\dfrac{t}{T}, & 0<t<T\\ 1 & t>T\end{cases}$$

is

(a) $\dfrac{-(1-e^{-sT})}{s^2T}$ (b) $\dfrac{(1-e^{-sT})}{s^2T}$ (c) $\dfrac{(1+e^{-sT})}{s^2T}$ (d) $\dfrac{(1-e^{sT})}{s^2T}$

Ans. : (b)

Solution: we can write

$$f(t)=[u_0(t)-u_T(t)]\frac{t}{T}+u_T(t)=[1-u_T(t)]\frac{t}{T}+u_T(t)=\frac{t}{T}-u_T(t)\frac{t}{T}+u_T(t)$$

Hence the transform of $f(t)$ is

$$L\{f(t)\}=L\left\{\frac{t}{T}\right\}-L\left\{u_T(t)\left[\frac{(t-T)+T}{T}\right]\right\}+L\{u_T(t)\}$$

$$=\frac{1}{s^2T}-\frac{e^{-sT}}{T}\left(\frac{1}{s^2}+\frac{T}{s}\right)+\frac{e^{-sT}}{s}=\frac{1-e^{-sT}}{s^2T}$$

Q80. The Fourier transform $\int_{-\infty}^{\infty}dxf(x)e^{ikx}$ of the function $f(x)=\dfrac{1}{x^2+2}$ is

(a) $\sqrt{2}\pi e^{-\sqrt{2}|k|}$ (b) $\sqrt{2}\pi\, e^{-\sqrt{2}k}$ (c) $\dfrac{\pi}{\sqrt{2}}e^{-\sqrt{2}k}$ (d) $\dfrac{\pi}{\sqrt{2}}e^{-\sqrt{2}|k|}$

Ans. : (d)

Solution: Fourier transform of $f(x)=\dfrac{1}{x^2+a^2}$, $a>0$ is $\int\dfrac{1}{x^2+a^2}e^{ikx}dx=\dfrac{\pi}{a}e^{-a|k|}$

Hence $\int\dfrac{1}{x^2+a^2}e^{ikx}dx=\dfrac{\pi}{\sqrt{2}}e^{-\sqrt{2}|k|}$

Q81. Given the values $\sin 45^0=0.7071$, $\sin 50^0=0.7660$, $\sin 55^0=0.8192$ and $\sin 60^0=0.8660$, the approximate value of $\sin 52^0$, computed by Newton's forward difference method, is

(a) 0.804 (b) 0.776 (c) 0.788 (d) 0.798

Ans. : (c)

Solution: Given -

x	$y=f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
45^0	$0\cdot7071$			
50^0	$0\cdot7660$	$0\cdot0589$	$-0\cdot0057$	$-0\cdot0007$
55^0	$0\cdot8192$	$0\cdot0532$	$-0\cdot0064$	
60^0	$0\cdot8660$	$0\cdot0468$		

$h=5$, $52^0 = x_0 + uh = 45^0 + uh$

$$u=\frac{52-45}{h}=\frac{52-45}{5}=\frac{7}{5}$$

From table we have

$$y_0 = 0\cdot7071,\ \Delta y_0 = 0\cdot0589\ ,\ \Delta^2 y_0 = -0\cdot0057$$

$$\Delta^3 y_0 = -0\cdot0007\ ,\ \Delta^4 y_0 = 0, ...$$

$$\therefore \sin\left(52^0\right) = y_0 + \frac{\Delta y_0}{\lfloor 1}u + \frac{\Delta^2 y_0}{\lfloor 2}u(u-1) + \frac{\Delta^3 y_0}{\lfloor 3}u(u-1)(u-2)+\cdots$$

$$=0\cdot7071+(0\cdot0589)\frac{7}{5}+\frac{(-0\cdot0057)}{2}\frac{7}{5}\left(\frac{7}{5}-1\right)+\left(\frac{-0\cdot0007}{6}\right)\frac{7}{5}\left(\frac{7}{5}-1\right)\left(\frac{7}{5}-2\right)+0$$

$$=0\cdot7071+0\cdot0825-\frac{0\cdot0399}{25}+\frac{0\cdot0049}{125}$$

$$=0\cdot7071+0\cdot0825-0\cdot0016+0\cdot0000=0\cdot7880$$

Q82. Let $f(x,t)$ be a solution of the heat equation $\frac{\partial f}{\partial t}=D\frac{\partial^2 f}{\partial x^2}$ in one dimension. The initial condition at $t=0$ is $f(x,0)=e^{-x^2}$ for $-\infty<x<\infty$. Then for all $t>0$, $f(x,t)$ is given by

[Useful integral: $\int_{-\infty}^{\infty} dx\, e^{-\alpha x^2}=\sqrt{\frac{\pi}{\alpha}}$]

(a) $\frac{1}{\sqrt{1+Dt}}e^{-\frac{x^2}{1+Dt}}$ (b) $\frac{1}{\sqrt{1+2Dt}}e^{-\frac{x^2}{1+2Dt}}$

(c) $\frac{1}{\sqrt{1+4Dt}}e^{-\frac{x^2}{1+4Dt}}$ (d) $e^{-\frac{x^2}{1+Dt}}$

Ans. : (c)

Solution: $\frac{\partial f}{\partial t}=D\frac{\partial^2 f}{\partial x^2}$, $t>0$

Let $D=c^2$, $\frac{\partial f}{\partial t}=c^2\frac{\partial^2 f}{\partial x^2}$

Initial condition $f(x,0)=e^{-x^2}=g(x)$, $-\infty<x<\infty$

Now, $g\left(x+2cz\sqrt{t}\right)=e^{-\left(x+2cz\sqrt{t}\right)^2}$

The solution $f(x,t)$ is given as

$$f(x,t)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\left(x+2cz\sqrt{t}\right)^2}\cdot e^{-z^2}dz=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\left(x^2+4c^2z^2t+4cxz\sqrt{t}\right)}e^{-z^2}dz$$

$$=\frac{e^{-x^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\left(4c^2z^2t+4cxz\sqrt{t}+z^2\right)}dz=\frac{e^{-x^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\left\{\left(1+4c^2t\right)z^2+\left(4cxz\sqrt{t}\right)\right\}}dz$$

$$=\frac{e^{-x^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\exp\left[-\left(1+4c^2t\right)\left\{z^2+2\cdot z\cdot\frac{2cx\sqrt{t}}{\left(1+4c^2t\right)}+\left(\frac{2cx\sqrt{t}}{1+4c^2t}\right)^2-\left(\frac{2cx\sqrt{t}}{1+4c^2t}\right)^2\right\}\right]dz$$

$$=\frac{e^{-x^2}}{\sqrt{\pi}}\int_{-\infty}^{+\infty}e^{-\left(1+4c^2t\right)\left\{z+\frac{2cx\sqrt{t}}{1+4tc^2}\right\}^2}\cdot e^{\frac{4c^2x^2t}{\left(1+4c^2t\right)}}dz=\frac{e^{-x^2}}{\sqrt{\pi}}e^{\frac{4c^2x^2t}{1+4c^2t}}\int_{-\infty}^{+\infty}e^{-\left(1+4c^2t\right)\left\{z+\frac{2cx\sqrt{t}}{1+4tc^2}\right\}^2}dz$$

$$=\frac{e^{\frac{-x^2}{1+4c^2t}}}{\sqrt{\pi}}\times2\times\frac{1}{2}\times\frac{\sqrt{\pi}}{\sqrt{1+4c^2t}}$$

$$f(x,t)=\frac{1}{\sqrt{1+4Dt}}e^{\frac{-x^2}{1+4Dt}}\qquad\left(c^2=D\right)$$

Q83. A stable asymptotic solution of the equation $x_{n+1}=1+\frac{3}{1+x_n}$ is $x=2$. If we take $x_n=2+\in_n$ and $x_{n+1}=2+\in_{n+1}$, where $\in_n$ and $\in_{n+1}$ are both small, the ratio $\frac{\in_{n+1}}{\in_n}$ is approximately

(a) $-\frac{1}{2}$ (b) $-\frac{1}{4}$ (c) $-\frac{1}{3}$ (d) $-\frac{2}{3}$

Ans. : (c)

Q84. The 2×2 identity matrix I and the Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ do not form a group under matrix multiplication. The minimum number of 2×2 matrices, which includes these four matrices, and form a group (under matrix multiplication) is

(a) 20 (b) 8 (c) 12 (d) 16

Ans. (d)

$\times$	I	$-I$	$i\times I$	$-i\times I$	σ_x	$-\sigma_x$	$i\sigma_x$	$-i\sigma_x$	σ_y	$-\sigma_y$	$i\sigma_y$	$-i\sigma_y$	σ_z	$-\sigma_z$	$i\sigma_z$	$-i\sigma_z$
I	I	$-I$	$i\times I$	$-i\times I$	σ_x	$-\sigma_x$	$i\sigma_x$	$-i\sigma_x$	σ_y	$-\sigma_y$	$i\sigma_y$	$-i\sigma_y$	σ_z	$-\sigma_z$	$i\sigma_z$	$-i\sigma_z$
$-I$	$-I$	I	$-i\times I$	$i\times I$	$-\sigma_x$	σ_x	$-i\sigma_x$	$i\sigma_x$	$-\sigma_y$	σ_y	$-i\sigma_y$	$i\sigma_y$	$-\sigma_z$	σ_z	$-i\sigma_z$	$i\sigma_z$
$i\times I$	$i\times I$	$-i\times I$	$-I$	I	$i\sigma_x$	$-i\sigma_x$	$-\sigma_x$	σ_x	$i\sigma_y$	$-i\sigma_y$	$-\sigma_y$	σ_y	$i\sigma_z$	$-i\sigma_z$	$-\sigma_z$	σ_z
$-i\times I$	$-i\times I$	$i\times I$	I	$-I$	$-i\sigma_x$	$i\sigma_x$	σ_x	$-\sigma_x$	$-i\sigma_y$	$i\sigma_y$	σ_y	$-\sigma_y$	$-i\sigma_z$	$i\sigma_z$	σ_z	$-\sigma_z$
σ_x	σ_x	$-\sigma_x$	$i\sigma_x$	$-i\sigma_x$	I	$-I$	$i\times I$	$-i\times I$	$i\sigma_z$	$-i\sigma_z$	$-\sigma_z$	σ_z	$-i\sigma_y$	$i\sigma_y$	σ_y	$-\sigma_y$
$-\sigma_x$	$-\sigma_x$	σ_x	$-i\sigma_x$	$i\sigma_x$	$-I$	I	$-i\times I$	$i\times I$	$-i\sigma_z$	$i\sigma_z$	σ_z	$-\sigma_z$	$i\sigma_y$	$-i\sigma_y$	$-\sigma_y$	σ_y
$i\sigma_x$	$i\sigma_x$	$-i\sigma_x$	$-\sigma_x$	σ_x	$i\times I$	$-i\times I$	$-I$	I	$-\sigma_z$	σ_z	$-i\sigma_z$	$i\sigma_z$	σ_y	$-\sigma_y$	$i\sigma_y$	$-i\sigma_y$
$-i\sigma_x$	$-i\sigma_x$	$i\sigma_x$	σ_x	$-\sigma_x$	$-i\times I$	$i\times I$	I	$-I$	σ_z	$-\sigma_z$	$i\sigma_z$	$-i\sigma_z$	$-\sigma_y$	σ_y	$-i\sigma_y$	$i\sigma_y$
σ_y	σ_y	$-\sigma_y$	$i\sigma_y$	$-i\sigma_y$	$-i\sigma_z$	$i\sigma_z$	σ_z	$-\sigma_z$	I	$-I$	$i\times I$	$-i\times I$	$i\sigma_x$	$-i\sigma_x$	$-\sigma_x$	σ_x
$-\sigma_y$	$-\sigma_y$	σ_y	$-i\sigma_y$	$i\sigma_y$	$i\sigma_z$	$-i\sigma_z$	$-\sigma_z$	σ_z	$-I$	I	$-i\times I$	$i\times I$	$-i\sigma_x$	$i\sigma_x$	σ_x	$-\sigma_x$
$i\sigma_y$	$i\sigma_y$	$-i\sigma_y$	$-\sigma_y$	σ_y	σ_z	$-\sigma_z$	$i\sigma_z$	$-i\sigma_z$	$i\times I$	$-i\times I$	$-I$	I	$-\sigma_x$	σ_x	$-i\sigma_x$	$i\sigma_x$
$-i\sigma_y$	$-i\sigma_y$	$i\sigma_y$	σ_y	$-\sigma_y$	$-\sigma_z$	σ_z	$-i\sigma_z$	$i\sigma_z$	$-i\times I$	$i\times I$	I	$-I$	σ_x	$-\sigma_x$	$i\sigma_x$	$-i\sigma_x$
σ_z	σ_z	$-\sigma_z$	$i\sigma_z$	$-i\sigma_z$	$i\sigma_y$	$-i\sigma_y$	$-\sigma_y$	σ_y	$-i\sigma_x$	$i\sigma_x$	σ_x	$-\sigma_x$	I	$-I$	$i\times I$	$-i\times I$

$-\sigma_z$	$-\sigma_z$	σ_z	$-i\sigma_z$	$i\sigma_z$	$-i\sigma_y$	$i\sigma_y$	σ_y	$-\sigma_y$	$i\sigma_x$	$-i\sigma_x$	$-\sigma_x$	σ_x	$-I$	I	$-i\times I$	$i\times I$
$i\sigma_z$	$i\sigma_z$	$-i\sigma_z$	$-\sigma_z$	σ_z	$-\sigma_y$	σ_y	$-i\sigma_y$	$i\sigma_y$	σ_x	σ_x	$i\sigma_x$	$-i\sigma_x$	$i\times I$	$-i\times I$	$-I$	I
$-i\sigma_z$	$-i\sigma_z$	$i\sigma_z$	σ_z	$-\sigma_z$	σ_y	$-\sigma_y$	$i\sigma_y$	$-i\sigma_y$	σ_x	σ_x	$-i\sigma_x$	$i\sigma_x$	$-i\times I$	$i\times I$	I	$-I$

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Q85. Which of the following can not be the eigen values of a real 3×3 matrix

(a) $2i, 0, -2i$ (b) $1,1,1$ (c) $e^{i\theta}, e^{-i\theta}, 1$ (d) i, 1, 0

Ans. : (d)

Solution: If the matrix is real then the complex eigen values always occurs with its complex conjugate. In option (d) if i is an eigen value then $-i$ must also be an eigen value. But $-i$ is not given in option, hence option (d) is incorrect.

Q86. Let $u(x,y)=e^{ax}\cos(by)$ be the real part of a function $f(z)=u(x,y)+iv(x,y)$ of the complex variable $z=x+iy$, where a,b are real constants and $a\neq0$. The function $f(z)$ is complex analytic everywhere in the complex plane if and only if

(a) $b=0$ (b) $b=\pm a$ (c) $b=\pm2\pi a$ (d) $b=a\pm2\pi$

Ans. : (b)

Solution: The function $f(z)$ will be analytic everywhere in the complex plane if and only if it satisfies the Cauchy Riemann equation in that region.

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

Hence $ae^{ax}\cos(by)=\dfrac{\partial v}{\partial y}$ (i)

and $be^{ax}\sin(by)=\dfrac{\partial v}{\partial x}$ (ii)

From equation (i)

$$v(x,y)=\frac{ae^{ax}\sin(by)}{b}+c(y) \quad \text{(iii)}$$

Differentiating partially with x gives

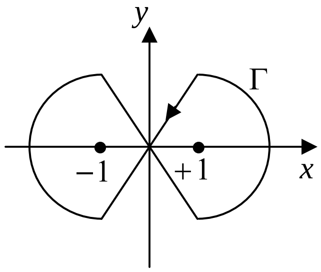
$$\frac{\partial v}{\partial x}=\frac{a^2e^{ax}\sin(by)}{b} \quad \text{(iv)}$$

From equation (iii) and (iv)

$$be^{ax}\sin(by)=\frac{a^2e^{ax}\sin(by)}{b}$$

$$\Rightarrow b^2=a^2 \Rightarrow b=\pm a$$

Q87. The integral $\oint_{\Gamma} \frac{ze^{i\pi z/2}}{z^2-1}dz$ along the closed contour Γ shown in the figure is



(a) 0 (b) 2π (c) -2π (d) $4\pi i$

Ans. : (c)

Solution: $f(z) = \frac{ze^{iz\pi/2}}{(z+1)(z-1)}$

For $z=+1$ anti-clockwise

$$I = 2\pi i \lim_{z\to 1} \frac{ze^{i\pi z/2}}{(z+1)} = \frac{2\pi i}{2}e^{i\pi/2} = \pi i e^{i\pi/2}$$

For $z=-1$

$$I = -2\pi i \lim_{z\to -1} \frac{ze^{i\pi z/2}}{(z-1)} = -2\pi i \times \frac{(-1)e^{-i\pi/2}}{(-2)} = -\pi i e^{-i\pi/2}$$

$$\text{Integral} = \pi i \frac{\left(e^{i\pi/2} - e^{-i\pi/2}\right)}{2i} \times 2i = 2\pi i^2 \sin\frac{\pi}{2} = -2\pi$$

Q88. The function $y(x)$ satisfies the differential equation $x\frac{dy}{dx} + 2y = \frac{\cos \pi x}{x}$. If $y(1)=1$, the value of $y(2)$ is

(a) π (b) 1 (c) $1/2$ (d) $1/4$

Ans. : (d)

Solution: The given differential equation can be written as

$$\frac{dy}{dx} + \frac{2}{x}y = \frac{\cos \pi x}{x^2}$$

This is a linear differential equation with Integrating factor $= e^{\int \frac{2}{x}dx} = x^2$

Hence $y.x^2 = \int x^2 . \frac{\cos \pi x}{x^2}dx + c \Rightarrow y = \frac{\sin \pi x}{\pi x^2} + \frac{c}{x^2}$

when $x=1, y=1$ hence $c=1 \Rightarrow y=\dfrac{\sin \pi x}{\pi x^2}+\dfrac{1}{x^2}$

hence, when $x=2, y=\dfrac{1}{4}$

Q89. The random variable $x(-\infty<x<\infty)$ is distributed according to the normal distribution $P(x)=\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}$. The probability density of the random variable $y=x^2$ is

(a) $\dfrac{1}{\sqrt{2\pi\sigma^2 y}}e^{-y/2\sigma^2},\ 0\le y<\infty$

(b) $\dfrac{1}{2\sqrt{2\pi\sigma^2 y}}e^{-y/2\sigma^2},\ 0\le y<\infty$

(c) $\dfrac{1}{\sqrt{2\sigma^2}}e^{-y/2\sigma^2},\ 0\le y<\infty$

(d) $\dfrac{1}{\sqrt{2\pi\sigma^2 y}}e^{-y/\sigma^2},\ 0\le y<\infty$

Ans. : (a)

Solution: $p(x)=\dfrac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}},\ -\infty<x<\infty$

$$\int_{-\infty}^{\infty} p(x)dx=1 \Rightarrow \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}dx=1$$

$$2\int_0^{\infty}\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{x^2}{2\sigma^2}}dx=1 \Rightarrow \frac{2\times 1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}e^{-\frac{x^2}{2\sigma^2}}dx=1$$

put $x^2=y$ $\quad dy=2xdx,\ dx=\dfrac{1}{2\sqrt{y}}$

$$2\times\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}e^{\frac{-y}{2\sigma^2}}\frac{1}{2\sqrt{y}}dy=\frac{1}{\sqrt{2\pi\sigma^2}}\int_0^{\infty}\frac{1}{\sqrt{y}}e^{\frac{-y}{2\sigma^2}}dy\cdot$$

$$f(y)=\frac{1}{\sqrt{2\pi y\sigma^2}}e^{-\frac{y}{2\sigma^2}},\quad 0<y<\infty\cdot$$

Q90. The Green's function satisfying

$$\frac{d^2}{dx^2}g(x,x_0)=\delta(x-x_0)$$

with the boundary conditions $g(-L,x_0)=0=g(L,x_0)$, is

(a) $\begin{cases}\frac{1}{2L}(x_0-L)(x+L), & -L\le x<x_0\\ \frac{1}{2L}(x_0+L)(x-L), & x_0\le x\le L\end{cases}$

(b) $\begin{cases}\frac{1}{2L}(x_0+L)(x+L), & -L\le x<x_0\\ \frac{1}{2L}(x_0-L)(x-L), & x_0\le x\le L\end{cases}$

(c) $\begin{cases}\frac{1}{2L}(L-x_0)(x+L), & -L\le x<x_0\\ \frac{1}{2L}(x_0+L)(L-x), & x_0\le x\le L\end{cases}$

(d) $\frac{1}{2L}(x-L)(x+L), \quad -L\le x\le L$

Ans. : (a)

Solution: $\frac{d^2}{dx^2}g(x,x_0)=\delta(x-x_0)$

boundary conditions:

$$g(-L,x_0)=0=g(L,x_0)$$

The homogeneous equation for Green's function is

$$\frac{d^2}{dx^2}g(x,x_0)=0$$

Solution of above equation is

$$g(x,x_0)=\begin{cases}Ax+B, & x<x_0\\ Cx+D, & x>x_0\end{cases}$$

Applying boundary condition

$$g(-L,x_0)=0\Rightarrow -AL+B=0\Rightarrow AL=B$$

$$g(L,x_0)=0\Rightarrow CL+D=0\Rightarrow -CL=D$$

Hence, $g(x,x_0)=\begin{cases}A(x+L), & x<x_0\\ C(x-L), & x>x_0\end{cases}$

From continuity of Green's function at $x=x_0$ we have

$$A(x_0+L)=C(x_0-L)$$

$$A = C\left(\frac{x_0 - L}{x_0 + L}\right)$$

From discontinuity of derivative of Green's function $\frac{\partial g}{\partial x}$ at $x = x_0$

We have

$$\left|\frac{\partial g}{\partial x}\right|_{x=x_0^+} - \left.\frac{\partial g}{\partial x}\right|_{x=x_0^-} = 1$$

$$\left|\frac{\partial g}{\partial x}\right|_{x=x_0^+} = C, \left.\frac{\partial g}{\partial x}\right|_{x=x_0^-} = A$$

$$\therefore C - A = 1 \Rightarrow C = A + 1$$

Thus, the required solution of Green's function is given by

$$g(x, x_0) = \begin{cases} \frac{(x_0 - L)(x + L)}{2L}, & x < x_0 \\ \frac{(x_0 + L)(x - L)}{2L}, & x > x_0 \end{cases}$$

Q91. Let $\sigma_x, \sigma_y, \sigma_z$ be the Pauli matrices and $x'\sigma_x + y'\sigma_y + z'\sigma_z = \exp\left(\frac{i\theta\sigma_z}{2}\right) \times$ $\left[x\sigma_x + y\sigma_y + z\sigma_z\right]\exp\left(-\frac{i\theta\sigma_z}{2}\right)$.

Then the coordinates are related as follows

(a) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (b) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(c) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} & 0 \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ (d) $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Ans. : (b)

Solution: $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Hence, $x\sigma_x + y\sigma_y + z\sigma_z = \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}$

$x'\sigma_x + y'\sigma_y + z'\sigma_z = \begin{pmatrix} z' & x^1 - iy' \\ x'+iy' & -z' \end{pmatrix}$

$\exp\left(\frac{i\theta\sigma_z}{z}\right) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$ and $\exp\left(\frac{-i\theta\sigma_z}{2}\right) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$

Hence, $\begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}\begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix}\begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$

$\Rightarrow \begin{pmatrix} z' & x'-iy' \\ x'+iy' & -z' \end{pmatrix} = \begin{pmatrix} z & e^{i\theta}(x-iy) \\ e^{-i\theta}(x+iy) & -z \end{pmatrix}$

Hence, $z' = z$ and $x' - iy' = e^{i\theta}(x-iy)$

Thus $x' - iy' = [(\cos\theta)x + (\sin\theta)y] - i[(\cos\theta)y - (\sin\theta)x]$

Thus $x' = (\cos\theta)x + (\sin\theta)y$

And $y' = (-\sin\theta)x + (\cos\theta)y$

Thus, $\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Q92. Which of the following sets of 3×3 matrices (in which a and b are real numbers) forms a group under matrix multiplication?

(a) $\left\{\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ b & 0 & 1 \end{pmatrix};\ a,b\in\mathbb{R}\right\}$

(b) $\left\{\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix};\ a,b\in\mathbb{R}\right\}$

(c) $\left\{\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix};\ a,b\in\mathbb{R}\right\}$

(d) $\left\{\begin{pmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ a,b\in\mathbb{R}\right\}$

Ans. : (c)

Solution: In order to form the group the required matrix must satisfy the following conditions.

(a) For any three matrices $A, B, C \in G$ $\quad A(BC)=(AB)C$

(b) There must exist an identity element I such that $AI = IA = A$

(c) There must exist an inverse element for each element belonging to G.

(d) For any two matrices $A, B \in G$, the matrix $AB \in G$ (closer property).

All the given matrices satisfy property (a), if we put $a=0$ and $b=0$, we see that property (b) also holds for all the given matrices.

If we put $a=1$ and $b=1$, in option (a) , two rows becomes identical and matrix in option (a) is non-invertible. If we put $a=1$ and $b=1$, in option (d), two rows becomes identical and the matrix is non-invertible.

Now only option (b) and (c) remains. For option (b) take two matrices and multiply

$$\begin{pmatrix} 1 & a_1 & 0 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a_2 & 0 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & a_1+a_2 & a_1b_2 \\ 0 & 1 & b_1+b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

we see that the resulting matrix does not satisfy closure property. For option (3) take two matrices and multiply

$$\begin{pmatrix} 1 & 0 & a_1 \\ 0 & 1 & b_1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & a_2 \\ 0 & 1 & b_2 \\ 0 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & a_2+a_1 \\ 0 & 1 & b_2+b_1 \\ 0 & 0 & 1 \end{pmatrix}$$

we see that the resulting matrix satisfy the closure property. Hence the correct option is (c)

Q93. A random variable n obeys Poisson statistics. The probability of finding $n=0$ is 10^{-6}. The expectation value of n is nearest to

(a) 14 (b) 10^6 (c) e (d) 10^2

Ans. : (a)

Solution: In Poisson's statistics the probability of finding the value n is given by $P(n)=\frac{\mu^n}{n!}e^{-\mu}$

The mean of Poisson's statistics is μ. From the question

$$P(0)=10^{-6} \Rightarrow 10^{-6}=\frac{\mu^0}{0!}e^{-\mu} \Rightarrow e^{-\mu}=10^{-6}$$

Talking Log of both sides, $-\mu=-6\ln 10 \Rightarrow \mu=6\ln 10$

Hence the expectation value of n is $\mu=6\times 2.30=13.8\approx 14$

NET/JRF (DEC - 2017)

Q94. Let A be a non-singular 3×3 matrix, the columns of which are denoted by the vectors $\vec{a},\vec{b}$ and $\vec{c}$, respectively. Similarly, $\vec{u},\vec{v}$ and $\vec{w}$ denote the vectors that form the corresponding columns of $\left(A^T\right)^{-1}$. Which of the following is true?

(a) $\vec{u}\cdot\vec{a}=0,\ \vec{u}\cdot\vec{b}=0,\ \vec{u}\cdot\vec{c}=1$

(b) $\vec{u}\cdot\vec{a}=0,\ \vec{u}\cdot\vec{b}=1,\ \vec{u}\cdot\vec{c}=0$

(c) $\vec{u}\cdot\vec{a}=1,\ \vec{u}\cdot\vec{b}=0,\ \vec{u}\cdot\vec{c}=0$

(d) $\vec{u}\cdot\vec{a}=0,\ \vec{u}\cdot\vec{b}=0,\ \vec{u}\cdot\vec{c}=0$

Ans. : (c)

Solution: We can take any 3×3 non singular matrix in order to avoid long calculation.

$$\text{Take } A=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\\ \downarrow & \downarrow & \downarrow\\ \vec{a} & \vec{b} & \vec{c}\end{bmatrix}\Rightarrow\left(A^T\right)^{-1}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/3\\ \downarrow & \downarrow & \downarrow\\ \vec{u} & \vec{v} & \vec{w}\end{bmatrix}$$

We see that

$\vec{u}.\vec{a}=1.1+0.0+0.0=1$

$\vec{u}.\vec{b}=1.0+0.2+0.0=0$

$\vec{u}.\vec{c}=1.0+0.0+0.3=0$

Q95. Consider the real function $f(x)=1/\left(x^2+4\right)$. The Taylor expansion of $f(x)$ about $x=0$ converges

(a) for all values of x

(b) for all values of x except $x=\pm 2$

(c) in the region $-2<x<2$

(d) for $x>2$ and $x<-2$

Ans. : (c)

Solution: $f(x)=\dfrac{1}{x^2+4}=\dfrac{1}{4\left(1+\dfrac{x^2}{4}\right)}$

Thus the Taylor's series of $f(x)$ is $\dfrac{1}{4}$ times the binomial series of $\left(1+\dfrac{x^2}{4}\right)^{-1}$

Now, the binomial series converges if $\left|\dfrac{x^2}{4}\right|<1\Rightarrow|x|^2<4\Rightarrow\left(|x|-2\right)\left(|x|+2\right)<0$

Since $|x|+2$ is always greater than 0,

Hence $|x|-2<0 \Rightarrow -2<x<2$

Q96. Consider the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & b & 2c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The condition for existence of a non-trivial solution and the corresponding normalised solution (upto a sign) is

(a) $b=2c$ and $(x,y,z)=\frac{1}{\sqrt{6}}(1,-2,1)$

(b) $c=2b$ and $(x,y,z)=\frac{1}{\sqrt{6}}(1,1,-2)$

(c) $c=b+1$ and $(x,y,z)=\frac{1}{\sqrt{6}}(2,-1,-1)$

(d) $b=c+1$ and $(x,y,z)=\frac{1}{\sqrt{6}}(1,-2,1)$

Ans. : (d)

Solution: We know that the matrix equation, $AX=0$, where A is the given matrix and X is a column vector has a non-zero solution if and only if $|A|=0$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & b & 2c \end{vmatrix} = 0 \Rightarrow 4c-3b-2c+6+b-4=0$$

$$\Rightarrow 2c-2b+2=0 \Rightarrow b=c+1$$

we do not need to perform further calculation.

Q97. Consider the differential equation $\frac{dy}{dt}+ay=e^{-bt}$ with the initial condition $y(0)=0$. Then the Laplace transform $Y(s)$ of the solution $y(t)$ is

(a) $\frac{1}{(s+a)(s+b)}$ (b) $\frac{1}{b(s+a)}$ (c) $\frac{1}{a(s+b)}$ (d) $\frac{e^{-a}-e^{-b}}{b-a}$

Ans. : (a)

Solution: Given $\frac{dy}{dt}+ay=e^{-bt}$

Taking Laplace transform of both sides

We obtain

$$L\left\{\frac{dy}{dt}\right\}+aL\{y(t)\}=L\{e^{-bt}\}\Rightarrow sY(s)-y(0)+aY(s)=\frac{1}{s+b}$$

Since, $y(0)=0$, we obtain

$$(s+a)Y(s)=\frac{1}{s+b}\Rightarrow Y(s)=\frac{1}{(s+a)(s+b)}$$

Q98. The number of linearly independent power series solutions, around $x=0$, of the second order linear differential equation $x\frac{d^2y}{dx^2}+\frac{dy}{dx}+xy=0$, is

(a) 0 (this equation does not have a power series solution)

(b) 1

(c) 2

(d) 3

Ans. : (b)

Q99. Consider an element $U(\varphi)$ of the group $SU(2)$, where φ is any one of the parameters of the group. Under an infinitesimal change $\varphi\to\varphi+\delta\varphi$, it changes as $U(\varphi)\to U(\varphi)+\delta U(\varphi)=(1+X(\delta\varphi))U(gj)$. To order $\delta\varphi$, the matrix $X(\delta\varphi)$ should always be

(a) positive definite (b) real symmetric (c) hermitian (d) anti-hermitian

Ans. : (d)

Solution: Since, $U(\phi)=e^{-im\phi}$

$$U(\phi+d\phi)=e^{-im\phi}.e^{-im(d\phi)}$$

$$\because U(\phi)+\delta U(\phi)=(1+X(d\phi))U(\phi)$$

$$\therefore\ 1+X(d\phi)=e^{-im(d\phi)} \text{ or, } 1+X(d\phi)=1-im(d\phi)+...$$

or, $Xd(\phi)=-im(d\phi)$

here, m is one of the Pauli spin matrices, since Pauli matrices are hermitian, taken complex conjugate, so matrix should anti-hermitian.

Hence correct option is (d)

Q100. The differential equation $\frac{dy(x)}{dx} = \alpha x^2$, with the initial condition $y(0) = 0$, is solved using Euler's method. If $y_E(x)$ is the exact solution and $y_N(x)$ the numerical solution obtained using n steps of equal length, then the relative error $\left|\frac{(y_N(x) - y_E(x))}{y_E(x)}\right|$ is proportional to

(a) $\frac{1}{n^2}$ (b) $\frac{1}{n^3}$ (c) $\frac{1}{n^4}$ (d) $\frac{1}{n}$

Ans. : (d)

Solution: $\frac{dy}{dx} = \alpha x^2,\ y(0) = 0$

$y_E = \frac{\alpha x^3}{3}$, but $x = n\hbar$

Exact solution, $y_E = \frac{\alpha n^3 h^3}{3}$

Numerically, $f(x, y) = \alpha x^2$

Euler's method, $y_i = y_{i-1} + hf(x_{i-1}, y_{i-1})$

$y_1 = 0$, $y_2 = \alpha h^3$ $y_3 = 5\alpha h^3$

$$y_n = \frac{(n-1)n(2n-1)}{6}\alpha h^3$$

Since, $0, 5, 14, 30, ...$ different from square terms

At, $x_0 = 0$ $x_1 = x_0 + h = h$ $x_2 = x_0 + 2h = 2h$ $x_3 = x_0 + 3h = 3h$

$x_{n-1} = x_0 + (n-1)h = (n-1)h$. Now, $x_n = nh$

$f(x_0, y_0) = 0$, $f(x_1, y_1) = \alpha h^2$, $f(x_2, y_2) = 4\alpha h^2$

$f(x_{n-1}, y_{n-1}) = \alpha(n-1)^2 h^2$

$$\left|\frac{(y_N - y_E)}{y_E}\right| = \left|\frac{\frac{(n-1)n(2n-1)\alpha h^3}{6} - \frac{\alpha n^3 h^3}{3}}{\frac{\alpha n^3 h^3}{3}}\right|$$

By solving, $\left|\frac{y_N - y_E}{y_E}\right| \propto \frac{1}{n}$

Q101. The interval $[0,1]$ is divided into n parts of equal length to calculate the integral $\int_0^1 e^{i2\pi x}dx$ using the trapezoidal rule. The minimum value of n for which the result is exact, is

(a) 2 (b) 3 (c) 4 (d) ∞

Ans. : (a)

Solution: $\int_0^1 e^{i2nx}dx = 0$, exact value

Now, $nh = 1-0,\ h = \frac{1}{n}$

$y = f(x) = e^{i2nx}$, Let $n = 2$, then $x_0 = 0,\ y_0 = 1$, $x_1 = \frac{1}{2}$, $y_1 = -1$ and $x_2 = 1,\ y_2 = 1$

$$I = \frac{h}{2}\left[y_0 + 2\left(y_1 + ... + y_{n-1}\right) + y_n\right] = \frac{1}{4}\left[1 + 2(-1) + 1\right]$$

$I = 0$. So, $n = 2$

Q102. The generating function $G(t,x)$ for the Legendre polynomials $P_n(t)$ is

$$G(t,x) = \frac{1}{\sqrt{1-2xt+x^2}} = \sum_{n=0}^{\infty} x^n P_n(t), \quad \text{for } |x| < 1$$

If the function $f(x)$ is defined by the integral equation $\int_0^x f(x')dx' = xG(1,x)$, it can be expressed as

(a) $\sum_{n,m=0}^{\infty} x^{n+m} P_n(1) P_m\left(\frac{1}{2}\right)$ (b) $\sum_{n,m=0}^{\infty} x^{n+m} P_n(1) P_m(1)$

(c) $\sum_{n,m=0}^{\infty} x^{n-m} P_n(1) P_m(1)$ (d) $\sum_{n,m=0}^{\infty} x^{n-m} P_n(0) P_m(1)$

Ans. : (b)

Solution: $G(t,x) = \frac{1}{\sqrt{1-2xt+x^2}} = \sum_{n=0}^{\infty} x^n P_n(t)$ for $|x| < 1$

$$G(1,x)=\frac{1}{\sqrt{1-2x+x^2}}=\sum_{n=0}^{\infty}x^nP_n(1)$$

$$=\frac{1}{\sqrt{(1-x)^2}}=\sum_{n=0}^{\infty}x^nP_n(1)=\frac{1}{1-x}\quad \text{Since } |x|<1$$

Now, $x\cdot\frac{1}{1-x}=\int_0^x f(x')\,dx'$

Differentiating both sides,

$$f(x)=\frac{d}{dx}\frac{x}{1-x}=\frac{1}{(1-x)^2}$$

NET/JRF (JUNE-2018)

Q103. Consider the following ordinary differential equation

$$\frac{d^2x}{dt^2}+\frac{1}{x}\left(\frac{dx}{dt}\right)^2-\frac{dx}{dt}=0$$

with the boundary conditions $x(t=0)=0$ and $x(t=1)=1$. The value of $x(t)$ at $t=2$ is

(a) $\sqrt{e-1}$ (b) $\sqrt{e^2+1}$ (c) $\sqrt{e+1}$ (d) $\sqrt{e^2-1}$

Ans. : (c)

Solution: The given equation can be written as

$$\frac{1}{x}\frac{d}{dt}\left(x\frac{dx}{dt}\right)-\frac{dx}{dt}=0\Rightarrow\frac{d}{dt}\left(x\frac{dx}{dt}\right)-x\frac{dx}{dt}=0$$

putting $y=x\frac{dx}{dt}$ gives

$$\frac{dy}{dt}-y=0\Rightarrow lny=t+lnc_1\Rightarrow y=c_1e^t$$

Since $x\frac{dx}{dt}=c_1e^t$ hence by integrating

$$\frac{x^2}{2}=c_1e^t+c_2 \qquad \text{(i)}$$

Using boundary conditions we obtain

$c_1+c_2=0$ and $c_1e+c_2=\frac{1}{2}$

Solving these equations we obtain $c_1 = \frac{1}{2(e-1)}$ and $c_2 = -\frac{1}{2(e-1)}$

Thus, $\frac{x^2}{2} = \frac{1}{2(e-1)}e^t - \frac{1}{2(e-1)}$

When $t = 2$, we obtain, $x^2 = \frac{e^2}{(e-1)} - \frac{1}{(e-1)} = \frac{(e^2-1)}{(e-1)} = e+1$

Therefore, $x(2) = \sqrt{e+1}$

Q104. What is the value of a for which $f(x, y) = 2x + 3(x^2 - y^2) + 2i(3xy + ay)$ is an analytic function of complex variable $z = x + iy$

(a) 1 (b) 0 (c) 3 (d) 2

Ans. : (a)

Solution: $f(x, y) = 2x + 3(x^2 - y^2) + 2i(3xy + \alpha y)$

$u = 2x + 3(x^2 - y^2)$, $v = 2(3xy + \alpha y)$

C-R conditions: $u_x = v_y, u_y = -v_x$,

$2 + 3(2x) = 2(3x + \alpha) \Rightarrow \alpha = 1 \Rightarrow -6y = -6y$

Q105. Consider the three vectors $\vec{v}_1 = 2\hat{i} + 3\hat{k}, \vec{v}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{v}_3 = 5\hat{i} + \hat{j} + a\hat{k}$ where $\hat{i}, \hat{j}$ and $\hat{k}$ are the standard unit vectors in a three-dimensional Euclidean space. These vectors will be linearly dependent if the value of a is

(a) $\frac{31}{4}$ (b) $\frac{23}{4}$ (c) $\frac{27}{4}$ (d) 0

Ans. : (a)

Solution: Given vector will be linearly dependent if the determinant of the matrix formed by taking these vectors as column is zero.

$$\begin{vmatrix} 2 & 1 & 5 \\ 0 & 2 & 1 \\ 3 & 2 & a \end{vmatrix} = 0 \Rightarrow 2(2a-2) - (-3) + 5(-6) = 0$$

$$\Rightarrow 4a - 4 + 3 - 30 = 0 \Rightarrow 4a - 31 = 0 \Rightarrow a = \frac{31}{4}$$

Q106. The Fourier transform $\int_{-\infty}^{\infty} dx f(x) e^{ikx}$ of the function $f(x) = e^{-|x|}$

(a) $-\frac{2}{1+k^2}$ (b) $-\frac{1}{2(1+k^2)}$ (c) $\frac{2}{1+k^2}$ (d) $\frac{2}{(2+k^2)}$

Ans. : (c)

Solution: $\int_{-\infty}^{+\infty} dx e^{-|x|} e^{ikx} = \int_{-\infty}^{+\infty} dx\, e^{-|x|} \cos kx dx$ odd function $\sin kx$ vanishes

$$\Rightarrow 2\int_0^\infty e^{-x} \cos kx\, dx = 2\frac{e^{-x}}{1+k^2}\left[-\cos kx + k \sin kx\right]_0^\infty$$

$$\because \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \sin bx\right]$$

$$\Rightarrow 2\int_0^\infty e^{-x} \cos kx dx = 2\frac{e^0}{1+k^2} = \frac{2}{1+k^2}$$

Q107. The value of the integral

$$\int_{-\pi/2}^{\pi/2} dx \int_{-1}^{+1} dy . \delta(\sin 2x) \delta(x-y) \text{ is}$$

(a) 0 (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 1

Ans. : (b)

Solution: $I = \int_{-\pi/2}^{\pi/2} dx \int_{-1}^{1} dy \delta(\sin 2x) \delta(x-y) = \int_{-\pi/2}^{\pi/2} dx\, \delta(\sin 2x) \int_{-1}^{1} \delta(y-x) dy$

If we assume that x lies between -1 and $+1$ then the second integral is 1 and the given integral becomes

$$I = \int_{-\pi/2}^{\pi/2} \delta(\sin 2x) dx$$

now $\delta(\sin 2x) = \sum_{n=-\infty}^{\infty} \frac{\delta\left(x - \frac{n\pi}{2}\right)}{\left|2 \cos 2\frac{n\pi}{2}\right|}$

Therefore, $I = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \delta(x) dx = \frac{1}{2}.1 = \frac{1}{2}$

Q108. Which of the following statements is true for a 3×3 real orthogonal matrix with determinant $+1$?

(a) the modulus of each of its eigenvalues need not be 1, but their product must be 1

(b) at least one of its eigenvalues is $+1$

(c) all of its eigenvalues must be real

(d) none of its eigenvalues must be real

Ans. : (b)

Solution: The characteristic equation of any 3×3 matrix is of thee form $\lambda^3+a\lambda^2+b\lambda+c=0$ which implies that at least one of the eigenvalues must be real. It is a proven fact that modulus of each eigenvalues of an orthogonal matrix is 1.

If all eigenvalues of 3×3 orthogonal matrix are real then only possibilities for eigenvalues are

$$\lambda_1=1,\lambda_2=1 \text{ and } \lambda_3=1 \text{ or } \lambda_1=-1,\lambda_2=-1,\lambda_3=1 \text{ or } \lambda_1=-1,\lambda_2=1,\lambda_3=-1$$

Thus we see that at least one eigenvalue is $+1$. Suppose one eigenvalues is real and other two eigenvalues are complex conjugates. Now

$$\lambda_1\lambda_2\lambda_3=1$$

$$\Rightarrow \lambda_1(a+ib)(a-ib)=1\Rightarrow \lambda_1(a^2+b^2)=1$$

Since a^2+b^2 is always positive hence $\lambda_1=1$.

In this case also we see that at least one eigenvalue must be $+1$

Q109. In the function $P_n(x)e^{-x^2}$ of a real variable x, $P_n(x)$ is polynomial of degree n. The maximum number of extrema that this function can have is

(a) $n+2$ (b) $n-1$ (c) $n+1$ (d) n

Ans. : (c)

Solution: $y=P_n(x)e^{-x^2}\Rightarrow P_n'(x)e^{-x^2}+P_n(x)e^{-x^2}(-2x)=0\Rightarrow P_n'(x)-2xP_n(x)=0$

$$P_0(x)=1, P_1(x)=2\Rightarrow P_0'(x)-2xP_0(x)=0\Rightarrow 0-2x.1=0$$

$x=0,\ 1$ extrema

$$P_1'(x)-2xP_1(x)=0$$

$1-2x.x=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$ i.e., 2 extrema.

Thus in general there are $(n+1)$ extrema.

Q110. The Green's function $G(x,x')$ for the equation $\frac{d^2y(x)}{dx^2}+y(x)=f(x)$, with the boundary values $y(0)=y\left(\frac{\pi}{2}\right)=0$, is

(a) $G(x,x')=\begin{cases} x\left(x'-\frac{\pi}{2}\right), & 0<x<x'<\frac{\pi}{2} \\ \left(x-\frac{\pi}{2}\right)x', & 0<x'<x<\frac{\pi}{2} \end{cases}$

(b) $G(x,x')=\begin{cases} -\cos x'\sin x, & 0<x<x'<\frac{\pi}{2} \\ -\sin x'\cos x, & 0<x'<x<\frac{\pi}{2} \end{cases}$

(c) $G(x,x')=\begin{cases} \cos x'\sin x, & 0<x<x'<\frac{\pi}{2} \\ \sin x'\cos x, & 0<x'<x<\frac{\pi}{2} \end{cases}$

(d) $G(x,x')=\begin{cases} x\left(\frac{\pi}{2}-x'\right), & 0<x<x'<\frac{\pi}{2} \\ x'\left(\frac{\pi}{2}-x\right), & 0<x'<x<\frac{\pi}{2} \end{cases}$

Ans. : (b)

Solution: $\frac{d^2y}{dx^2}+y=0 \Rightarrow m^2+1=0 \Rightarrow m=0\pm i$

$y_1(x)=\sin x(at), y_1'(x)=\cos x$

$y_2(x)=\cos x \quad \left(at\ \frac{\pi}{2}\right), y_2'(x)=-\sin x$

$A=\{P(x')[y_2'(x')y_1(x_1')-y_1'(x')]y_2(x')\}$

$\Rightarrow A=\{-\sin x'\sin x'-\cos'\cos x'\} \quad \because P(x)=1 \Rightarrow A=-1$

Thus $G(x,x') = \begin{cases} Ay_1(x)\,y_2(x'), & x < x' \\ Ay_2(x)\,y_1(x), & x > x' \end{cases} = \begin{cases} -\sin x \cos x', & 0 < x < x' < \dfrac{\pi}{2} \\ -\cos x \sin x', & 0 < x' < x < \dfrac{\pi}{2} \end{cases}$

Q111. The fractional error in estimating the integral $\int_0^1 x dx$ using Simpson's $\frac{1}{3}$ rule, using a step size 0.1, is nearest to

(a) 10^{-4} (b) 0 (c) 10^{-2} (d) 3×10^{-4}

Ans. : (b)

Solution: $I = \frac{h}{3}\left[y_0 + 2(y_2 + y_4 + ...) + 4(y_1 + y_3 + y_6 + ...) + y_n\right]$

$= \frac{0.1}{3}\left[0 + 2(0.2 + 0.4 + 0.6 + 0.8) + 4(0.1 + 0.3 + 0.5 + 0.7 + 0.9 + 1)\right]$

$= \frac{1}{30}\left[4 + 10 + 1\right] = \frac{15}{30} = 0.5$

fractional error $= \frac{\Delta I}{I_{true}} = \frac{0.5 - 0.5}{0.5} \approx 0$

y_0	0
y_1	0.1
y_2	0.2
y_3	0.3
y_4	0.4
y_5	0.5
y_6	0.6
y_7	0.7
y_8	0.8
y_9	0.9
y_{10}	1.0

NET/JRF (DEC-2018)

Q112. One of the eigenvalues of the matrix e^A is e^a, where $A = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}$. The product of the other two eigenvalues of e^A is

(a) e^{2a} (b) e^{-a} (c) e^{-2a} (d) 1

Ans. : (d)

Solution: Eigenvalues of matrix A are a, a and $-a$. The product of two other eigenvalues of A are $e^a a^{-a} = 1$

Alternativety

$e^{TraceA} = e^{\lambda_1+\lambda_2+\lambda_3} = \det e^A$

$\Rightarrow e^{\lambda_1}.e^{\lambda_2+\lambda_3} = \det e^A \Rightarrow e^a.e^{\lambda_2}.e^{\lambda_3} = e^a$

$\Rightarrow e^{\lambda_2}.e^{\lambda_3} = 1$

Q113. The polynomial $f(x) = 1 + 5x + 3x^2$ is written as linear combination of the Legendre polynomials

$\left(P_0(x) = 1, P_1(x), P_2(x) = \frac{1}{2}(3x^2 - 1) \right)$ as $f(x) = \sum_n c_n P_n(x)$. The value of c_0 is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 2 (d) 4

Ans. : (c)

Solution: $f(x) = 1 + 5x + 3x^2$

$1 = P_0(x) \qquad x = P_1(x)$

$x^2 = \frac{1}{3}(2P_2(x) + 1)$

$f(x) = P_0(x) + 5P_1(x) + 2P_2(x) + P_0(x)$

$= 2P_0(x) + 5P_1(x) + 2P_2(x)$

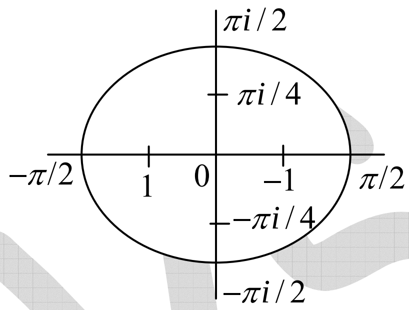
$= c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) \quad c_0 = 2$

Q114. The value of the integral $\oint_C \frac{dz}{z}\frac{\tanh 2z}{\sin \pi z}$, where C is a circle of radius $\frac{\pi}{2}$, traversed counter-clockwise, with centre at $z=0$, is

(a) 4 (b) $4i$ (c) $2i$ (d) 0

Ans. : (b)

Solution: $\oint_C \frac{dz}{z}\frac{\tanh 2z}{\sin \pi z}dz$



$z = 0, 1, -1, \frac{\pi i}{4}, \frac{-\pi i}{4}$

$$f(z) = \frac{2z - \frac{1}{3}(2z)^3 + \frac{2}{15}(2z)^5 ....}{z\left(\pi z - \frac{\pi^3 z^3}{3!} + ...\right)}$$

$$\frac{2}{\pi z}\left(1 - \frac{1}{2}z^2 + ...\right)\left(1 - \frac{\pi^2 z^2}{2!} + ...\right)$$

$$b_1 = \frac{2}{\pi}$$

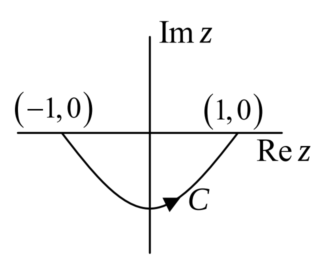
As $\operatorname{Re} z = 1$, $\frac{\tanh^2}{-\pi}$ and $\operatorname{Re} z = -1$, $\frac{\tanh^2}{-\pi}$

$$\operatorname{Re} z = \frac{i\pi}{4} = -\frac{1}{\pi}\left(2\operatorname{cosec} h \frac{\pi^2}{4}\right)$$

$$\operatorname{Re} z = \frac{-i\pi}{4} = -\frac{1}{\pi}\left(2\operatorname{cosec} h \frac{\pi^2}{4}\right)$$

$I = 2\pi i \Sigma R = 4i$ only when 0 lies inside, otherwise wrong question.

Q115. The integral $I = \int_C e^z dz$ is evaluated from the point $(-1, 0)$ to $(1, 0)$ along the contour C, which is an arc of the parabola $y = x^2 - 1$, as shown in the figure.



The value of I is

(a) 0 (b) $2\sinh 1$ (c) $e^{2i}\sinh 1$ (d) $e + e^{-1}$

Ans. : (b)

Solution: $\int_C f(z)dz = 2\pi i\Sigma R$

$$\int_C f(z)dz + \int_1^{-1} e^x dx = 0$$

$$\int_C f(z)dz = -\int_1^{-1} e^x dx = \int_1^{-1} e^x dx = \frac{(e^1 - e^{-1})}{2} \cdot 2 = 2\sinh 1$$

Q116. In terms of arbitrary constants A and B, the general solution to the differential equation

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 3y = 0 \text{ is}$$

(a) $y = \frac{A}{x} + Bx^3$ (b) $y = Ax + \frac{B}{x^3}$

(c) $y = Ax + Bx^3$ (d) $y = \frac{A}{x} + \frac{B}{x^3}$

Ans. : (d)

Solution: The given equation is Euler-Cauchy differential equation. The characteristic equation of

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 6y = 0$$

is, $m^2 + 4m + 6 = 0 \Rightarrow m = -3$ or $m = -1$

Thus, $y_1 = x^{-1} = \frac{1}{x}$ and $y_2 = x^2 = \frac{1}{x^3}$

Therefore the general solution is

$$y = \frac{A}{x} + \frac{B}{x^3}$$

Q117. The Green's function $G(x, x')$ for the equation $\frac{d^2y(x)}{dx^2} = f(x)$, with the boundary values $y(0) = 0$ and $y(1) = 0$, is

(a) $G(x, x') = \begin{cases} \frac{1}{2}x(1 - x'), & 0 < x < x' < 1 \\ \frac{1}{2}x'(1 - x) & 0 < x' < x < 1 \end{cases}$

(b) $G(x, x') = \begin{cases} x(x' - 1), & 0 < x < x' < 1 \\ x'(1 - x) & 0 < x' < x < 1 \end{cases}$

(c) $G(x,x')=\begin{cases} -\frac{1}{2}x(1-x'), & 0<x<x'<1 \\ \frac{1}{2}x'(1-x) & 0<x'<x<1 \end{cases}$ (d) $G(x,x')=\begin{cases} x(x'-1), & 0<x<x'<1 \\ x'(x-1) & 0<x'<x<1 \end{cases}$

Ans. : (d)

Solution: $\frac{d^2y}{dx^2}=f(x)$

$p(x')=1$

$x_1=1, y_2=x$

$y_1=x, y_2=1-x \qquad w=\begin{vmatrix} x & 1-x \\ 1 & -1 \end{vmatrix}=-1$

$A=-1$

$$G(x,x')=\begin{cases} A & y_1 & y_2' \\ A & y_1' & y_2 \end{cases}=\begin{cases} x & (x'-1) & 0<x<x'<1 \\ x' & (x-1) & 0<x'<x<1 \end{cases}$$

Q118. A 4×4 complex matrix A satisfies the relation $A^{\dagger}A=4I$, where I is the 4×4 identity matrix. The number of independent real parameters of A is

(a) 32 (b) 10 (c) 12 (d) 16

Ans. : (d)

Solution: Given that $A^{\dagger}A=4I \Rightarrow \frac{1}{4}(A^{\dagger}A)=I$

Let $A=2B$ then

$A^{\dagger}=2B^{\dagger}$

Therefore, $B^{\dagger}B=I$

This shows that B is a unitary matrix. The number of independent real parameters needed to specify an $n\times n$ unitary matrix is n^2. Thus, the number of independent parameter needed to specify matrix B is $4^2=16$.

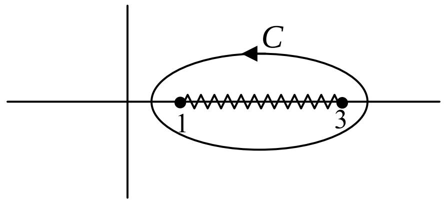
Now, the number of independent parameters needed to specify matrix A is same as that of matrix B.

Thus the number of independent parameters needed to specify A is 16

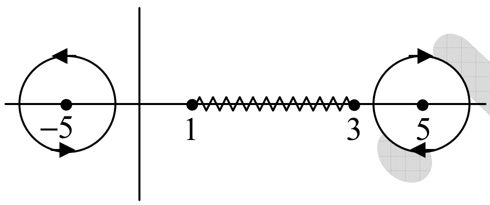
Q119. The contour C of the following integral

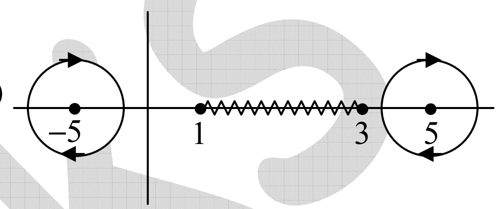
$$\oint_C dz \frac{\sqrt{(z-1)(z-3)}}{(z^2-25)^3}$$

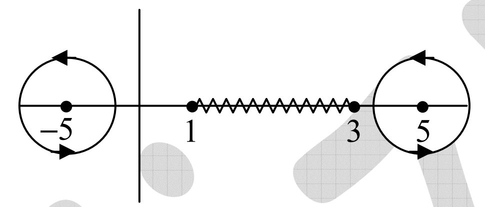
in the complex z plane is shown in the figure below.

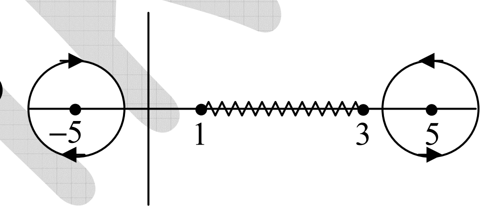


This integral is equivalent to an integral along the contours

(a)


(b)


(c)


(d)


Ans. : (c)

Solution: $z=1,3$ are branch points ∞ is not a branch point 1 branch cut 3

Q120. The value of the integral $\int_0^1 x^2 dx$, evaluated using the trapezoidal rule with a step size of 0.2, is

(a) 0.30 (b) 0.39 (c) 0.34 (d) 0.27

Ans. : (c)

Solution: $I=\frac{0.2}{2}\left[0+2(0.04+0.16+0.36+0.64)+1\right]$

$=0.1(2.4+1)=0.34$

	x	$f(x)$
x_0	0	0
x_1	0.2	0.04
x_2	0.4	0.16
x_3	0.6	0.36
x_4	0.8	0.64
x_5	1.0	1.00