CLASSICAL MECHANICS SOLUTIONS

NET/JRF (JUNE-2011)

Q1. A particle of unit mass moves in a potential $V(x) = ax^2 + \frac{b}{x^2}$, where a and b are positive constants. The angular frequency of small oscillations about the minimum of the potential is

(a) $\sqrt{8b}$ (b) $\sqrt{8a}$ (c) $\sqrt{8a/b}$ (d) $\sqrt{8b/a}$

Ans. : (b)

Solution: $V(x) = ax^2 + \frac{b}{x^2} \Rightarrow \frac{\partial V}{\partial x} = 0 \Rightarrow 2ax - \frac{2b}{x^3} = 0 \Rightarrow ax^4 - b = 0 \Rightarrow x_0 = \left(\frac{b}{a}\right)^{\frac{1}{4}}$.

Since $\omega = \sqrt{\frac{k}{m}}$, $m = 1$ and $k = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}$ where x_0 is stable equilibrium point.

Hence $k = \frac{\partial^2 V}{\partial x^2} = 2a + \frac{6b}{x_0^4} = 2a + \frac{6b}{b/a} = 8a$ at $x = x_0 = \left(\frac{b}{a}\right)^{\frac{1}{4}}$.

Thus, $\omega = \sqrt{8a}$.

Q2. The acceleration due to gravity (g) on the surface of Earth is approximately 2.6 times that on the surface of Mars. Given that the radius of Mars is about one half the radius of Earth, the ratio of the escape velocity on Earth to that on Mars is approximately

(a) 1.1 (b) 1.3 (c) 2.3 (d) 5.2

Ans. : (c)

Solution: Escape velocity $= \sqrt{2gR}$

$$\frac{\text{Escape velocity of Earth}}{\text{Escape velocity of Mass}} = \sqrt{\frac{g_e R_e}{g_m R_m}} = 2.3 \quad \text{where } \frac{R_e}{R_m} = 2 \text{ and } \frac{g_e}{g_m} = 2.6.$$

Q3. The Hamiltonian of a system with n degrees of freedom is given by $H(q_1,.....q_n; p_1,.......p_n; t)$, with an explicit dependence on the time t. Which of the following is correct?

(a) Different phase trajectories cannot intersect each other.

(b) H always represents the total energy of the system and is a constant of the motion.

(c) The equations $\dot{q}_i = \partial H/\partial p_i, \dot{p}_i = -\partial H/\partial q_i$ are not valid since H has explicit time dependence.

(d) Any initial volume element in phase space remains unchanged in magnitude under time evolution.

Ans. : (a)

Q4. The Lagrangian of a particle of charge e and mass m in applied electric and magnetic fields is given by $L = \frac{1}{2}m\vec{v}^2 + e\vec{A}\cdot\vec{v} - e\phi$, where $\vec{A}$ and ϕ are the vector and scalar potentials corresponding to the magnetic and electric fields, respectively. Which of the following statements is correct?

(a) The canonically conjugate momentum of the particle is given by $\vec{p} = m\vec{v}$

(b) The Hamiltonian of the particle is given by $H = \frac{\vec{p}^2}{2m} + \frac{e}{m}\vec{A}\cdot\vec{p} + e\phi$

(c) L remains unchanged under a gauge transformation of the potentials

(d) Under a gauge transformation of the potentials, L changes by the total time derivative

Ans. : (d)

Solution: $\square^2 V + \frac{\partial L}{\partial t} = -\frac{\rho}{\varepsilon_0}$

Q5. Consider the decay process $\tau^- \rightarrow \pi^- + \nu_\tau$ in the rest frame of the τ^-. The masses of the τ^-, π^- *and* ν_τ are M_τ, M_π and zero respectively.

A. The energy of π^- is

(a) $\frac{\left(M_\tau^2 - M_\pi^2\right)c^2}{2M_\tau}$ (b) $\frac{\left(M_\tau^2 + M_\pi^2\right)c^2}{2M_\tau}$ (c) $\left(M_\tau - M_\pi\right)c^2$ (d) $\sqrt{M_\tau M_\pi}c^2$

Ans. : (b)

Solution: $\tau^- \rightarrow \pi^- + \nu_\tau$

From conservation of energy $\mathrm{M_\tau c^2 = E_\pi + E_\nu}$.

$E_\pi^2 = p^2c^2 + M_\pi^2c^4$ and $E_\nu^2 = p^2c^2$ since momentum of π^- and ν_τ is same.

$$M_\tau c^2 = E_\pi + E_\nu,\; M_\pi{}^2c^4 = E_\pi^2 - E_\nu^2 \Rightarrow E_\pi - E_\nu = \frac{M_\pi^2c^4}{M_\tau c^2}$$

$$E_\pi - E_\nu = \frac{M_\pi^2c^2}{M_\tau} \text{ and } E_\pi + E_\nu = M_\tau c^2 \Rightarrow E_\pi = \frac{\left(M_\tau^2 + M_\pi^2\right)c^2}{2M_\tau}.$$

B. The velocity of π^- is

(a) $\frac{\left(M_\tau^2 - M_\pi^2\right)c}{M_\tau^2 + M_\pi^2}$ (b) $\frac{\left(M_\tau^2 + M_\pi^2\right)c}{M_\tau^2 - M_\pi^2}$ (c) $\frac{M_\pi c}{M_\tau}$ (d) $\frac{M_\tau c}{M_\pi}$

Ans. : (a)

Solution: Velocity of π^-

$$E_\pi = \frac{\left(M_\tau^2 + M_\pi^2\right)c^2}{2M_\tau} = \frac{M_\pi c^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow \left(1-\frac{v^2}{c^2}\right) = \frac{4M_\pi^2 M_\tau^2}{\left(M_\tau^2 + M_\pi^2\right)^2}$$

$$\Rightarrow \frac{v^2}{c^2} = 1 - \frac{4M_\pi^2 M_\tau^2}{\left(M_\tau^2 + M_\pi^2\right)^2} \Rightarrow \frac{v^2}{c^2} = \frac{M_\tau^4 + M_\pi^4 + 2M_\tau^2 M_\pi^2 - 4M_\pi^2 M_\tau^2}{\left(M_\tau^2 + M_\pi^2\right)^2} \Rightarrow v = \left(\frac{M_\tau^2 - M_\pi^2}{M_\tau^2 + M_\pi^2}\right)c\,.$$

Q6. The Hamiltonian of a particle of unit mass moving in the xy-plane is given to be: $H = xp_x - yp_y - \frac{1}{2}x^2 + \frac{1}{2}y^2$ in suitable units. The initial values are given to be $(x(0), y(0)) = (1,1)$ and $(p_x(0), p_y(0)) = \left(\frac{1}{2}, -\frac{1}{2}\right)$. During the motion, the curves traced out by the particles in the xy-plane and the $p_x p_y$ – plane are

(a) both straight lines

(b) a straight line and a hyperbola respectively

(c) a hyperbola and an ellipse, respectively

(d) both hyperbolas

Ans. : (d)

Solution: $H = xp_x - yp_y - \frac{1}{2}x^2 + \frac{1}{2}y^2$

Solving Hamiltonion equation of motion

$$\frac{\partial H}{\partial x} = -\dot{p}_x \Rightarrow p_x - x = -\dot{p}_x \text{ and } \frac{\partial H}{\partial y} = -\dot{p}_y \Rightarrow -p_y + y = -\dot{p}_y\,.$$

$$\frac{\partial H}{\partial p_x} = \dot{x} \Rightarrow x = \dot{x} \text{ and } \frac{\partial H}{\partial p_y} = \dot{y} \Rightarrow -y = \dot{y}\,.$$

After solving these four differential equation and eliminating time t and using boundary condition one will get $\Rightarrow x = \frac{1}{y}$ and $p_x = \frac{1}{2}\frac{1}{p_y}$

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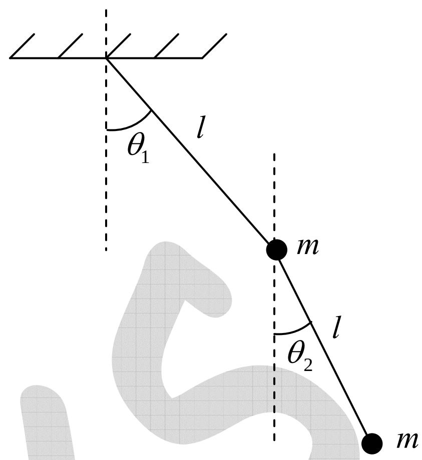
Q7. A double pendulum consists of two point masses m attached by strings of length l as shown in the figure: The kinetic energy of the pendulum is



(a) $\frac{1}{2}ml^2\left[\dot{\theta}_1^2+\dot{\theta}_2^2\right]$

(b) $\frac{1}{2}ml^2\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right]$

(c) $\frac{1}{2}ml^2\left[\dot{\theta}_1^2+2\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right]$

(d) $\frac{1}{2}ml^2\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1+\theta_2)\right]$

Ans. : (b)

Solution: Let co-ordinate (x_1, y_1) and (x_2, y_2). $K.E.=\frac{1}{2}m\left(\dot{x}_1^2+\dot{y}_1^2\right)+\frac{1}{2}m\left(\dot{x}_2^2+\dot{y}_2^2\right)$

$$x_1=l\sin\theta_1,\ y_1=l\cos\theta_1 \quad \Rightarrow \dot{x}_1=l\cos\theta_1\dot{\theta}_1,\ \dot{y}_1=-l\sin\theta_1\dot{\theta}_1$$

$$x_2=l\sin\theta_1+l\sin\theta_2,\ y_2=l\cos\theta_1+l\cos\theta_2$$

$$\Rightarrow \dot{x}_2=l\cos\theta_1\dot{\theta}_1+l\cos\theta_2\dot{\theta}_2,\ \dot{y}_2=l\left(-\sin\theta_1\dot{\theta}_1\right)+l\left(-\sin\theta_2\right)\dot{\theta}_2$$

Put the value of $\dot{x}_1, \dot{y}_1, \dot{x}_2, \dot{y}_2$ in K.E equation, one will get

$$T=\frac{1}{2}ml^2\left[2\dot{\theta}_1^2+\dot{\theta}_2^2+2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)\right].$$

Q8. A constant force F is applied to a relativistic particle of rest mass m. If the particle starts from rest at $t=0$, its speed after a time t is

(a) Ft/m (b) $c\tanh\left(\frac{Ft}{mc}\right)$ (c) $c\left(1-e^{-Ft/mc}\right)$ (d) $\frac{Fct}{\sqrt{F^2t^2+m^2c^2}}$

Ans. : (d)

Solution: $\frac{dp}{dt}=F \Rightarrow p=Ft+c$. At $t=0, p=0$ so, $c=0$

$$\Rightarrow p=Ft \Rightarrow \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}}=Ft \Rightarrow u=\frac{\left(\frac{F}{m}\right)t}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}}=\frac{Fct}{\sqrt{F^2t^2+m^2c^2}}.$$

Q9. The potential of a diatomic molecule as a function of the distance r between the atoms is given by $V(r) = -\frac{a}{r^6} + \frac{b}{r^{12}}$. The value of the potential at equilibrium separation between the atoms is:

(a) $-4a^2/b$ (b) $-2a^2/b$ (c) $-a^2/2b$ (d) $-a^2/4b$

Ans. : (d)

Solution: $V(r) = -\frac{a}{r^6} + \frac{b}{r^{12}}$, for equilibrium $\frac{\partial V}{\partial r} = 0 \Rightarrow -(-6)\frac{\mathrm{a}}{\mathrm{r}^7} - \frac{12\mathrm{b}}{\mathrm{r}^{13}} = 0 \Rightarrow \frac{1}{r^7}\left[6a - \frac{12b}{r^6}\right] = 0$

$$\Rightarrow 6a - \frac{12b}{r^6} = 0 \Rightarrow r = \left(\frac{12b}{6a}\right)^{\frac{1}{6}} \Rightarrow r = \left(\frac{2b}{a}\right)^{\frac{1}{6}}$$

$$\Rightarrow V\left(r = \left(\frac{2b}{a}\right)^{\frac{1}{6}}\right) = -\frac{a}{\left(\frac{2b}{a}\right)} + \frac{b}{\left(\frac{2b}{a}\right)^2} = -\frac{a^2}{2b} + \frac{a^2}{4b} = -\frac{a^2}{4b}.$$

Q10. Two particles of identical mass move in circular orbits under a central potential $V(r) = \frac{1}{2}kr^2$. Let l_1 and l_2 be the angular momenta and r_1, r_2 be the radii of the orbits respectively. If $\frac{l_1}{l_2} = 2$, the value of $\frac{r_1}{r_2}$ is:

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) 2 (d) $1/2$

Ans. : (a)

Solution: $V_{eff} = \frac{l^2}{2mr^2} + \frac{1}{2}kr^2$, where l is angular momentum.

Condition for circular orbit $\frac{\partial V_{eff}}{\partial r} = 0 \Rightarrow -\frac{l^2}{mr^3} + kr = 0 \Rightarrow l^2 \propto r^4 \Rightarrow l \propto r^2$.

Thus $\frac{l_1}{l_2} = \left(\frac{r_1}{r_2}\right)^2 \Rightarrow \frac{r_1}{r_2} = \sqrt{\frac{l_1}{l_2}} \Rightarrow \frac{r_1}{r_2} = \sqrt{2}$ since $\frac{l_1}{l_2} = 2$.

Q11. A particle of mass m moves inside a bowl. If the surface of the bowl is given by the equation $z = \frac{1}{2}a(x^2 + y^2)$, where a is a constant, the Lagrangian of the particle is

(a) $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 - gar^2)$ (b) $\frac{1}{2}m[(1 + a^2r^2)\dot{r}^2 + r^2\dot{\phi}^2]$

(c) $\frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 - gar^2)$ (d) $\frac{1}{2}m[(1 + a^2r^2)\dot{r}^2 + r^2\dot{\phi}^2 - gar^2]$

Ans. : (d)

Solution: $L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$, where $z = \frac{1}{2}a\left(x^2 + y^2\right)$.

It has cylindrical symmetry. Thus $x = r\cos\phi$, $y = r\sin\phi$, $z = \frac{1}{2}a\left(r^2\right)$.

$\dot{x} = \dot{r}\cos\phi - r\sin\phi\dot{\phi}$, $\dot{y} = \dot{r}\sin\phi + r\cos\phi\dot{\phi}$ and $\dot{z} = a\left(r\dot{r}\right)$.

So, $L = \frac{1}{2}m\left[\left(1 + a^2r^2\right)\dot{r}^2 + r^2\dot{\phi}^2 - gar^2\right]$.

Q12. A planet of mass m moves in the inverse square central force field of the Sun of mass M. If the semi-major and semi-minor axes of the orbit are a and b, respectively, the total energy of the planet is:

(a) $-\frac{GMm}{a+b}$

(b) $-GMm\left(\frac{1}{a} + \frac{1}{b}\right)$

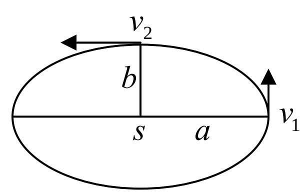
(c) $-\frac{GMm}{a}\left(\frac{1}{b} - \frac{1}{a}\right)$

(d) $-GMm\left(\frac{a-b}{(a+b)^2}\right)$

Ans. : (a)

Solution: Assume Sun is at the centre of elliptical orbit.

Conservation of energy $\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$



Conservation of momentum $L = mv_1a = mv_2b$

$v_2 = v_1\left(\frac{a}{b}\right)$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2 - v_1^2\frac{a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2 - a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}$$

$$E = \frac{1}{2}mv_1^2 - \frac{GMm}{a} = GMm\frac{b}{a}\frac{1}{(b+a)} - \frac{GMm}{a}$$

$$= \frac{GMm}{a}\left(\frac{b}{(b+a)} - 1\right) = \frac{GMm}{a}\left(\frac{b-b-a}{(b+a)}\right) = -\frac{GMm}{(b+a)}$$

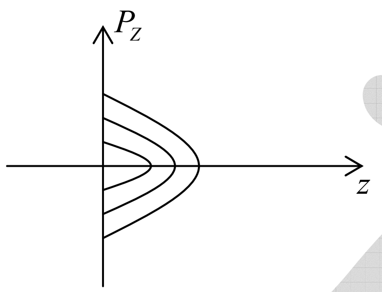
Q13. An annulus of mass M made of a material of uniform density has inner and outer radii a and b respectively. Its principle moment of inertia along the axis of symmetry perpendicular to the plane of the annulus is:

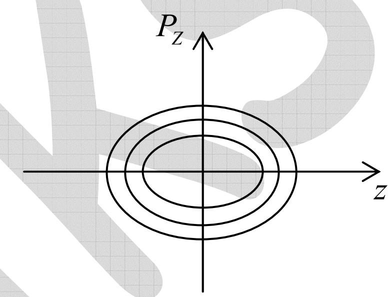
(a) $\frac{1}{2}M\frac{\left(b^4+a^4\right)}{\left(b^2-a^2\right)}$ (b) $\frac{1}{2}M\pi\left(b^2-a^2\right)$

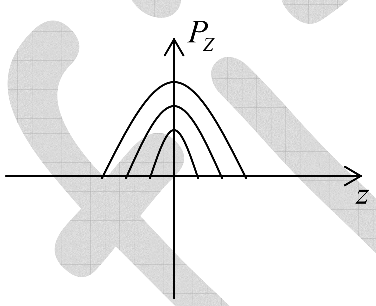
(c) $\frac{1}{2}M\left(b^2-a^2\right)$ (d) $\frac{1}{2}M\left(b^2+a^2\right)$

Ans. : (d)

Q14. The trajectory on the zp_z - plane (phase-space trajectory) of a ball bouncing perfectly elastically off a hard surface at $z = 0$ is given by approximately by (neglect friction):

(a)


(b)


(c)


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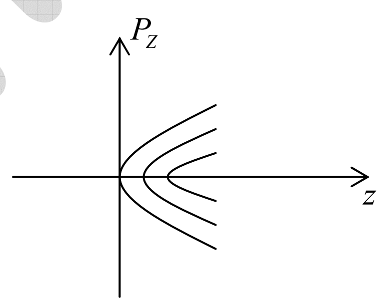

Ans. : (a)

Solution: $H = \frac{P_z^2}{2m} + mgz$ and $E = \frac{P_z^2}{2m} + mgz$.

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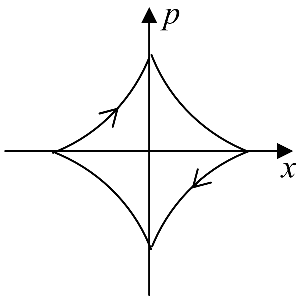
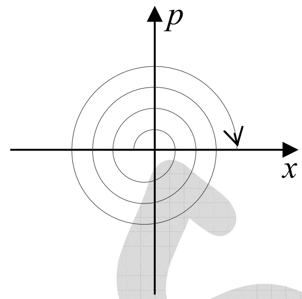
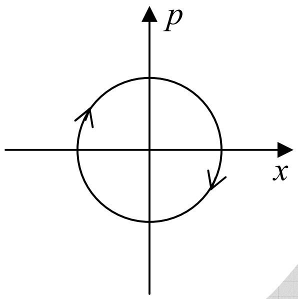
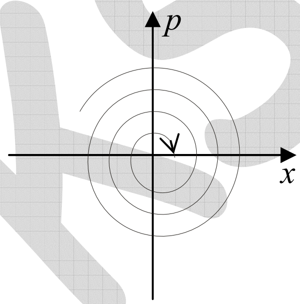
Q15. The bob of a simple pendulum, which undergoes small oscillations, is immersed in water. Which of the following figures best represents the phase space diagram for the pendulum?

(a)


(b)


(c)


(d)


Ans. : (d)

Solution: When simple pendulum oscillates in water it is damped oscillation so amplitude continuously decrease and finally it stops.

Q16. Two events separated by a (spatial) distance $9\times10^9 m$, are simultaneous in one inertial frame. The time interval between these two events in a frame moving with a constant speed $0.8\,c$ (where the speed of light $c=3\times10^8\,m/s$) is

(a) $60s$ (b) $40s$ (c) $20s$ (d) $0s$

Ans. : (b)

Solution: $x_2' - x_1' = 9\times10^9 m$ and $t_2' - t_1' = 0$. Then

$$t_2 - t_1 = \left(\frac{t_2' + \frac{v}{c^2}x_2'}{\sqrt{1-\frac{v^2}{c^2}}}\right) - \left(\frac{t_1^1 + \frac{v}{c^2}x_1'}{\sqrt{1-\frac{v^2}{c^2}}}\right) \Rightarrow t_2 - t_1 = \frac{t_2' - t_1'}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{v}{c^2}\frac{\left(x_2' - x_1'\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{v}{c^2}\frac{\left(x_2' - x_1'\right)}{\sqrt{1-\frac{v^2}{c^2}}}.$$

Put $v = 0.8c \quad \Rightarrow t_2 - t_1 \cong 40\sec$

Q17. If the Lagrangian of a particle moving in one dimensions is given by $L = \frac{\dot{x}^2}{2x} - V(x)$ the Hamiltonian is

(a) $\frac{1}{2}xp^2 + V(x)$ (b) $\frac{\dot{x}^2}{2x} + V(x)$ (c) $\frac{1}{2}\dot{x}^2 + V(x)$ (d) $\frac{p^2}{2x} + V(x)$

Ans. : (a)

Solution: Since $H = p_x\dot{x} - L$ and $\frac{\partial L}{\partial \dot{x}} = p_x \Rightarrow \frac{\dot{x}}{x} = p_x \Rightarrow \dot{x} = p_x x$.

$$\mathrm{H} = \mathrm{p_x}\dot{\mathrm{x}} - \frac{\dot{\mathrm{x}}^2}{2\mathrm{x}} + \mathrm{V}(\mathrm{x}) \Rightarrow \mathrm{H} = \mathrm{p_x} \times \mathrm{p_x x} - \frac{(\mathrm{p_x x})^2}{2\mathrm{x}} + \mathrm{V}(\mathrm{x}) \Rightarrow H = \frac{p_x^2 x}{2} + V(x).$$

Q18. A horizontal circular platform mutes with a constant angular velocity Ω directed vertically upwards. A person seated at the centre shoots a bullet of mass m horizontally with speed v. The acceleration of the bullet, in the reference frame of the shooter, is

(a) $2v\Omega$ to his right (b) $2v\Omega$ to his left

(c) $v\Omega$ to his right (d) $v\Omega$ to his left

Ans. : (a)

Solution: Velocity of bullet $= v\hat{j}$, Angular velocity= $\Omega\hat{k}$. There will be coriolis force $\vec{F} = 2m(\vec{v} \times \vec{\Omega})$.

$\vec{F} = 2m\Omega v\hat{i} \Rightarrow \vec{a} = 2v\Omega\hat{i}$.

Q19. The Poisson bracket $\{|\vec{r}|, |\vec{p}|\}$ has the value

(a) $|\vec{r}||\vec{p}|$ (b) $\hat{r} \cdot \hat{p}$ (c) 3 (d) 1

Ans. : (b)

Solution: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $|\vec{r}| = (x^2 + y^2 + z^2)^{1/2}$, $p = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$,

$|\vec{p}| = (p_x^2 + p_y^2 + p_z^2)^{1/2}$

$$\{|\vec{r}|, |\vec{p}|\} = \left(\frac{\partial|\vec{r}|}{\partial x} \cdot \frac{\partial|\vec{p}|}{\partial p_x} - \frac{\partial|\vec{r}|}{\partial p_x} \cdot \frac{\partial|\vec{p}|}{\partial x}\right) + \left(\frac{\partial|\vec{r}|}{\partial y} \cdot \frac{\partial|\vec{p}|}{\partial p_y} - \frac{\partial|\vec{r}|}{\partial p_y} \cdot \frac{\partial|\vec{p}|}{\partial y}\right) + \left(\frac{\partial|\vec{r}|}{\partial z} \cdot \frac{\partial|\vec{p}|}{\partial p_z} - \frac{\partial|\vec{r}|}{\partial p_z} \cdot \frac{\partial|\vec{p}|}{\partial y}\right)$$

$$= \frac{x}{|\vec{r}|}\frac{p_x}{|\vec{p}|} + \frac{y}{|\vec{r}|}\frac{p_y}{|\vec{p}|} + \frac{z}{|\vec{r}|}\frac{p_z}{|\vec{p}|} = \frac{\vec{r} \cdot \vec{p}}{|\vec{r}||\vec{p}|} = (\hat{r} \cdot \hat{p})$$

Q20. Consider the motion of a classical particle in a one dimensional double-well potential $V(x)=\frac{1}{4}(x^2-2)^2$. If the particle is displaced infinitesimally from the minimum on the x-axis (and friction is neglected), then

(a) the particle will execute simple harmonic motion in the right well with an angular frequency $\omega=\sqrt{2}$

(b) the particle will execute simple harmonic motion in the right well with an angular frequency $\omega=2$

(c) the particle will switch between the right and left wells

(d) the particle will approach the bottom of the right well and settle there

Ans. : (b)

Solution: $V(x)=\frac{1}{4}(x^2-2)^2 \Rightarrow \frac{\partial V}{\partial x}=\frac{2}{4}(x^2-2)\times 2x=0 \Rightarrow x=0, x=\pm\sqrt{2}$.

$\frac{\partial^2 V}{\partial x^2}=3x^2-2$. At $x=0$, $\frac{\partial^2 V}{\partial x^2}<0$ so V is maximum. Thus it is unstable point

$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=\pm\sqrt{2}}=4$ and it is stable equilibrium point with $\omega=\sqrt{\frac{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=x_0}}{\mu}}=2 \quad \because \mu=1$.

Q21. What is proper time interval between the occurrence of two events if in one inertial frame events are separated by $7.5\times10^8\, m$ and occur $6.5\, s$ apart?

(a) $6.50\, s$ (b) $6.00\, s$ (c) $5.75\, s$ (d) $5.00\, s$

Ans. : (b)

Solution: Proper time interval

$$\Delta t=\sqrt{(\Delta t')^2-\frac{r^2}{c^2}}=\sqrt{(6.5)^2-\left(\frac{7.5}{3}\right)^2}=6 \text{ sec.}$$

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Q22. A solid cylinder of height H, radius R and density ρ, floats vertically on the surface of a liquid of density ρ_0. The cylinder will be set into oscillatory motion when a small instantaneous downward force is applied. The frequency of oscillation is

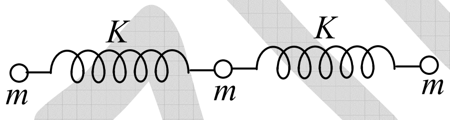
(a) $\frac{\rho g}{\rho_0 H}$ (b) $\frac{\rho}{\rho_0}\sqrt{\frac{g}{H}}$ (c) $\sqrt{\frac{\rho g}{\rho_0 H}}$ (d) $\sqrt{\frac{\rho_0 g}{\rho H}}$

Ans. : (d)

Solution: From Newton's law of motion $ma = mg - \rho_0 Agh$ where A is area of cross section, $m = \rho AH$.

$$\Rightarrow \rho AHa = \rho AHg - \rho_0 Agh \Rightarrow a = 1 - \frac{\rho_0 gh}{\rho H} \Rightarrow \omega = \sqrt{\frac{\rho_0 g}{\rho H}}$$

Q23. Three particles of equal mass (m) are connected by two identical massless springs of stiffness constant (K) as shown in the figure



If x_1, x_2 and x_3 denote the horizontal displacement of the masses from their respective equilibrium positions the potential energy of the system is

(a) $\frac{1}{2}K\left[x_1^2 + x_2^2 + x_3^2\right]$ (b) $\frac{1}{2}K\left[x_1^2 + x_2^2 + x_3^2 - x_2\left(x_1 + x_3\right)\right]$

(c) $\frac{1}{2}K\left[x_1^2 + 2x_2^2 + x_3^2 - 2x_2\left(x_1 + x_3\right)\right]$ (d) $\frac{1}{2}K\left[x_1^2 + 2x_2^2 - 2x_2\left(x_1 + x_3\right)\right]$

Ans. : (c)

Solution: $V = \frac{1}{2}K\left(x_2 - x_1\right)^2 + \frac{1}{2}K\left(x_3 - x_2\right)^2$,

$$V = \frac{1}{2}K\left(x_2^2 + x_1^2 - 2x_2x_1\right) + \frac{1}{2}K\left(x_3^2 + x_2^2 - 2x_3x_2\right) \Rightarrow V = \frac{1}{2}K\left[x_1^2 + 2x_2^2 + x_3^2 - 2x_2\left(x_1 + x_3\right)\right]$$

Q24. A planet of mass m moves in the gravitational field of the Sun (mass M). If the semi-major and semi-minor axes of the orbit are a and b respectively, the angular momentum of the planet is

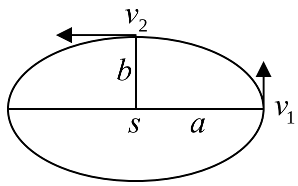
(a) $\sqrt{2GMm^2(a+b)}$ (b) $\sqrt{2GMm^2(a-b)}$ (c) $\sqrt{\frac{2GMm^2ab}{a-b}}$ (d) $\sqrt{\frac{2GMm^2ab}{a+b}}$

Ans. : (d)

Solution: Assume Sun is at the centre of elliptical orbit.

Conservation of energy $\frac{1}{2}mv_1^2 - \frac{GMm}{a} = \frac{1}{2}mv_2^2 - \frac{GMm}{b}$

Conservation of momentum $L = mv_1a = mv_2b$



$$v_2 = v_1\left(\frac{a}{b}\right)$$

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_2^2 = \frac{GMm}{a} - \frac{GMm}{b} \Rightarrow \frac{1}{2}m\left(v_1^2 - v_1^2\frac{a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right)$$

$$\frac{1}{2}mv_1^2\left(\frac{b^2-a^2}{b^2}\right) = GMm\left(\frac{b-a}{ab}\right) \Rightarrow \frac{1}{2}mv_1^2 = GMm\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}$$

$$v_1 = \sqrt{2GM\left(\frac{b}{a}\right)\cdot\frac{1}{(b+a)}}$$

$$L = mv_1a = m\sqrt{2GM\left(\frac{b}{a}\right)\cdot\left(\frac{1}{b+a}\right)}\cdot a = m\sqrt{\frac{2GMab}{(b+a)}} \Rightarrow L = \sqrt{\frac{2GMm^2ab}{a+b}}$$

Q25. The Hamiltonian of a simple pendulum consisting of a mass m attached to a massless string of length l is $H = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$. If L denotes the Lagrangian, the value of $\frac{dL}{dt}$ is:

(a) $-\frac{2g}{l}p_\theta\sin\theta$

(b) $-\frac{g}{l}p_\theta\sin 2\theta$

(c) $\frac{g}{l}p_\theta\cos\theta$

(d) $lp_\theta^2\cos\theta$

Ans. : (a)

Solution: $\frac{dL}{dt} = [L,H] + \frac{\partial L}{\partial t}$ where $H = \frac{p_\theta^2}{2ml^2} + mgl(1-\cos\theta)$.

$$L = \sum_i p_i\dot{q}_i - H = p_\theta\dot{\theta} - H\ ,\ \dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{p_\theta}{ml^2}\ ,\ \Rightarrow L = \frac{ml^2\dot{\theta}^2}{2} - mgl(1-\cos\theta).$$

Hence we have to calculate $[L,H]$ which is only defined into phase space i.e. p_θ and θ.

Then $\Rightarrow L = \dfrac{p_\theta^2}{2ml^2} - mgl(1-\cos\theta)$

$$[L,H] = \frac{\partial L}{\partial \theta}\times\frac{\partial H}{\partial p_\theta} - \frac{\partial L}{\partial p_\theta}\times\frac{\partial H}{\partial \theta} = -\frac{2g}{l}p_\theta \sin\theta \text{ and } \frac{\partial L}{\partial t} = 0 \Rightarrow \frac{dL}{dt} = -\frac{2g}{l}p_\theta\sin\theta$$

Q26. Two bodies of equal mass m are connected by a massless rigid rod of length l lying in the xy-plane with the centre of the rod at the origin. If this system is rotating about the z-axis with a frequency ω, its angular momentum is

(a) $ml^2\omega/4$ (b) $ml^2\omega/2$ (c) $ml^2\omega$ (d) $2ml^2\omega$

Ans. : (b)

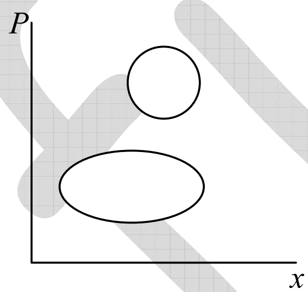
Solution: Since rod is massless i.e. $M = 0$.

Moment of inertia of the system $I = m_1r_1^2 + m_2r_2^2$, $m_1 = m_2 = m$ and $r_1 = r_2 = \dfrac{l}{2}$

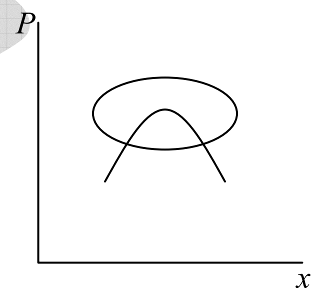
$I = \dfrac{ml^2}{4} + \dfrac{ml^2}{4} \Rightarrow I = \dfrac{ml^2}{2}$. Angular momentum, $J = I\omega$ and $J = \dfrac{ml^2\omega}{2}$.

Q27. Which of the following set of phase-space trajectories is not possible for a particle obeying Hamilton's equations of motion?

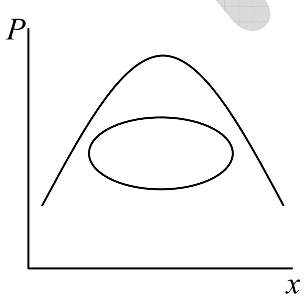
(a)



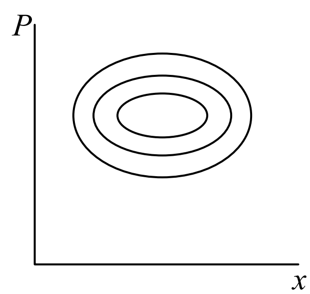
(b)



(c)



(d)



Ans. : (b)

Solution: Phase curve does not cut each other

Q28. The muon has mass $105\,MeV/c^2$ and mean life time $2.2\,\mu s$ in its rest frame. The mean distance traversed by a muon of energy $315\,MeV$ before decaying is approximately,

(a) $3\times10^5\,km$ (b) $2.2\,cm$ (c) $6.6\,\mu m$ (d) $1.98\,km$

Ans. : (d)

Solution: Since $E = 315MeV$ and $m_0 = 105\dfrac{MeV}{c^2}$.

$$E = mc^2 \Rightarrow E = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 315 = \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow 315 = \frac{105}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow v = 0.94c.$$

Now, $t = \dfrac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$, $t_0 = 2.2\mu s \Rightarrow t = \dfrac{2.2\times10^{-6}}{\sqrt{1-\frac{8}{9}}} \Rightarrow t = 6.6\,\mu s$

Now the distance traversed by muon is $vt = 0.94c\times 6.6\times10^{-6} = 1.86\,km$.

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Q29. The area of a disc in its rest frame S is equal to 1 (in some units). The disc will appear distorted to an observer O moving with a speed u with respect to S along the plane of the disc. The area of the disc measured in the rest frame of the observer O is (c is the speed of light in vacuum)

(a) $\left(1-\frac{u^2}{c^2}\right)^{1/2}$ (b) $\left(1-\frac{u^2}{c^2}\right)^{-1/2}$ (c) $\left(1-\frac{u^2}{c^2}\right)$ (d) $\left(1-\frac{u^2}{c^2}\right)^{-1}$

Ans. : (a)

Solution: Area of disc from S frame is 1 i.e. $\pi a^2 = 1$ or $\pi a\cdot a = 1$

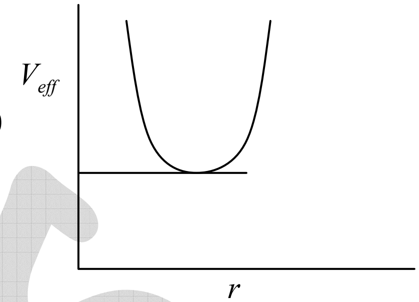
Area of disc from S' frame is $\pi a\cdot b = \pi a\cdot a\sqrt{1-\frac{u^2}{c^2}} = 1\cdot\sqrt{1-\frac{u^2}{c^2}} = \sqrt{1-\frac{u^2}{c^2}}$

where $b = a\sqrt{1-\frac{u^2}{c^2}}$.

Q30. A planet of mass m and an angular momentum L moves in a circular orbit in a potential, $V(r) = -k/r$, where k is a constant. If it is slightly perturbed radially, the angular frequency of radial oscillations is

(a) $mk^2/\sqrt{2}L^3$ (b) mk^2/L^3 (c) $\sqrt{2}mk^2/L^3$ (d) $\sqrt{3}mk^2/L^3$

Ans. : (b)



Solution: $V_{eff} = \frac{L^2}{2mr^2} - \frac{k}{r}$. For circular orbit $\frac{\partial V_{eff}}{\partial r} = -\frac{L^2}{mr^3} + \frac{k}{r^2} = 0$

$\Rightarrow \frac{L^2}{mr^3} = \frac{k}{r^2}$. Thus $r = r_0 = \frac{L^2}{mk} \Rightarrow \omega = \sqrt{\frac{k}{m}}$,

$$k = \left.\frac{d^2V_{eff}}{dr^2}\right|_{r=r_0} = +\left.\frac{3L^2}{mr^4} - \frac{2k}{r^3}\right|_{r=r_0} = \frac{3L^2}{m\left(\frac{L^2}{mk}\right)^4} - \frac{2k}{\left(\frac{L^2}{mk}\right)^3} = \frac{3m^3k^4}{L^6} - \frac{2m^3k^4}{L^6} = \frac{m^3k^4}{L^6}$$

$$\omega = \sqrt{\frac{\left.\frac{d^2V}{dr^2}\right|_{r=r_0}}{m}} \Rightarrow \omega = \frac{mk^2}{L^3}.$$

Q31. The number of degrees of freedom of a rigid body in d space-dimensions is

(a) $2d$ (b) 6 (c) $d(d+1)/2$ (d) $d!$

Ans. : (c)

Q32. A system is governed by the Hamiltonian

$$H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_x - bx)^2$$

where a and b are constants and p_x, p_y are momenta conjugate to x and y respectively. For what values of a and b will the quantities $(p_x - 3y)$ and $(p_y + 2x)$ be conserved?

(a) $a = -3, b = 2$ (b) $a = 3, b = -2$

(c) $a = 2, b = -3$ (d) $a = -2, b = 3$

Ans. : (d)

Solution: Poisson bracket $[p_x - 3y, H] = 0$ and $[p_y + 2y, H] = 0$

$p_y(b-3) + x(3b - b^2) = 0$ and $p_x(a+2) - y(2a + a^2) = 0$

$\Rightarrow a = -2, b = 3$

Q33. The Lagrangian of a particle of mass m moving in one dimension is given by

$$L = \frac{1}{2}m\dot{x}^2 - bx$$

where b is a positive constant. The coordinate of the particle $x(t)$ at time t is given by: (in following c_1 and c_2 are constants)

(a) $-\frac{b}{2m}t^2 + c_1 t + c_2$ (b) $c_1 t + c_2$

(c) $c_1 \cos\left(\frac{bt}{m}\right) + c_2 \sin\left(\frac{bt}{m}\right)$ (d) $c_1 \cosh\left(\frac{bt}{m}\right) + c_2 \sinh\left(\frac{bt}{m}\right)$

Ans. : (a)

Solution: Equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}(m\dot{x}) + b = 0 \Rightarrow m\ddot{x} + b = 0 \Rightarrow m\ddot{x} = -b$

$$\frac{d^2x}{dt^2} = -\frac{b}{m} \Rightarrow \frac{dx}{dt} = -\frac{b}{m}t + c_1 \Rightarrow x = -\frac{b}{m}\frac{t^2}{2} + c_1 t + c_2$$

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Q34. Let A, B and C be functions of phase space variables (coordinates and momenta of a mechanical system). If $\{,\}$ represents the Poisson bracket, the value of $\{A,\{B,C\}\} - \{\{A,B\},C\}$ is given by

(a) 0 (b) $\{B,\{C,A\}\}$ (c) $\{A,\{C,B\}\}$ (d) $\{\{C,A\},B\}$

Ans. : (d)

Solution: We know that Jacobi identity equation

$$\{A,\{B,C\}\} + \{B,\{C,A\}\} + \{C,\{A,B\}\} = 0$$

Now $\{A,\{B,C\}\} - \{\{A,B\},C\} = -\{B,\{C,A\}\} = \{\{C,A\},B\}$

Q35. A particle moves in a potential $V = x^2 + y^2 + \frac{z^2}{2}$. Which component(s) of the angular momentum is/are constant(s) of motion?

(a) None (b) L_x, L_y and L_z (c) only L_x and L_y (d) only L_z

Ans. : (d)

Solution: A particle moves in a potential $V = x^2 + y^2 + \frac{z^2}{2}$

$$V(r,\theta,\phi) = r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + \frac{r^2}{2}\cos^2\theta$$

$$V(r,\theta,\phi) = r^2 \sin^2\theta + \frac{r^2}{2}\cos^2\theta$$

Now ϕ is cyclic-co-ordinate (p_ϕ) i.e L_z is constant of motion.

Q36. The Hamiltonian of a relativistic particle of rest mass m and momentum p is given by $H = \sqrt{p^2 + m^2} + V(x)$, in units in which the speed of light $c = 1$. The corresponding Lagrangian is

(a) $L = m\sqrt{1+\dot{x}^2} - V(x)$

(b) $L = -m\sqrt{1-\dot{x}^2} - V(x)$

(c) $L = \sqrt{1+m\dot{x}^2} - V(x)$

(d) $L = \frac{1}{2}m\dot{x}^2 - V(x)$

Ans. : (b)

Solution: $H = \sqrt{p^2 + m^2} + V(x) \Rightarrow \frac{\partial H}{\partial p} = \dot{x} = \frac{1}{2}\frac{2p}{\left(p^2+m^2\right)^{\frac{1}{2}}} \Rightarrow \dot{x}\left(p^2+m^2\right)^{1/2} = p$

$$\Rightarrow p = \frac{\dot{x}m}{\sqrt{1-\dot{x}^2}}$$

Now $L = \sum \dot{x}p - H = \dot{x}p - H = \dot{x}p - \sqrt{p^2+m^2} - V(x)$

Put value $p = \frac{\dot{x}m}{\sqrt{1-\dot{x}^2}} \quad \Rightarrow L = -m\sqrt{1-\dot{x}^2} - V(x)$

Q37. A pendulum consists of a ring of mass M and radius R suspended by a massless rigid rod of length l attached to its rim. When the pendulum oscillates in the plane of the ring, the time period of oscillation is

(a) $2\pi\sqrt{\frac{l+R}{g}}$

(b) $\frac{2\pi}{\sqrt{g}}\left(l^2+R^2\right)^{1/4}$

(c) $2\pi\sqrt{\frac{2R^2+2Rl+l^2}{g(R+l)}}$

(d) $\frac{2\pi}{\sqrt{g}}\left(2R^2+2Rl+l^2\right)^{1/4}$

Ans. : (c)

Solution: The moment of inertia about pivotal point is given by

$I = I_{c.m} + Md^2 = MR^2 + M(l+R)^2$

If ring is displaced by angle θ then potential energy is $-Mg(l+R)\cos\theta$

The Lagrangian is given by

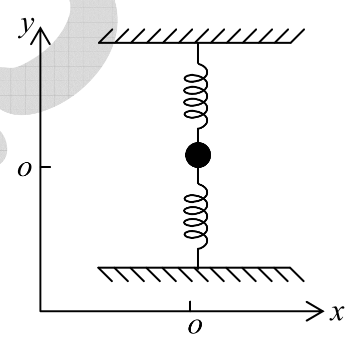
$$L = \frac{1}{2}I\dot{\theta}^2 - V(\theta) = \frac{1}{2}(MR^2 + M(l+R)^2)\dot{\theta}^2 + Mg(l+R)\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = 0 \Rightarrow (MR^2 + M(l+R)^2)\ddot{\theta} + Mg(l+R)\sin\theta = 0$$

For small oscillation $\sin\theta = \theta \Rightarrow (MR^2 + M(l+R)^2)\ddot{\theta} + Mg(l+R)\theta = 0$

Time period is given by $2\pi\sqrt{\frac{2R^2 + 2Rl + l^2}{g(R+l)}}$.

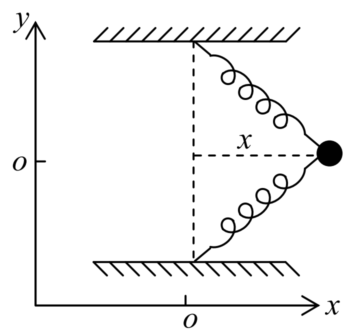
Q38. Consider a particle of mass m attached to two identical springs each of length l and spring constant k (see the figure). The equilibrium configuration is the one where the springs are unstretched. There are no other external forces on the system. If the particle is given a small displacement along the x-axis, which of the following describes the equation of motion for small oscillations?



(a) $m\ddot{x} + \frac{kx^3}{l^2} = 0$ (b) $m\ddot{x} + kx = 0$ (c) $m\ddot{x} + 2kx = 0$ (d) $m\ddot{x} + \frac{kx^2}{l} = 0$

Ans. : (a)

Solution: The lagrangian of system is given by



$$L = \frac{1}{2}m\dot{x}^2 - V(x)$$

The potential energy is given by

$$V(x) = \frac{k}{2}\left[\left(x^2 + l^2\right)^{\frac{1}{2}} - l\right]^2 + \frac{k}{2}\left[\left(x^2 + l^2\right)^{\frac{1}{2}} - l\right]^2$$

$$V(x) = k\left[\left(x^2 + l^2\right)^{\frac{1}{2}} - l\right]^2$$

For small oscillation one can approximate potential by Taylor expansion

$$V(x) = kl^2\left[\left(\left(1+\frac{x^2}{l^2}\right)^{\frac{1}{2}} - 1\right)\right]^2 \Rightarrow V(x) = kl^2\left[\left(1+\frac{1}{2}\frac{x^2}{l^2} - \frac{1}{8}\frac{x^4}{l^4}\right) - 1\right]^2$$

$$V(x) = \frac{kl^2}{4}\left(\frac{x^2}{l^2}\right)^2 \Rightarrow V(x) = k\left(\frac{x^4}{4l^2}\right).$$

So Lagrangian of system is given by $L = \frac{1}{2}m\dot{x}^2 - k\left(\frac{x^4}{4l^2}\right)$

The Lagranges equation of motion $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = 0 \Rightarrow m\ddot{x} + \frac{kx^3}{l^2} = 0$.

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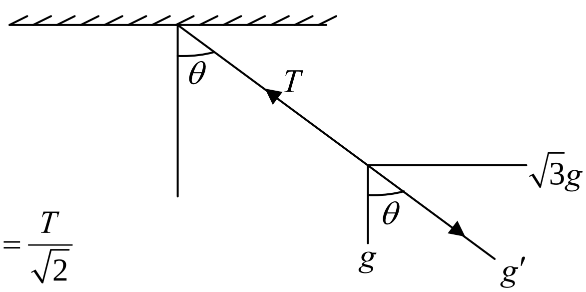
Q39. The time period of a simple pendulum under the influence of the acceleration due to gravity g is T. The bob is subjected to an additional acceleration of magnitude $\sqrt{3}\,g$ in the horizontal direction. Assuming small oscillations, the mean position and time period of oscillation, respectively, of the bob will be

(a) 0° to the vertical and $\sqrt{3}T$ (b) 30° to the vertical and $T/2$

(c) 60° to the vertical and $T/\sqrt{2}$ (d) 0° to the vertical and $T/\sqrt{3}$

Ans. : (c)

Solution: $T = 2\pi\sqrt{\frac{l}{g}}$



$g' = \sqrt{3g^2 + g^2} = \sqrt{4g^2} = 2g$

$T' = 2\pi\sqrt{\frac{l}{2g}} \Rightarrow T' = 2\pi\sqrt{\frac{l}{g}}\cdot\frac{1}{\sqrt{2}} \Rightarrow T' = \frac{T}{\sqrt{2}}$

$T\cos\theta = mg$, $T\sin\theta = \sqrt{3}mg \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = 60^{\circ}$

Q40. A particle of mass m and coordinate q has the Lagrangian $L=\frac{1}{2}m\dot{q}^2-\frac{\lambda}{2}q\dot{q}^2$, where λ is a constant. The Hamiltonian for the system is given by

(a) $\frac{p^2}{2m}+\frac{\lambda qp^2}{2m^2}$ (b) $\frac{p^2}{2(m-\lambda q)}$

(c) $\frac{p^2}{2m}+\frac{\lambda qp^2}{2(m-\lambda q)^2}$ (d) $\frac{p\dot{q}}{2}$

Ans. : (b)

Solution: $H=\sum \dot{q}p-L$ where $L=\frac{1}{2}m\dot{q}^2-\frac{\lambda}{2}q\dot{q}^2$

$$\frac{\partial L}{\partial \dot{q}}=p=m\dot{q}-\lambda q\dot{q}\Rightarrow p=\dot{q}(m-\lambda q)\Rightarrow \dot{q}=\frac{p}{m-\lambda q}$$

$$\Rightarrow H=\dot{q}p-L=\frac{p^2}{(m-\lambda q)}-\frac{1}{2}m\frac{(p^2)}{(m-\lambda q)^2}+\frac{\lambda}{2}q\cdot\frac{p^2}{(m-\lambda q)^2}$$

$$\Rightarrow H=\dot{q}p-L=\frac{p^2}{(m-\lambda q)}-\frac{p^2}{2(m-\lambda q)^2}(m-\lambda q)$$

$$\Rightarrow H=\dot{q}p-L=\frac{p^2}{(m-\lambda q)}-\frac{p^2}{2(m-\lambda q)}\quad\Rightarrow H=\frac{p^2}{2(m-\lambda q)}$$

Q41. The coordinates and momenta $x_i,\ p_i(i=1,2,3)$ of a particle satisfy the canonical Poisson bracket relations $\{x_i,\ p_j\}=\delta_{ij}$. If $C_1=x_2p_3+x_3p_2$ and $C_2=x_1p_2-x_2p_1$ are constants of motion, and if $C_3=\{C_1,\ C_2\}=x_1p_3+x_3p_1$, then

(a) $\{C_2,\ C_3\}=C_1$ and $\{C_3,\ C_1\}=C_2$ (b) $\{C_2,\ C_3\}=-C_1$ and $\{C_3,\ C_1\}=-C_2$

(c) $\{C_2,\ C_3\}=-C_1$ and $\{C_3,\ C_1\}=C_2$ (d) $\{C_2,\ C_3\}=C_1$ and $\{C_3,\ C_1\}=-C_2$

Ans. : (d)

Solution: $C_1=x_2p_3+x_3p_2,\ C_2=x_1p_2-x_2p_1,\ C_3=x_1p_3+x_3p_1$

$$\{C_2,C_3\}=\left(\frac{\partial C_2}{\partial x_1}\frac{\partial C_3}{\partial p_1}-\frac{\partial C_2}{\partial p_1}\frac{\partial C_3}{\partial x_1}\right)+\left(\frac{\partial C_2}{\partial x_2}\frac{\partial C_3}{\partial p_2}-\frac{\partial C_2}{\partial p_2}\frac{\partial C_3}{\partial x_2}\right)+\left(\frac{\partial C_2}{\partial x_3}\frac{\partial C_3}{\partial p_3}-\frac{\partial C_2}{\partial p_3}\frac{\partial C_3}{\partial x_3}\right)$$

$$\{C_2,C_3\}=\left(p_2x_3-(-x_2)p_3\right)+\left(0-x_1\cdot 0\right)+\left(0\cdot x_1-0\cdot p_1\right)=\left(p_2x_3+x_2p_3\right)=C_1$$

$$\{C_3, C_1\} = \left(\frac{\partial C_3}{\partial x_1}\frac{\partial C_1}{\partial p_1} - \frac{\partial C_3}{\partial p_1}\frac{\partial C_1}{\partial x_1}\right) + \left(\frac{\partial C_3}{\partial x_2}\frac{\partial C_1}{\partial p_2} - \frac{\partial C_3}{\partial p_2}\frac{\partial C_1}{\partial x_2}\right) + \left(\frac{\partial C_3}{\partial x_3}\frac{\partial C_1}{\partial p_3} - \frac{\partial C_3}{\partial p_3}\frac{\partial C_1}{\partial x_3}\right)$$

$$\{C_3, C_1\} = (p_3 \cdot 0 - x_3 \cdot 0) + (0 \cdot x_3 - 0 \cdot p_3) + (p_1 x_2 - x_1 p_2) = -(x_1 p_2 - x_2 p_1) = -C_2$$

Q42. The recently-discovered Higgs boson at the LHC experiment has a decay mode into a photon and a Z boson. If the rest masses of the Higgs and Z boson are $125\ \text{GeV/c}^2$ and $90\ \text{GeV/c}^2$ respectively, and the decaying Higgs particle is at rest, the energy of the photon will approximately be

(a) $35\sqrt{3}\,GeV$ (b) $35\,GeV$ (c) $30\,GeV$ (d) $15\,GeV$

Ans. : (c)

Solution: $H_B \to P_H + Z_B$

From conservation of momentum $0 = \vec{P}_1 + \vec{P}_2 \Rightarrow \vec{P}_1 = -\vec{P}_2 \Rightarrow |P_1| = |P_2|$

Now $E_{H_B} = E_{P_H} + E_{Z_B} \Rightarrow E_{P_H} + E_{Z_B} = M_{H_B}c^2$

$E_{P_H}^2 = P_1^2c^2 + 0$ and $E_{Z_B}^2 = P_2^2c^2 + M_{Z_B}^2c^4$

$$\Rightarrow \left(E_{Z_B} - E_{P_H}\right)\left(E_{Z_B} + E_{P_H}\right) = M_{Z_B}^2c^4 \quad \because |P_1| = |P_2|$$

$$\Rightarrow E_{Z_B} - E_{P_H} = \frac{M_{Z_B}^2c^4}{M_{H_B}c^2} = \frac{M_{Z_B}^2c^2}{M_{H_B}} \qquad \because E_{Z_B} + E_{P_H} = M_{H_B}c^2$$

$$\Rightarrow 2E_{P_H} = M_{H_B}c^2 - \frac{M_{z_B}^2c^2}{M_{H_B}} \Rightarrow E_{P_H} = \frac{\left(M^2{}_{H_B} - M_{z_B}^2\right)c^2}{M_{H_B}}$$

$$\Rightarrow E_{P_H} = \left(\frac{125\times125 - 90\times90}{2\times125}\right)\times\frac{c^4}{c^4} = 30.1GeV$$

Q43. A canonical transformation relates the old coordinates (q, p) to the new ones (Q, P) by the relations $Q = q^2$ and $P = p/2q$. The corresponding time independent generating function is

(a) P/q^2 (b) q^2P (c) q^2/P (d) qP^2

Ans. : (b)

Solution: $Q = q^2; P = p/2q$

$$\frac{\partial F_2}{\partial q} = p \Rightarrow \frac{\partial F_2}{\partial q} = P \cdot 2q \Rightarrow F_2 = q^2 P + f(P)$$

$$\frac{\partial F_2}{\partial P} = Q = q^2 \Rightarrow F_2 = q^2 P + f(q)$$

comparing both side $f(q) = f(P) = 0 \Rightarrow F_2 = q^2 P$

NET/JRF (DEC-2014)

Q44. The equation of motion of a system described by the time-dependent Lagrangian

$$L = e^{\gamma t}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] \text{ is}$$

(a) $m\ddot{x} + \gamma m\dot{x} + \frac{dV}{dx} = 0$ (b) $m\ddot{x} + \gamma m\dot{x} - \frac{dV}{dx} = 0$

(c) $m\ddot{x} - \gamma m\dot{x} + \frac{dV}{dx} = 0$ (d) $m\ddot{x} + \frac{dV}{dx} = 0$

Ans. : (a)

Solution: $\because L = e^{\gamma t}\left[\frac{1}{2}m\dot{x}^2 - V(x)\right] \Rightarrow \frac{\partial L}{\partial \dot{x}} = e^{\gamma t} m\dot{x}$ and $\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x}e^{\gamma t}$

$$\because \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow \frac{d}{dt}\left(e^{\gamma t} m\dot{x}\right) + \frac{\partial V}{\partial x}e^{\gamma t} = m\ddot{x}e^{\gamma t} + m\dot{x}\gamma e^{\gamma t} + \frac{\partial V}{\partial x}e^{\gamma t} = 0$$

$$\left(m\ddot{x} + m\gamma\dot{x} + \frac{\partial V}{\partial x}\right)e^{\gamma t} = 0 \Rightarrow m\ddot{x} + \gamma m\dot{x} + \frac{\partial V}{\partial x} = 0$$

Q45. A particle of mass m is moving in the potential $V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$ where a, b are positive constants. The frequency of small oscillations about a point of stable equilibrium is

(a) $\sqrt{a/m}$ (b) $\sqrt{2a/m}$ (c) $\sqrt{3a/m}$ (d) $\sqrt{6a/m}$

Ans. : (b)

Solution: $\because V(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$

$$\frac{\partial V}{\partial x} = 0 \Rightarrow -ax + bx^3 = 0 \Rightarrow x\left[-a + bx^2\right] = 0 \Rightarrow x = \pm\left(\frac{a}{b}\right)^{\frac{1}{2}}, 0$$

$\because \frac{\partial^2 V}{\partial x^2} = -a + 3bx^2 \quad \Rightarrow At \quad x = 0, \frac{\partial^2 V}{\partial x^2} = -a$ (Negative so it is unstable point)

$\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=\pm\left(\frac{a}{b}\right)^{\frac{1}{2}}} = -a + 3b\frac{a}{b} = 2a$ (Positive so it is stable point)

$$\Rightarrow \omega = \sqrt{\frac{\frac{\partial^2 V}{\partial x^2}}{m}} = \sqrt{\frac{2a}{m}}$$

Q46. The radius of Earth is approximately 6400 km. The height h at which the acceleration due to Earth's gravity differs from g at the Earth's surface by approximately 1% is

(a) 64 km (b) 48 km (c) 32 km (d) 16 km

Ans. : (c)

Solution: $\frac{g}{g'} = 1 + \frac{2h}{R} \Rightarrow \frac{g}{g'} - 1 = \frac{2h}{R} \Rightarrow \frac{\Delta g}{g'} = \frac{2h}{R} \Rightarrow h = 32\ k.m.$

Q47. According to the special theory of relativity, the speed v of a free particle of mass m and total energy E is:

(a) $v = c\sqrt{1 - \frac{mc^2}{E}}$ (b) $v = \sqrt{\frac{2E}{m}}\left(1 + \frac{mc^2}{E}\right)$

(c) $v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2}$ (d) $v = c\left(1 + \frac{mc^2}{E}\right)$

Ans. : (c)

Solution: $E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow 1 - \frac{v^2}{c^2} = \left(\frac{mc^2}{E}\right)^2 \Rightarrow \frac{v^2}{c^2} = 1 - \frac{m^2c^4}{E^2} \Rightarrow v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2}$

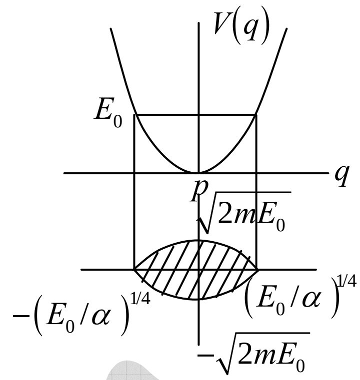
Q48. The Hamiltonian of a classical particle moving in one dimension is $H = \frac{p^2}{2m} + \alpha q^4$ where α is a positive constant and p and q are its momentum and position respectively. Given that its total energy $E \le E_0$ the available volume of phase space depends on E_0 as

(a) $E_0^{3/4}$ (b) E_0

(c) $\sqrt{E_0}$ (d) is independent of E_0

Ans. : (a)

Solution: $H = \frac{p^2}{2m} + \alpha q^4$



Phase area $= \oint p \cdot dq$

$$A = \oint p \cdot dq = \pi\sqrt{2mE} \times \left(\frac{E}{\alpha}\right)^{\frac{1}{4}}$$

$$A \propto E_0^{1/2} \cdot E_0^{1/4} \Rightarrow A \propto E_0^{3/4}$$

Q49. A mechanical system is described by the Hamiltonian $H(q, p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$. As a result of the canonical transformation generated by $F(q, Q) = -\frac{Q}{q}$, the Hamiltonian in the new coordinate Q and momentum P becomes

(a) $\frac{1}{2m}Q^2P^2 + \frac{m\omega^2}{2}Q^2$

(b) $\frac{1}{2m}Q^2P^2 + \frac{m\omega^2}{2}P^2$

(c) $\frac{1}{2m}P^2 + \frac{m\omega^2}{2}Q^2$

(d) $\frac{1}{2m}Q^2P^4 + \frac{m\omega^2}{2}P^{-2}$

Ans. : (d)

Solution: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$, $F = F_1(q, Q) = -\frac{Q}{q}$

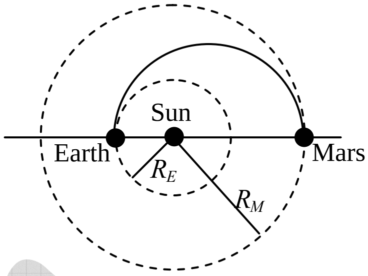
$$\Rightarrow \frac{\partial F_1}{\partial q} = p \Rightarrow \frac{Q}{q^2} = p \qquad \text{(i)}$$

$$\Rightarrow \frac{\partial F_1}{\partial Q} = -P \Rightarrow -\frac{1}{q} = -P \Rightarrow q = \frac{1}{P} \qquad \text{(ii)}$$

From equation (i) and (ii) $\Rightarrow p = QP^2 \qquad \because q = \frac{1}{P}$

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \frac{Q^2P^4}{2m} + \frac{1}{2}m\omega^2\left(\frac{1}{P^2}\right) = \frac{1}{2m}Q^2P^4 + \frac{1}{2}m\omega^2 P^{-2}$$

Q50. The probe *Mangalyaan* was sent recently to explore the planet Mars. The inter-planetary part of the trajectory is approximately a half-ellipse with the Earth (at the time of launch), Sun and Mars (at the time the probe reaches the destination) forming the major axis. Assuming that the orbits of Earth and Mars are approximately circular with radii R_E and R_M, respectively, the velocity (with respect to the Sun) of the probe during its voyage when it is at a distance $r(R_E << r << R_M)$ from the Sun, neglecting the effect of Earth and Mars, is



(a) $\sqrt{2GM\dfrac{(R_E+R_M)}{r(R_E+R_M-r)}}$ (b) $\sqrt{2GM\dfrac{(R_E+R_M-r)}{r(R_E+R_M)}}$

(c) $\sqrt{2GM\dfrac{R_E}{rR_M}}$ (d) $\sqrt{\dfrac{2GM}{r}}$

Ans. : (b)

Solution: Total energy $E=-K/2a$ where $2a$ major axis and $2a=R_E+R_M$.

$$\frac{1}{2}mv^2-\frac{GMm}{r}=-\frac{GMm}{(R_E+R_M)}\Rightarrow v=\sqrt{2GM\frac{(R_E+R_M-r)}{r(R_E+R_M)}}$$

NET/JRF (JUNE-2015)

Q51. A particle moves in two dimensions on the ellipse $x^2 + 4y^2 = 8$. At a particular instant it is at the point $(x, y) = (2,1)$ and the x-component of its velocity is 6 (in suitable units). Then the y-component of its velocity is

(a) -3 (b) -2 (c) 1 (d) 4

Ans. (a)

Solution: $\because x^2 + 4y^2 = 8 \Rightarrow 2x\frac{dx}{dt} + 8y\frac{dy}{dt} = 0$

$$\Rightarrow \quad 2xv_x + 8y\, v_y = 0 \Rightarrow 2\times2\times6 + 8\times1\times v_y = 0 \Rightarrow v_y = -3$$

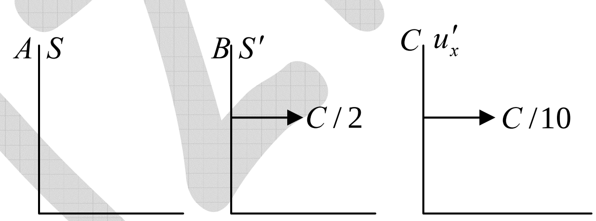
Q52. Consider three inertial frames of reference A, B and C. the frame B moves with a velocity $\frac{c}{2}$ with respect to A, and C moves with a velocity $\frac{c}{10}$ with respect to B in the same direction. The velocity of C as measured in A is

(a) $\frac{3c}{7}$ (b) $\frac{4c}{7}$ (c) $\frac{c}{7}$ (d) $\frac{\sqrt{3}c}{7}$

Ans. (b)

Solution: $v = \frac{c}{2}, \quad u_x' = \frac{c}{10}$

$$u_x = \frac{u_x' + v}{1 + \frac{u' v_x}{c^2}} = \frac{4c}{7}$$



Q53. If the Lagrangian of a dynamical system in two dimensions is $L = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y}$, then its Hamiltonian is

(a) $H = \frac{1}{m}p_x p_y + \frac{1}{2m}p_y^2$ (b) $H = \frac{1}{m}p_x p_y + \frac{1}{2m}p_x^2$

(c) $H = \frac{1}{m}p_x p_y - \frac{1}{2m}p_y^2$ (d) $H = \frac{1}{m}p_x p_y - \frac{1}{2m}p_x^2$

Ans. (c)

Solution: $L = \frac{1}{2}m\dot{x}^2 + m\dot{x}\dot{y} \Rightarrow \frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{y} = p_x$ (i)

$\Rightarrow \quad \dfrac{\partial L}{\partial \dot{y}} = m\dot{x} = p_y \qquad$ or $\quad \dot{x} = \dfrac{p_y}{m} \qquad$ (ii)

put $\dot{x} = \dfrac{p_y}{m}$ in equation (i)$\Rightarrow p_y + m\dot{y} = p_x \Rightarrow \dot{y} = \dfrac{p_x - p_y}{m}$

$$H = p_x\dot{x} + p_y\dot{y} - L = p_x\dot{x} + p_y\dot{y} - \frac{1}{2}m\dot{x}^2 - m\dot{x}\dot{y}$$

put value of $\dot{x}$ and $\dot{y} \Rightarrow H = \dfrac{p_x p_y}{m} - \dfrac{p_y^2}{2m}$

Q54. A particle of mass m moves in the one dimensional potential $V(x) = \dfrac{\alpha}{3}x^3 + \dfrac{\beta}{4}x^4$ where $\alpha, \beta > 0$. One of the equilibrium points is $x = 0$. The angular frequency of small oscillations about the other equilibrium point is

(a) $\dfrac{2\alpha}{\sqrt{3m\beta}}$ (b) $\dfrac{\alpha}{\sqrt{m\beta}}$ (c) $\dfrac{\alpha}{\sqrt{12m\beta}}$ (d) $\dfrac{\alpha}{\sqrt{24m\beta}}$

Ans. (b)

Solution: $V(x) = \dfrac{\alpha}{3}x^3 + \dfrac{\beta}{4}x^4 \Rightarrow \dfrac{\partial V}{\partial x} = \alpha x^2 + \beta x^3 = 0 \Rightarrow x_0 = -\dfrac{\alpha}{\beta}$

Spring constant $k = \left.\dfrac{\partial^2 V}{\partial x^2}\right|_{x=x_0} = \dfrac{\alpha^2}{\beta}(+ve) \Rightarrow \omega = \sqrt{\dfrac{k}{m}} = \dfrac{\alpha}{\sqrt{\beta m}}$

Q55. A particle of unit mass moves in the xy-plane in such a way that $\dot{x}(t) = y(t)$ and $\dot{y}(t) = -x(t)$. We can conclude that it is in a conservative force-field which can be derived from the potential

(a) $\dfrac{1}{2}(x^2 + y^2)$ (b) $\dfrac{1}{2}(x^2 - y^2)$ (c) $x + y$ (d) $x - y$

Ans. (a)

Solution: $\because \dot{x} = y$ and $\dot{y} = -x$

$\Rightarrow \quad \ddot{x} = \dot{y} = -x \quad$ and $\ddot{y} = -\dot{x} = -y$

$\Rightarrow \quad \ddot{x} + x = 0 \quad$ and $\ddot{y} + y = 0$

that is possible for $L = \dfrac{1}{2}m\dot{x}^2 + \dfrac{1}{2}m\dot{y}^2 - \dfrac{1}{2}(x^2 + y^2) \Rightarrow V = \dfrac{1}{2}(x^2 + y^2)$

Q56. A particle moves in one dimension in the potential $V=\frac{1}{2}k(t)x^2$, where $k(t)$ is a time dependent parameter. Then $\frac{d}{dt}\langle V\rangle$, the rate of change of the expectation value $\langle V\rangle$ of the potential energy is

(a) $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle+\frac{k}{2m}\langle xp+px\rangle$ (b) $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle+\frac{1}{2m}\langle p^2\rangle$

(c) $\frac{k}{2m}\langle xp+px\rangle$ (d) $\frac{1}{2}\frac{dk}{dt}\langle x^2\rangle$

Ans. (a)

Solution: $H=\frac{p^2}{2m}+\frac{1}{2}k(t)x^2$

$$\frac{d}{dt}\langle V\rangle=\langle[V,H]\rangle+\left\langle\frac{\partial V}{\partial t}\right\rangle\Rightarrow\left[\frac{1}{2}k(t)x^2,\frac{p^2}{2m}+\frac{1}{2}k(t)x^2\right]+\frac{x^2}{2}\frac{\partial k}{\partial t}=[V,H]$$

$$\frac{d}{dt}\langle V\rangle=\frac{1}{2}k(t)\cdot 2\left\langle\frac{xp+px}{2m}\right\rangle+\left\langle\frac{x^2}{2}\right\rangle\frac{\partial k}{\partial t}=\left\langle\frac{x^2}{2}\right\rangle\frac{\partial k}{\partial t}+\frac{1}{2m}k(t)\langle xp+px\rangle$$

Q57. Let q and p be the canonical coordinate and momentum of a dynamical system. Which of the following transformations is canonical?

1. $Q_1=\frac{1}{\sqrt{2}}q^2$ and $P_1=\frac{1}{\sqrt{2}}p^2$

2. $Q_2=\frac{1}{\sqrt{2}}(p+q)$ and $P_2=\frac{1}{\sqrt{2}}(p-q)$

(a) neither 1 nor 2 (b) both 1 and 2

(c) only 1 (d) only 2

Ans. (d)

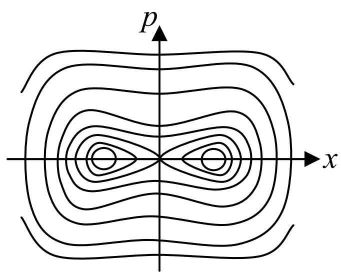
Solution: For $A: Q_1=\frac{q^2}{\sqrt{2}}, P_1=\frac{P^2}{\sqrt{2}}$

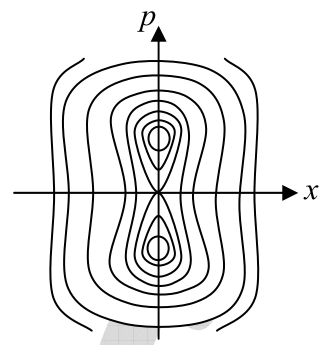
$$[Q_1,P_1]=\frac{\partial Q_1}{\partial q}.\frac{\partial P_1}{\partial p}-\frac{\partial Q_1}{\partial p}.\frac{\partial P_1}{\partial q}\neq 1 \qquad \text{(Not canonical)}$$

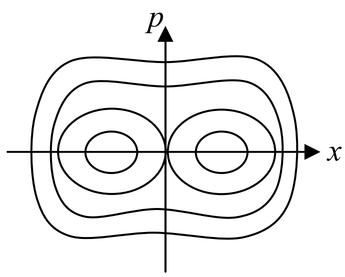
For $B: Q_2=\frac{1}{\sqrt{2}}(p+q), P_2=\frac{1}{\sqrt{2}}(p-q)$

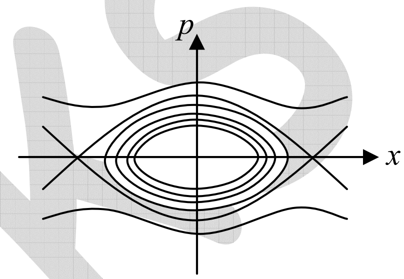
$$[Q_2,p_2]=1 \qquad \text{(canonical)}$$

Q58. Which of the following figures is a schematic representation of the phase space trajectories (i.e., contours of constant energy) of a particle moving in a one-dimensional potential $V(x) = \frac{-1}{2}x^2 + \frac{1}{4}x^4$

(a)


(b)


(c)


(d)


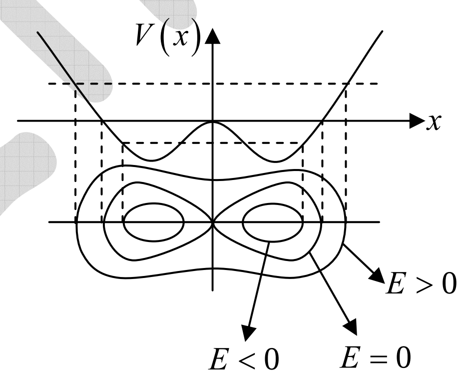
Ans. (a)

Solution: $V(x) = \frac{-x^2}{2} + \frac{x^4}{4}$

$$\frac{\partial V}{\partial x} = 0 \Rightarrow x = 0, x = \pm 1$$

$$\frac{\partial^2 V}{\partial x^2} = -ve \text{ for } x = 0 \quad \text{(unstable point)}$$

$= +$ ve for $x = \pm 1$ (stable point)



NET/JRF (DEC-2015)

Q59. Two masses m each, are placed at the points $(x,y)=(a,a)$ and $(-a,-a)$ and two masses, $2m$ each, are placed at the points $(a,-a)$ and $(-a,a)$. The principal moments of inertia of the system are

(a) $2m^2, 4ma^2$ (b) $4ma^2, 8ma^2$ (c) $4ma^2, 4ma^2$ (d) $8ma^2, 8ma^2$

Ans. : (b)

Solution: $I_{xx}=\sum_i m_i\left(y_i^2+z_i^2\right)=\sum m_i y_i^2 \qquad \because z_i=0$

$\Rightarrow I_{xx}=ma^2+ma^2+2ma^2+2ma^2 \Rightarrow I_{xx}=6ma^2$

Similarly, $I_{yy}=6ma^2$ and $I_{zz}=12ma^2$

$I_{xz}=I_{zx}=0, I_{yz}=I_{zy}=0$

$I_{xy}=I_{yx}=-m_i\sum_i x_i y_i=-ma^2-ma^2+2ma^2+2ma^2 \Rightarrow I_{xy}=I_{yx}=2ma^2$

Moment of inertia tensor

$$I=\begin{pmatrix}6ma^2 & 2ma^2 & 0\\ 2ma^2 & 6ma^2 & 0\\ 0 & 0 & 12ma^2\end{pmatrix}$$

Eigen value of matrices is principal moment of inertia, which is given by

$\lambda_1=4ma^2=I_x,\ \lambda_2=8ma^2=I_y,\ \lambda_3=12ma^2=I_z$

So, $I_x=4ma^2$ and $I_y=8ma^2$

Q60. The Lagrangian of a system is given by

$$L=\frac{1}{2}m\dot{q}_1^2+2m\dot{q}_2^2-k\left(\frac{5}{4}q_1^2+2q_2^2-2q_1q_2\right)$$

where m and k are positive constants. The frequencies of its normal modes are

(a) $\sqrt{\frac{k}{2m}},\sqrt{\frac{3k}{m}}$ (b) $\sqrt{\frac{k}{2m}}\left(13\pm\sqrt{73}\right)$ (c) $\sqrt{\frac{5k}{2m}},\sqrt{\frac{k}{m}}$ (d) $\sqrt{\frac{k}{2m}},\sqrt{\frac{6k}{m}}$

Ans. : (a)

Solution: $L=\frac{1}{2}m\dot{q}_1^2+2m\dot{q}_2^2-k\left[\frac{5}{4}q_1^2+2q_2^2-2q_1q_2\right]$

$$L=\frac{1}{2}m\dot{q}_1^2+\frac{4}{2}m\dot{q}_2^2-\frac{k}{2}\left[\frac{10}{4}q_1^2+4q_2^2-2q_1q_2-2q_2q_1\right]$$

$$T=\begin{pmatrix} m & 0 \\ 0 & 4m \end{pmatrix}, \qquad V=\begin{pmatrix} \frac{10}{4}k & -2k \\ -2k & 4k \end{pmatrix}$$

The secular equation $\left|V-\omega^2 m\right|=0$

$$\begin{pmatrix} \frac{10}{4}k-\omega^2 m & -2k \\ -2k & 4k-\omega^2 4m \end{pmatrix}=0,\ \left(\frac{10}{4}k-\omega^2 m\right)\left(4k-4\omega^2 m\right)-4k^2=0$$

$$\Rightarrow 10k^2-10\omega^2 km-4\omega^2 km+4\omega^4 m^2-4k^2=0$$

$$\Rightarrow 3k^2-7\omega^2 km+2\omega^4 m^2=0 \Rightarrow 3k^2-6\omega^2 km-\omega^2 km+2\omega^4 m^2=0$$

$$\Rightarrow \left(k-2\omega^2 m\right)\left(3k-\omega^2 m\right)=0 \Rightarrow \omega=\sqrt{\frac{k}{2m}},\quad \omega=\sqrt{\frac{3k}{m}}$$

Q61. Consider a particle of mass m moving with a speed v. If T_R denotes the relativistic kinetic energy and T_N its non-relativistic approximation, then the value of $\frac{\left(T_R-T_N\right)}{T_R}$ for $v=0.01\,c$, is

(a) 1.25×10^{-5} (b) 5.0×10^{-5} (c) 7.5×10^{-5} (d) 1.0×10^{-4}

Ans. : None of the options is correct.

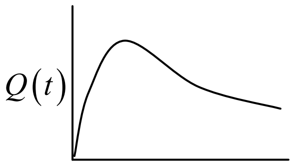
Solution: $T_N=\frac{1}{2}m_0v^2$, $T_R=mc^2-m_0c^2=\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}-m_0c^2$ $\quad(\because\ v=0.01c)$

Now, $$\frac{\left(T_R-T_N\right)}{T_R}=1-\frac{T_N}{T_R}=1-\frac{\frac{1}{2}m_0v^2}{\frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}}-m_0c^2}=1-\frac{\frac{v^2}{2}}{\frac{c^2}{\sqrt{1-\frac{v^2}{c^2}}}-c^2}=1-\frac{\frac{(0.01)^2}{2}}{\frac{1}{\sqrt{1-(0.01)^2}}-1}$$

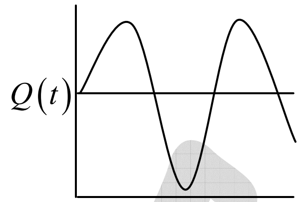
$$\frac{T_R-T_N}{T_R}=0.75$$

Q62. A canonical transformation $(p,q)\rightarrow(P,Q)$ is performed on the Hamiltonian $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$ via the generating function, $F=\frac{1}{2}m\omega q^2\cot Q$. If $Q(0)=0$, which of the following graphs shows schematically the dependence of $Q(t)$ on t?

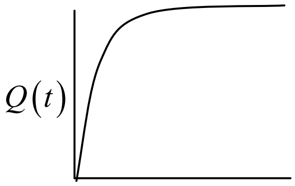
(a)



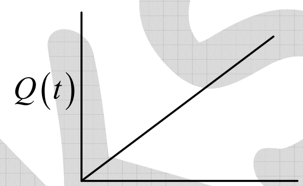
(b)



(c)



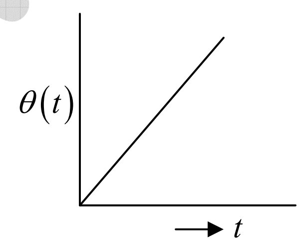
(d)



Ans. : (d)

Solution: $H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$, $F_1=\frac{1}{2}m\omega q^2\cot Q$

$\frac{\partial F_1}{\partial q}=p$, $\frac{\partial F_1}{\partial Q}=-P$, $K=H+\frac{\partial F_1}{\partial t}$



$$\frac{\partial F_1}{\partial q}=p=m\omega q\cot Q \quad \text{.....(i)}$$

$$\frac{\partial F_1}{\partial Q}=-P\Rightarrow-\frac{1}{2}m\omega q^2\operatorname{cosec}^2Q=-P$$

$$\Rightarrow\frac{1}{2}m\omega q^2=\frac{P}{\operatorname{cosec}^2Q} \quad \text{.....(ii)}$$

From (i) and (ii)

$$p=\sqrt{2m\omega P}\cos Q$$

$$\Rightarrow K=H+\frac{\partial F_1}{\partial t}; \quad \because\frac{\partial F_1}{\partial t}=0$$

$\Rightarrow K=H=\frac{p^2}{2m}+\frac{1}{2}m\omega^2q^2$ put the value of p and q

$\Rightarrow K=P\omega$ using equation of motion Q and P

$$\frac{\partial K}{\partial Q} = -\dot{P} = 0 \Rightarrow \dot{P} = 0 \Rightarrow P = \text{ constant}$$

$$\frac{\partial K}{\partial P} = \dot{Q} \Rightarrow \frac{\partial K}{\partial P} = \omega = \dot{Q} \Rightarrow \dot{Q} = \text{constant} \qquad (\because P = \text{constant})$$

$$\dot{Q} = \omega \Rightarrow Q = \omega t + \alpha \quad (\text{from boundary condition } \alpha = 0)$$

$$\Rightarrow Q = \omega t$$

Therefore, option (d) is correct.

Q63. The Lagrangian of a particle moving in a plane s given in Cartesian coordinates as

$$L = \dot{x}\dot{y} - x^2 - y^2$$

In polar coordinates the expression for the canonical momentum p_r (conjugate to the radial coordinate r) is

(a) $\dot{r}\sin\theta + r\dot{\theta}\cos\theta$

(b) $\dot{r}\cos\theta + r\dot{\theta}\sin\theta$

(c) $2\dot{r}\cos\theta - r\dot{\theta}\sin 2\theta$

(d) $\dot{r}\sin 2\theta + r\dot{\theta}\cos 2\theta$

Ans. : (d)

Solution: $L = \dot{x}\dot{y} - x^2 - y^2 = \dot{x}\dot{y} - (x^2 + y^2)$

$$x = r\cos\theta,\ y = r\sin\theta \Rightarrow \dot{x} = \dot{r}\cos\theta - r\sin\theta\dot{\theta},\ \dot{y} = \dot{r}\sin\theta + r\cos\theta\,\dot{\theta}$$

$$L = \dot{r}^2\sin\theta\cos\theta - r^2\sin\theta\cos\theta\,\dot{\theta}^2 + \dot{r}\,r\cos^2\theta\dot{\theta} - \dot{r}r\sin^2\theta\dot{\theta}$$

$$P_r = \frac{\partial L}{\partial \dot{r}} \Rightarrow 2\dot{r}\sin\theta\cos\theta + r\dot{\theta}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\Rightarrow P_r = \dot{r}\sin 2\theta + r\dot{\theta}\cos 2\theta$$

NET/JRF (JUNE-2016)

Q64. Let (x,t) and (x',t') be the coordinate systems used by the observers O and O', respectively. Observer O' moves with a velocity $v=\beta c$ along their common positive x-axis. If $x_+ = x+ct$ and $x_- = x-ct$ are the linear combinations of the coordinates, the Lorentz transformation relating O and O' takes the form

(a) $x'_+ = \frac{x_- - \beta x_+}{\sqrt{1-\beta^2}}$ and $x'_- = \frac{x_+ - \beta x_-}{\sqrt{1-\beta^2}}$

(b) $x'_+ = \sqrt{\frac{1+\beta}{1-\beta}}x_+$ and $x'_- = \sqrt{\frac{1-\beta}{1+\beta}}x_-$

(c) $x'_+ = \frac{x_+ - \beta x_-}{\sqrt{1-\beta^2}}$ and $x'_- = \frac{x_- - \beta x_+}{\sqrt{1-\beta^2}}$

(d) $x'_+ = \sqrt{\frac{1-\beta}{1+\beta}}x_+$ and $x'_- = \sqrt{\frac{1+\beta}{1-\beta}}x_-$

Ans. : (d)

Solution: $x'_+ = x' + ct'$

$$= \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{c\left(t-\frac{vx}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x\left(1-\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{ct\left(1-\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = x\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} + ct\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = \sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}(x+ct)$$

$$x'_+ = \sqrt{\frac{1-\beta}{1+\beta}}\, x_+$$

$$x'_- = x' - ct' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{c\left(t-\frac{vx}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{x\left(1+\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{ct\left(1+\frac{v}{c}\right)}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x'_- = x\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - ct\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \Rightarrow x'_- = \sqrt{\frac{1+\beta}{1-\beta}}(x-ct) \Rightarrow x'_- = \sqrt{\frac{1+\beta}{1-\beta}}x_-$$

Q65. A ball of mass m, initially at rest, is dropped from a height of 5 meters. If the coefficient of restitution is 0.9, the speed of the ball just before it hits the floor the second time is approximately (take $g = 9.8\ m/s^2$)

(a) $9.80\ m/s$ (b) $9.10\ m/s$ (c) $8.91\ m/s$ (d) $7.02\ m/s$

Ans. : (c)

Solution: velocity just before hitting first time is

$v_1 = \sqrt{2gh} = \sqrt{2\times 9.8\times 5} = 9.89\ m/s$

After hitting velocity will be $= ev_1 = 0.9\times 9.89$

$v_2 = 8.9\ m/s$

velocity hitting before second time will be same as v_2

Q66. The Hamiltonian of a system with generalized coordinate and momentum (q,p) is $H = p^2q^2$. A solution of the Hamiltonian equation of motion is (in the following A and B are constants)

(a) $p = Be^{-2At}$, $q = \frac{A}{B}e^{2At}$ (b) $p = Ae^{-2At}$, $q = \frac{A}{B}e^{-2At}$

(c) $p = Ae^{At}$, $q = \frac{A}{B}e^{-At}$ (d) $p = 2Ae^{-A^2t}$, $q = \frac{A}{B}e^{A^2t}$

Ans. : (a)

Solution: $H = p^2q^2$

From Hamilton's equation

$$\frac{\partial H}{\partial q} = -\dot{p} \Rightarrow \frac{dp}{dt} = -2p^2q \qquad \text{(i)}$$

$$\frac{\partial H}{\partial p} = \dot{q} \Rightarrow \frac{dq}{dt} = 2pq^2 \qquad \text{(ii)}$$

from equations (i) and (ii)

$$\frac{dp}{p} = -\frac{dq}{q}$$

Integrating both sides, $\ln p = -\ln q + \ln A$

$$pq = A \qquad \text{(iii)}$$

from equation (i)

$$\frac{dp}{dt} = -2p^2q = -2pA$$

$$\Rightarrow \quad \int \frac{dp}{p} = -\int 2A dt + \ln B \Rightarrow \ln\frac{p}{B} = -2At \Rightarrow p = Be^{-2At}$$

Putting this value of p in equation (iii) gives $q = \frac{A}{B}e^{2At}$

Hence, the correct option is (a)

Q67. A canonical transformation $(q,p)\to(Q,P)$ is made through the generating function $F(q,P)=q^2P$ on the Hamiltonian

$$H(q,p)=\frac{p^2}{2\alpha q^2}+\frac{\beta}{4}q^4$$

where α and β are constants. The equations of motion for (Q,P) are

(a) $\dot{Q}=\frac{P}{\alpha}$ and $\dot{P}=-\beta Q$

(b) $\dot{Q}=\frac{4P}{\alpha}$ and $\dot{P}=\frac{-\beta Q}{2}$

(c) $\dot{Q}=\frac{P}{\alpha}$ and $\dot{P}=-\frac{2P^2}{Q}-\beta Q$

(d) $\dot{Q}=\frac{2P}{\alpha}$ and $\dot{P}=-\beta Q$

Ans. : (b)

Solution: $F(q,P)=q^2P$

This is F_2 type generating function so

$$\frac{\partial F_2}{\partial q}=p \ \& \ \frac{\partial F_2}{\partial P}=Q$$

$$p=2qP \ \& \ Q=q^2 \Rightarrow q=(Q)^{\frac{1}{2}} \ \& \ p=2(Q)^{\frac{1}{2}}P$$

$$H(Q,P)=\frac{4QP^2}{2\alpha Q}+\frac{\beta}{4}Q^2=\frac{2P^2}{\alpha}+\frac{\beta Q^2}{4}$$

$$\Rightarrow \frac{\partial H}{\partial P}=\dot{Q}\Rightarrow \dot{Q}=\frac{4P}{\alpha} \text{ and } \frac{\partial H}{\partial Q}=-\dot{P}\Rightarrow \dot{P}=-\frac{\beta Q}{2}$$

Q68. The Lagrangian of a system moving in three dimensions is

$$L=\frac{1}{2}m\dot{x}_1^2+m\left(\dot{x}_2^2+\dot{x}_3^2\right)-\frac{1}{2}kx_1^2-\frac{1}{2}k\left(x_2+x_3\right)^2$$

The independent constants of motion is/are

(a) energy alone

(b) only energy, one component of the linear momentum and one component of the angular momentum

(c) only energy, one component of the linear momentum

(d) only energy, one component of the angular momentum

Ans. : (a)

Solution: The motion is in $3D$. So don't get confine with $x_1, x_2 x_3$ they are actually x, y, z

Langrangian is then

$$L = \frac{1}{2}m\dot{x}^2 + m\left(\dot{y}^2 + \dot{z}^2\right) - \frac{1}{2}kx^2 - \frac{1}{2}k\left(y+z\right)^2, \text{ when } \frac{\partial L}{\partial x} \neq 0, \frac{\partial L}{\partial y} \neq 0, \frac{\partial L}{\partial z} \neq 0$$

So, not any component at Linear momentum is conserve.

Now transform the Lagrangian to Hamiltonian

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{4m} + \frac{P_z^2}{4m} + \frac{1}{2}kx^2 + \frac{1}{2}k\left(y+z\right)^2$$

$\frac{\partial H}{\partial t} = 0$ so energy is conserved

Now let us assume $L_x = yP_z - zP_y$

$$\frac{dL_x}{dt} = \left[L_x, H\right] + \frac{\partial L_x}{\partial t}$$

$$\left[L_x, H\right] = \left[yP_z - zP_y, H\right] = \left[y, H\right]P_z + y\left[P_z, H\right] - \left[z, H\right]P_y - z\left[P_y, H\right]$$

$$\Rightarrow \left[L_x, H\right] = \left[y, \frac{P_y^2}{4m}\right]P_z + y\left[P_z, \frac{1}{2}k\left(y+z\right)^2\right] - \left[z, \frac{P_z^2}{4m}\right]P_y - z\left[P_y, \frac{1}{2}k\left(y+z\right)^2\right]$$

$$= 2P_y\frac{P_z}{4m} + y\left[0 - \frac{1}{2}k.2\left(y+z\right)\right] - \left[2P_y\frac{P_z}{4m}\right] - z\left[0 - \frac{1}{2}k.2\left(y+z\right)\right]$$

$$= -k\left(y^2 + yz\right) + k\left(z^2 + yz\right) = -k\left[y^2 - z^2\right] = k\left[z^2 - y^2\right]$$

$\Rightarrow \frac{dL_x}{dt} \neq 0$. Similarly $\frac{dL_y}{dt} \neq 0$ and $\Rightarrow \frac{dL_z}{dt} \neq 0$

Q69. For a particle of energy E and momentum p (in a frame F), the rapidity y is defined as $y = \frac{1}{2}\ln\left(\frac{E + p_3 c}{E - p_3 c}\right)$. In a frame F' moving with velocity $v = \left(0, 0, \beta c\right)$ with respect to F, the rapidity y' will be

(a) $y' = y + \frac{1}{2}\ln\left(1 - \beta^2\right)$

(b) $y' = y - \frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$

(c) $y' = y + \ln\left(\frac{1+\beta}{1-\beta}\right)$

(d) $y' = y + 2\ln\left(\frac{1+\beta}{1-\beta}\right)$

Ans. : (b)

Solution: $y=\frac{1}{2}\ln\left(\frac{E+p_3c}{E-p_3c}\right)$

Then $y'=\frac{1}{2}\ln\left(\frac{E'+p_3'c}{E'-p_3'c}\right)$

Where $p_3'=\gamma\left(p_3-v\left(\frac{E}{c^2}\right)\right)$ $\qquad E'=\gamma\left(E-vp_3\right)$

Put the value of p_3' and E' one will get $y'=\frac{1}{2}\ln\left(\frac{(E+p_3c)-\frac{v}{c}(E+p_3c)}{(E-p_3c)+\frac{v}{c}(E-p_3c)}\right)$

$$\frac{1}{2}\ln\left(\frac{(E+p_3c)(1-\beta)}{(E-p_3c)(1+\beta)}\right)\Rightarrow\frac{1}{2}\ln\left(\frac{(E+p_3c)}{(E-p_3c)}\right)+\frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right)$$

$$y+\frac{1}{2}\ln\left(\frac{1-\beta}{1+\beta}\right)\Rightarrow y-\frac{1}{2}\ln\left(\frac{1+\beta}{1-\beta}\right)$$

NET/JRF (DEC-2016)

Q70. A ball of mass m is dropped from a tall building with zero initial velocity. In addition to gravity, the ball experiences a damping force of the form $-\gamma\upsilon$, where υ is its instantaneous velocity and γ is a constant. Given the values $m=10\,kg, \gamma=10\,kg/s$ and $g\approx 10\,m/s^2$ the distance travelled (in metres) in time t in seconds, is

(a) $10\left(t+1-e^{-t}\right)$ (b) $10\left(t-1+e^{-t}\right)$

(c) $5t^2-\left(1-e^{t}\right)$ (d) $5t^2$

Ans. : (b)

Solution: $m\frac{d^2x}{dt^2}=mg-\gamma\frac{dx}{dt}$

Putting the values of m,γ and g and simplifying we obtain

$$\frac{d^2x}{dt^2}+\frac{dx}{dt}=10$$

The general solution of this equation is $x(t)=c_1+c_2t+10e^{-t}$

Using the initial conditions $x(0)=0, x'(0)=0$

We obtain, $c_1=-10$ and $c_2=10$

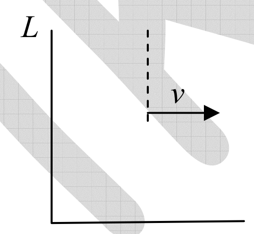
Hence the required solution is $x(t)=10(t-1+e^{-t})$

Q71. A relativistic particle moves with a constant velocity v with respect to the laboratory frame. In time τ, measured in the rest frame of the particle, the distance that it travels in the laboratory frame is

(a) $v\tau$ (b) $\frac{c\tau}{\sqrt{1-\frac{v^2}{c^2}}}$ (c) $v\tau\sqrt{1-\frac{v^2}{c^2}}$ (d) $\frac{v\tau}{\sqrt{1-\frac{v^2}{c^2}}}$

Ans. : (d)

Solution: From Particle $x_1'=0$ $x_2'=0$ $t_{initial}=t_1'$ $t_{final}=t_2'$



$$x_1=\frac{x_1'+vt_1'}{\sqrt{1-v^2/c^2}},\ x_2=\frac{x_2'+vt_2'}{\sqrt{1-v^2/c^2}}$$

$$x_2-x_1=\frac{x_2'-x_1'}{\sqrt{1-v^2/c^2}}+\frac{v(t_2'-t_1')}{\sqrt{1-v^2/c^2}}$$

$$\Delta x=\frac{v(t_2'-t_1')}{\sqrt{1-v^2/c^2}}=\frac{v\tau}{\sqrt{1-v^2/c^2}}$$

Q72. A particle in two dimensions is in a potential $V(x,y)=x+2y$. Which of the following (apart from the total energy of the particle) is also a constant of motion?

(a) p_y-2p_x (b) p_x-2p_y

(c) p_x+2p_y (d) p_y+2p_x

Ans. : (a)

Solution: $V(x,y)=x+2y$

$$H=\frac{p_x^2}{2m}+\frac{p_y^2}{2m}+x+2y$$

$$\frac{d(p_y-2p_x)}{dt}=[p_y-2p_x,H]+\frac{\partial}{\partial t}(p_y-2p_x)$$

$$=[p_y-2p_x,H]=[p_y-2p_x,x+2y]=[p_y,2y]-[2p_x,x]=-2+2=0$$

Q73. The dynamics of a particle governed by the Lagrangian

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - kx\dot{x}t \text{ describes}$$

(a) an undamped simple harmonic oscillator

(b) a damped harmonic oscillator with a time varying damping factor

(c) an undamped harmonic oscillator with a time dependent frequency

(d) a free particle

Ans. : (d)

Solution: $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 - kx\,\dot{x}\,t$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} - kxt, \frac{\partial L}{\partial x} = -kx - k\dot{x}t$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} - k\dot{x}t - kx + kx + k\dot{x}t = 0 \Rightarrow m\ddot{x} = 0$$

So motion is equivalent to free particle

Q74. The parabolic coordinates (ξ, η) are related to the Cartesian coordinates (x, y) by $x = \xi\eta$ and $y = \frac{1}{2}(\xi^2 - \eta^2)$. The Lagrangian of a two-dimensional simple harmonic oscillator of mass m and angular frequency ω is

(a) $\frac{1}{2}m\left[\dot{\xi}^2 + \dot{\eta}^2 - \omega^2(\xi^2 + \eta^2)\right]$

(b) $\frac{1}{2}m(\xi^2 + \eta^2)\left[(\dot{\xi}^2 + \dot{\eta}^2) - \frac{1}{4}\omega^2(\xi^2 + \eta^2)\right]$

(c) $\frac{1}{2}m(\xi^2 + \eta^2)\left[\dot{\xi}^2 + \dot{\eta}^2 - \frac{1}{2}\omega^2\xi\eta\right]$

(d) $\frac{1}{2}m(\xi^2 + \eta^2)\left[\dot{\xi}^2 + \dot{\eta}^2 - \frac{1}{4}\omega^2\right]$

Ans. : (b)

Solution: For two dimensional Harmonic oscillation

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}m\omega^2(x^2 + y^2)$$

$$x = \xi\eta, \qquad y = \frac{1}{2}(\xi^2 - \eta^2)$$

$$\dot{x} = \dot{\xi}\eta + \xi\dot{\eta}, \quad \dot{y} = \xi\dot{\xi} - \eta\dot{\eta}$$

$$L=\frac{1}{2}m\left[\left(\dot{\xi}\eta+\xi\dot{\eta}\right)^2+\left(\xi\dot{\xi}-\eta\dot{\eta}\right)^2\right]-\frac{1}{2}m\omega^2\left[\xi^2\eta^2+\frac{1}{4}\left(\xi^2-\eta^2\right)^2\right]$$

$$L=\frac{1}{2}m\left(\dot{\xi}^2\eta^2+\xi^2\dot{\eta}^2+\xi^2\dot{\xi}^2+\eta^2\dot{\eta}^2\right)-\frac{1}{8}m\omega^2\left(\xi^4+\eta^4+2\xi^2\eta^2\right)$$

$$=\frac{1}{2}m\left(\xi^2+\eta^2\right)\left(\dot{\eta}^2+\dot{\xi}^2\right)-\frac{1}{8}m\omega^2\left(\xi^2+\eta^2\right)^2$$

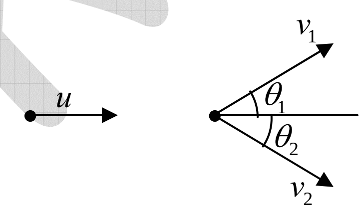
$$=\frac{1}{2}m\left(\xi^2+\eta^2\right)\left[\dot{\eta}^2+\dot{\xi}^2-\frac{1}{4}\omega^2\left(\xi^2+\eta^2\right)\right]$$

Q75. After a perfectly elastic collision of two identical balls, one of which was initially at rest, the velocities of both the balls are non zero. The angle θ between the final, velocities (in the lab frame) is

(a) $\theta=\frac{\pi}{2}$ (b) $\theta=\pi$ (c) $0<\theta\le\frac{\pi}{2}$ (d) $\frac{\pi}{2}<\theta\le\pi$

Ans. : (a)

Solution: Angle between two particle $\theta_1+\theta_2=0$



Conservation of momentum

$$mu=mv_1\cos\theta_1+mv_2\cos\theta_2 \quad \text{(i)}$$

$$0=mv_1\sin\theta_1-mv_2\sin\theta_2 \quad \text{(ii)}$$

conservation of kinetic energy

$$\frac{1}{2}mu^2=\frac{1}{2}mv_1^2+\frac{1}{2}mv_2^2 \quad \text{(iii)}$$

From (i) and (ii)

$$u^2=v_1^2+v_2^2+2v_1v_2\left(\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\right)$$

$$u^2=v_1^2+v_2^2+2v_1v_2\cos\left(\theta_1+\theta_2\right) \quad \text{(iv)}$$

$$u^2=v_1^2+v_2^2 \quad \text{(v)}$$

$$v_1^2+v_2^2=v_1^2+v_2^2+2v_1v_2\cos\left(\theta_1+\theta_2\right)$$

$$\Rightarrow \quad \cos\left(\theta_1+\theta_2\right)=0$$

$$\theta_1+\theta_2=\frac{\pi}{2}\Rightarrow\theta=\frac{\pi}{2}$$

Q76. Consider circular orbits in a central force potential $V(r) = -\frac{k}{r^n}$, where $k > 0$ and $0 < n < 2$. If the time period of a circular orbit of radius R is T_1 and that of radius $2R$ is T_2, then $\frac{T_2}{T_1}$

(a) $2^{\frac{n}{2}}$ (b) $2^{\frac{2}{3}n}$ (c) $2^{\frac{n}{2}+1}$ (d) 2^n

Ans. : (c)

Solution: $V_{eff} = \frac{J^2}{2mr^2} - \frac{k}{r^n}$, $\frac{\partial V_{eff}}{\partial r} = -\frac{J^2}{mr^3} + \frac{nk}{r^{n+1}} = 0$

$$\because J = mr^2\omega \quad \Rightarrow \frac{m^2\omega^2 r^4}{r^3} = \frac{nk}{r^{n+1}} \Rightarrow \omega^2 \propto \frac{1}{r^{n+2}} \Rightarrow \omega \propto r^{-(n+2)/2} \Rightarrow T \propto r^{\frac{n}{2}+1}$$

$$\frac{T_2}{T_1} = \left(\frac{2R}{R}\right)^{\frac{n+2}{2}} = 2^{\frac{n}{2}+1}$$

Q77. Consider a radioactive nucleus that is travelling at a speed $\frac{c}{2}$ with respect to the lab frame. It emits γ-rays of frequency ν_0 in its rest frame. There is a stationary detector, (which is not on the path of the nucleus) in the lab. If a γ-ray photon is emitted when the nucleus is closest to the detector, its observed frequency at the detector is

(a) $\frac{\sqrt{3}}{2}\nu_0$ (b) $\frac{1}{\sqrt{3}}\nu_0$ (c) $\frac{1}{\sqrt{2}}\nu_0$ (d) $\sqrt{\frac{2}{3}}\nu_0$

Ans. : (a)

Solution: $\nu = \nu_0\sqrt{1-\frac{v^2}{c^2}}$ (If detector is not in the path at nucleus)

$$\nu = \nu_0\sqrt{1-\frac{1}{4}} = \nu_0\frac{\sqrt{3}}{2}$$

NET/JRF (JUNE-2017)

Q78. The Hamiltonian for a system described by the generalised coordinate x and generalised momentum p is

$$H = \alpha x^2 p + \frac{p^2}{2(1+2\beta x)} + \frac{1}{2}\omega^2 x^2$$

where α, β and ω are constants. The corresponding Lagrangian is

(a) $\frac{1}{2}\left(\dot{x}-\alpha x^2\right)^2(1+2\beta x)-\frac{1}{2}\omega^2 x^2$ (b) $\frac{1}{2(1+2\beta x)}\dot{x}^2-\frac{1}{2}\omega^2 x^2-\alpha x^2\dot{x}$

(c) $\frac{1}{2}\left(\dot{x}^2-\alpha^2 x\right)^2(1+2\beta x)-\frac{1}{2}\omega^2 x^2$ (d) $\frac{1}{2(1+2\beta x)}\dot{x}^2-\frac{1}{2}\omega^2 x^2+\alpha x^2\dot{x}$

Ans. : (a)

Solution: $H = ax^2 p + \frac{p^2}{2(1+2\beta x)} + \frac{1}{2}\omega^2 x^2$.

$$\frac{\partial H}{\partial p} = \dot{x} \Rightarrow ax^2 + \frac{p}{(1+2\beta x)} \Rightarrow p = \left(\dot{x}-ax^2\right)(1+2\beta x)\cdot$$

$$L = \dot{x}P - H$$

$$= \dot{x}P - ax^2 P - \frac{p^2}{(1+2\beta x)} - \frac{1}{2}\omega^2 x^2\cdot$$

$$= x\left(\dot{x}-\alpha x^2\right)(1+2\beta x) - \alpha x^2\left(\dot{x}-\alpha x^2\right)(1+2\beta x) - \frac{\left(\dot{x}-\alpha x^2\right)^2(1+2\beta x)^2}{2(1+2\beta x)}$$

$$= (1+2\beta x)\left(\dot{x}-\alpha x^2\right)\left[x-\alpha x^2-\frac{\left(\dot{x}-\alpha x^2\right)}{2}\right]-\frac{1}{2}\omega^2 x^2$$

$$= (1+2\beta x)\left(\dot{x}-\alpha x^2\right)\frac{\left(\dot{x}-\alpha x^2\right)^2}{2}-\frac{1}{2}\omega^2 x^2 = (1+2\beta x)\frac{\left(\dot{x}-\alpha x^2\right)^2}{2}-\frac{1}{2}\omega^2 x^2$$

Q79. An inertial observer sees two events E_1 and E_2 happening at the same location but $6\,\mu s$ apart in time. Another observer moving with a constant velocity v (with respect to the first one) sees the same events to be $9\,\mu s$ apart. The spatial distance between the events, as measured by the second observer, is approximately

(a) 300 m (b) 1000 m (c) 2000 m (d) 2700 m

Ans. : (c)

Solution: $x_2^1 - x_1^1 = 0$, $t_2^1 - t_1^1 = 6\times10^{-6}$, $t_2 - t_1 = 9\times10^{-6}$, $x_2 - x_1 = ?$

$$t_2 - t_2 = 9\times10^{-6}$$

$$\left(\frac{t_2^1 + \frac{v}{c^2}x_2^1}{\sqrt{1-v^2/c^2}}\right) - \frac{\left(t_1' + \frac{v}{c^2}x_1'\right)}{\sqrt{1-v^2/c^2}} = 9\times10^{-6}$$

$$\frac{t_2' - t_1'}{\sqrt{1-v^2/c^2}} = 9\times10^{-6} \Rightarrow \frac{6\times10^{-6}}{\sqrt{1-v^2/c^2}} = 9\times10^{-6}$$

$$v = \sqrt{\frac{5}{9}}c \Rightarrow \sqrt{1-\frac{v^2}{c^2}} = 2/3\cdot$$

$$(x_2 - x_1) = \left(\frac{x_2' + vt_2'}{\sqrt{1-v^2/c^2}}\right) - \left(\frac{x_1' + vt_1'}{\sqrt{1-v^2/c^2}}\right)$$

$$\frac{v}{\sqrt{1-v^2/c^2}}(t_2' - t_1')$$

$$(x_2 - x_1) = \frac{\sqrt{5}}{3}c\times\frac{9}{6}\times\left(6\times10^{-6}\right) = \frac{\sqrt{5}}{3}\times3\times10^8\times\frac{9}{6}\times6\times10^{-6}$$

$$= 9\times\sqrt{5}\times10^2 = 20.12\times10^2 \simeq 2000m$$

Q80. A ball weighing 100 gm, released from a height of $5m$, bounces perfectly elastically off a plate. The collision time between the ball and the plate is $0.5s$. The average force on the plate is approximately

(a) $3N$ (b) $2N$ (c) $5N$ (d) $4N$

Ans. : (d)

Solution: $m = \frac{100}{1000} = 0.1kg$

$$mgh = \frac{1}{2}mv^2 \quad v = \sqrt{2gh}.$$

$$v = 10\ m/\text{sec}$$

change in momentum during collision, $(mv) - (-mv) = 2k.gm/\text{sec}$

$$f = \frac{\Delta P}{\Delta t} = \frac{2}{0.5} = 4N$$

Q81. A solid vertical rod, of length L and cross-sectional area A, is made of a material of Young's modulus Y. The rod is loaded with a mass M, and, as a result, extends by a small amount ΔL in the equilibrium condition. The mass is then suddenly reduced to $M/2$. As a result the rod will undergo longitudinal oscillation with an angular frequency

(a) $\sqrt{2YA/ML}$ (b) $\sqrt{YA/ML}$

(c) $\sqrt{2YA/M\Delta L}$ (d) $\sqrt{YA/M\Delta L}$

Ans. : (b)

Solution: $Y = \frac{Fl}{A\Delta l} \Rightarrow F = \frac{YA\Delta l}{l}$

For mass m, $mg = \frac{YA\Delta l}{l}$

For mass $\frac{m}{2} g = \frac{YA}{l}\left(\frac{\Delta l}{2}\right)$

Equation (i) and (ii) is for equilibrium condition

Change in force will generate acceleration

$$\Delta F = \left(\frac{mg}{2} - mg\right) = \frac{YA}{l}\left(\frac{\Delta l}{2} - \Delta l\right) = \frac{YA}{2}\frac{\Delta l}{l}$$

$$\frac{-m\omega^2 \Delta l}{2} = \frac{YA}{l}\frac{\Delta l}{2}$$

$$\omega = \sqrt{\frac{YA}{ml}}$$

Q82. The Lagrangian of a free relativistic particle (in one dimension) of mass m is given by $L = -m\sqrt{1-\dot{x}^2}$ where $\dot{x} = dx/dt$. If such a particle is acted upon by a constant force in the direction of its motion, the phase space trajectories obtained from the corresponding Hamiltonian are

(a) ellipses (b) cycloids (c) hyperbolas (d) parabolas

Ans. : (c)

Solution: $E^2 = p^2c^2 + m_0^2c^4$.

For constant force $\frac{dP}{dt} = F$, $P = Ft \Rightarrow t = \frac{P}{F}$

$$\frac{mu}{\sqrt{1-u^2/c^2}} = Ft \Rightarrow u = \frac{(F/m)t}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}}.$$

$$\frac{dx}{dt} = \frac{\left(\frac{F}{m}\right)t.}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}} \Rightarrow x = \frac{F}{m}\int_0^t \frac{tdt}{\sqrt{1+\left(\frac{Ft}{mc}\right)^2}} = \frac{mc^2}{F}\left[\sqrt{1+\left(\frac{Ft}{mc}\right)^2}-1\right]$$

$$\left(\frac{Fx}{mc^2}+1\right)^2 = \left(1+\frac{P^2}{m^2c^4}\right)$$

$$P^2 = F^2x^2 + 2mc^2Fx = \left(Fx+mc^2\right)^2 - m^2c^4$$

$\left(Fx+mc^2\right)^2 - P^2 = m^2c^4$, which is equation of hyperbola.

Q83. A Hamiltonian system is described by the canonical coordinate q and canonical momentum p. A new coordinate Q is defined as $Q(t) = q(t+\tau) + p(t+\tau)$, where t is the time and τ is a constant, that is, the new coordinate is a combination of the old coordinate and momentum at a shifted time. The new canonical momentum $P(t)$ can be expressed as

(a) $p(t+\tau) - q(t+\tau)$ (b) $p(t+\tau) - q(t-\tau)$

(c) $\frac{1}{2}\left[p(t-\tau) - q(t+\tau)\right]$ (d) $\frac{1}{2}\left[p(t+\tau) - q(t+\tau)\right]$

Ans. : (d)

Solution: Given $[q_1 p] = 1$

$Q(t) = q(t+\tau) + p(t+\tau)$

If P is new momentum, then $[Q, P] = 1$

For option (a), $[Q,P] = [q+p, p-q] = [q,p] - q[q,q] + [p,-q] + [p,p]$

$= (1) - (-1) = 2$

For option (b) $[Q,P] = [q+p, p-q] = 2$

For option (c) $[Q,P]=\frac{1}{2}[q+p,p-q]=\frac{2}{2}=1$ i.e. canonical transform

For option (d) $[Q,P]=\frac{1}{2}[q+p,\ p-q]=\frac{2}{2}=1$

Option (c) and (d) are correct. But from translation symmetry option (d) is more suitable.

Q84. The energy of a one-dimensional system, governed by the Lagrangian

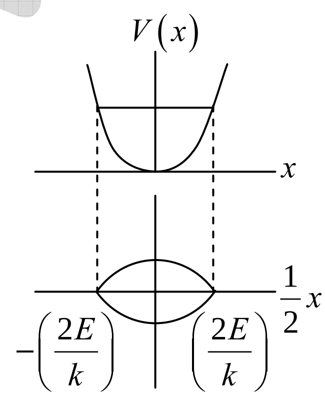
$$L=\frac{1}{2}m\dot{x}^2-\frac{1}{2}kx^{2n}$$

where k and n are two positive constants, is E_0. The time period of oscillation τ satisfies

(a) $\tau\propto k^{-\frac{1}{n}}$ (b) $\tau\propto k^{-\frac{1}{2n}}E_0^{\frac{1-n}{2n}}$ (c) $\tau\propto k^{-\frac{1}{2n}}E_0^{\frac{n-2}{2n}}$ (d) $\tau\propto k^{-\frac{1}{n}}E_0^{\frac{1+n}{2n}}$

Ans. : (b)

Solution: $\omega=\frac{\partial H}{\partial J}\Rightarrow T=\frac{\partial J}{\partial H}$, Time Period $T=\frac{\partial J}{\partial E}$, where J is Action variable



$$J=\oint Pdx=4\int_0^{\alpha}\sqrt{2m\left(E-V(x)\right)dx}$$

$$\alpha=\left(\frac{2E}{k}\right)^{1/2n}$$

$$J=4\int_0^{\left(\frac{2E}{k}\right)}\sqrt{2m\left(E-\frac{1}{2}kx^{2n}\right)}dn=4\sqrt{2mE}\int_0^{\left(\frac{2E}{k}\right)^{1/2n}}\sqrt{1-\frac{k}{2E}x^{2n}dx}$$

Don't try to solve integration rather try to make E independent.

$$J=4\sqrt{2mE}\left(\frac{2E}{k}\right)^{1/2n}\int_0^1\sqrt{1-t^{2n}}dt,\quad \text{Put }\left(\frac{k}{2E}\right)^{\frac{1}{2n}}x=t\ \Rightarrow dx=\left(\frac{2E}{k}\right)^{\frac{1}{2n}}dt$$

$$J=c4\sqrt{2m}\frac{1}{(k)^{1/2n}}E^{\left(\frac{1+n}{2n}\right)},\text{ where } c=\int_0^1\sqrt{1-t^{2n}}dt$$

$$T\propto\frac{\partial J}{\partial E}\propto(k)^{\frac{-1}{2n}}E^{\left(\frac{1+n}{2n}-1\right)}\propto(k)^{\frac{-1}{2n}}E^{\frac{1-n}{2n}}$$

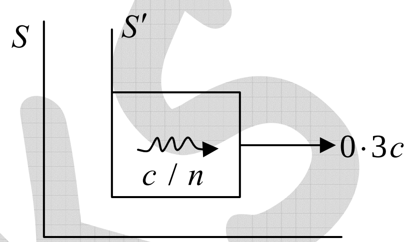
NET/JRF (DEC - 2017)

Q85. A light signal travels from a point A to a point B, both within a glass slab that is moving with uniform velocity (in the same direction as the light) with speed $0.3c$ with respect to an external observer. If the refractive index of the slab is 1.5, then the observer will measure the speed of the signal as

(a) $0.67c$ (b) $0.81c$ (c) $0.97c$ (d) c

Ans. : (b)

Solution: $v = 0.3c$,



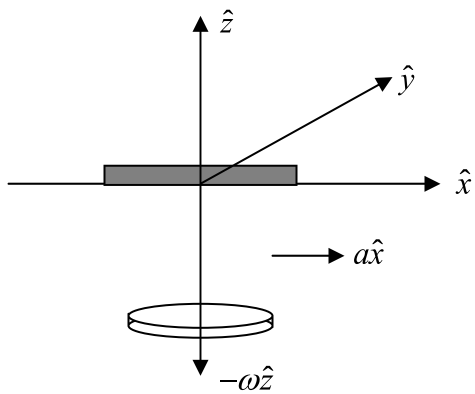
$$u'_x = \frac{c}{n} \quad n = 1.5$$

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} = \frac{0.3c + \frac{c}{n}}{1 + \frac{c}{n} \cdot \frac{0.3c}{c^2}}$$

$$u_x = 0.81c.$$

Q86. A disc of mass m is free to rotate in a plane parallel to the xy plane with an angular velocity $-\omega\hat{z}$ about a massless rigid rod suspended from the roof of a stationary car (as shown in the figure below). The rod is free to orient itself along any direction.

The car accelerates in the positive x-direction with an acceleration $a > 0$. Which of the following statements is true for the coordinates of the centre of mass of the disc in the reference frame of the car?



(a) only the x and the z coordinates change

(b) only the y and the z coordinates change

(c) only the x and the y coordinates change

(d) all the three coordinates change

Ans. : (d)

Q87. A cyclist, weighing a total of $80\,kg$ with the bicycle, pedals at a speed of $10\ m/s$. She stops pedalling at an instant which is taken to be $t=0$. Due to the velocity dependent frictional force, her velocity is found to vary as $v(t)=\dfrac{10}{\left(1+\dfrac{t}{30}\right)}\ m.s$, where t is measured in seconds. When the velocity drops to $8\ m/s$, she starts pedalling again to maintain a constant speed. The energy expended by her in 1 minute at this (new) speed, is

(a) $4\ kJ$ (b) $8\ kJ$ (c) $16\ kJ$ (d) $32\ kJ$

Ans. : (b)

Solution: The acceleration of cyclist is

$$a(t)=\frac{d}{dt}\left[\frac{300}{t+30}\right]=\frac{-300}{(t+30)^2}$$

Hence net force acting on the cyclist

$$F(t)=\frac{-300\times 80}{(t+30)^2}$$

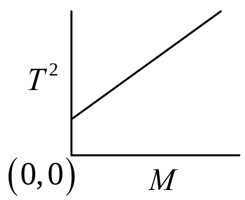
Since frictional force is the only force acting on the cyclist, this net force is equal to functional force. We can write

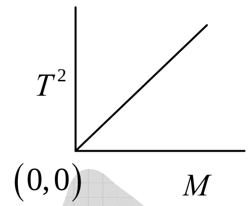
$$F(t)=\frac{-80}{300}\left(\frac{300}{t+30}\right)^2=\frac{-4}{15}\left[v(t)\right]^2$$

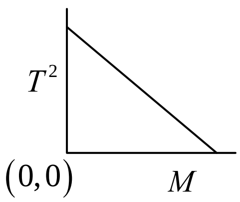
When the cyclist moves at a constant speed, the frictional force is $F(t)=\dfrac{-4}{15}(8)^2$.

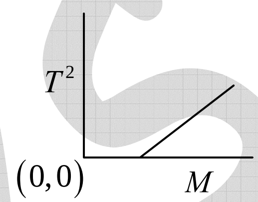
The displacement during this interval is 8×60. Thus the work done by frictional force $=\dfrac{-4}{15}\times 64\times 8\times 60$ is $-8.192\ KJ$. Hence in order to maintain constant speed the cyclist must supply an energy equal to $8.192\ KJ$.

Q88. The spring constant k of a spring of mass m_s is determined experimentally by loading the spring with mass M and recording the time period T, for a single oscillation. If the experiment is carried out for different masses, then the graph that correctly represents the result is

(a)


(b)


(c)


(d)


Ans. : (a)

Solution: The Langragian of system.

$$L = \frac{1}{2}\cdot\frac{m_s}{3}\dot{x}^2 + \frac{1}{2}M\dot{x}^2 - \frac{1}{2}kx^2, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial x} = 0 \Rightarrow \left(\frac{m_s}{3} + M\right)\ddot{x} = -kx$$

$$T = 2\pi\sqrt{\frac{M + \frac{m_s}{3}}{k}} \Rightarrow T^2 = 4\pi\frac{\left(M + \frac{m_s}{3}\right)}{k}$$

Q89. Consider a set of particles which interact by a pair potential $V = ar^6$ where r is the inter-particle separation and $a > 0$ is a constant. If a system of such particles has reached virial equilibrium, the ratio of the kinetic to the total energy of the system is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$

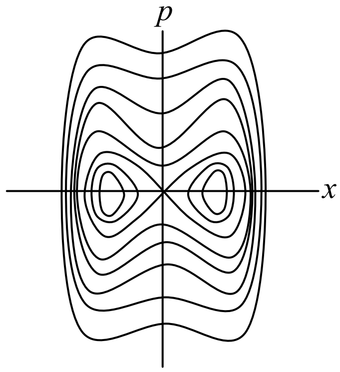
Ans. : (c)

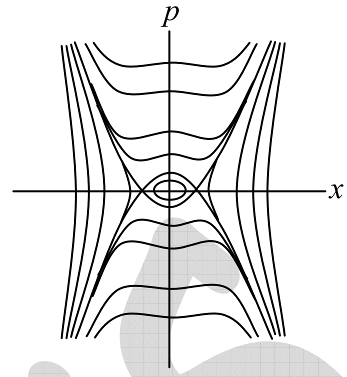
Solution: $V(r) = ar^n \ \langle T\rangle = \frac{n}{2}\langle V\rangle$

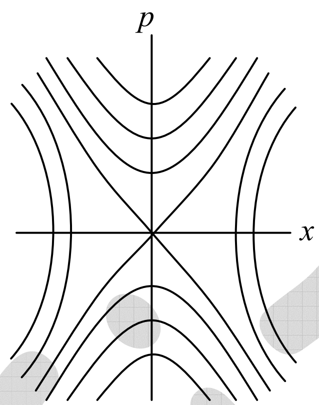
$V\langle r\rangle = a\,r^6$ and $\langle T\rangle = 3\langle V\rangle$

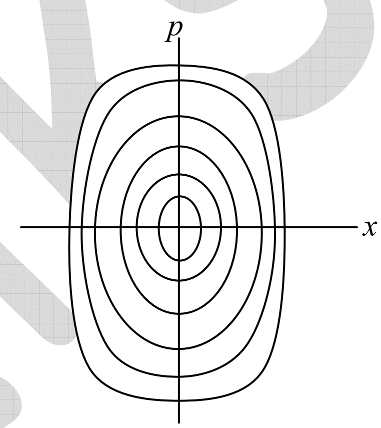
Then, $\langle E\rangle = \langle T\rangle + \langle V\rangle = \langle T\rangle + \frac{1}{3}\langle T\rangle \Rightarrow \langle E\rangle = \frac{4}{3}\langle T\rangle \Rightarrow \frac{\langle T\rangle}{\langle E\rangle} = \frac{3}{4}$

Q90. A particle moves in one dimension in a potential $V(x) = -k^2x^4 + \omega^2 x^2$ where k and ω are constants. Which of the following curves best describes the trajectories of this system in phase space?

(a) 

(b) 

(c) 

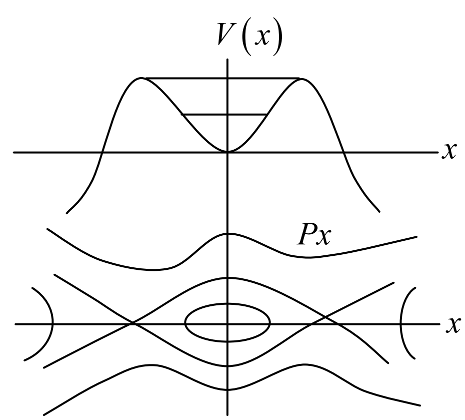
(d) 

Ans. : (c)

Solution: $V(x) = -k^2 x^4 + \omega^2 x^2$

For equation point

$\frac{\partial V}{\partial x} = 0 \Rightarrow -4k^2x^3 + 2\omega^2 x = 0$, $x = 0$ or $x^2 = \frac{\omega^2}{2k^2}$



Now, $\frac{d^2V}{dx^2} = -12k^2x^2 + 2\omega^2$ At, $x = 0$

$\frac{d^2V}{dx^2} = 2\omega^2, x = 0$ is minimum.

And, $\frac{d^2V}{dx^2} = -12k^2 \frac{\omega^2}{2k^2} + 2\omega^2 = -4\omega^2$, at $x^2 = \frac{\omega^2}{2k^2}$

Hence, $x = \pm\sqrt{\frac{\omega^2}{2k^2}}$ is maxima.

Q91. Let (x, p) be the generalized coordinate and momentum of a Hamiltonian system. If new variables (X, P) are defined by $X = x^{\alpha}\sinh(\beta p)$ and $P = x^{\gamma}\cosh(\beta p)$, where α, β and γ are constants, then the conditions for it to be a canonical transformation, are

(a) $\alpha = \frac{1}{2\beta}(\beta+1)$ and $\gamma = \frac{1}{2\beta}(\beta-1)$ (b) $\beta = \frac{1}{2\gamma}(\alpha+1)$ and $\gamma = \frac{1}{2\alpha}(\alpha-1)$

(c) $\alpha = \frac{1}{2\beta}(\beta-1)$ and $\gamma = \frac{1}{2\beta}(\beta+1)$ (d) $\beta = \frac{1}{2\gamma}(\alpha-1)$ and $\gamma = \frac{1}{2\alpha}(\alpha+1)$

Ans. : (c)

Solution: $X = x^{\alpha}\sinh(\beta p)$

$P = x^{\gamma}\cosh\beta p$

For canonical transformation

$$\frac{\partial X}{\partial x}.\frac{\partial P}{\partial p} - \frac{\partial X}{\partial p}.\frac{\partial P}{\partial X} = 1$$

$\alpha x^{\alpha-1}\sinh\beta p x^{\gamma}\beta\sinh\beta p$

$\gamma x^{\gamma-1}\cosh\beta p.x^{\alpha}\beta\cosh\beta b = 1$

$\beta x^{\alpha+\gamma-1}\left[\alpha\sin^2\text{h}\,\beta p - r\cos^2\text{h}\,\beta p\right] = 1$

$\alpha+\gamma-1=0$ (i)

$\beta\alpha\sin^2 h\beta p - \beta v\cos^2 h\beta p = \cos^2 h\beta p - \sin^2 h\beta p$

equating the coefficient on both side

$\beta\alpha = -1$ (ii)

$\beta\gamma = -1$ (iii)

From (ii) and (iii) $\alpha = \gamma$

And from (i) $\alpha = \gamma = \frac{1}{2}$, $\beta = -2$ (Not convinced with any answer)

NET/JRF (JUNE-2018)

Q92. Two particles A and B move with relativistic velocities of equal magnitude v, but in opposite directions, along the x-axis of an inertial frame of reference. The magnitude of the velocity of A, as seen from the rest frame of B, is

(a) $\dfrac{2v}{\left(1-\dfrac{v^2}{c^2}\right)}$ (b) $\dfrac{2v}{\left(1+\dfrac{v^2}{c^2}\right)}$ (c) $2v\sqrt{\dfrac{c-v}{c+v}}$ (d) $\dfrac{2v}{\sqrt{1-\dfrac{v^2}{c^2}}}$

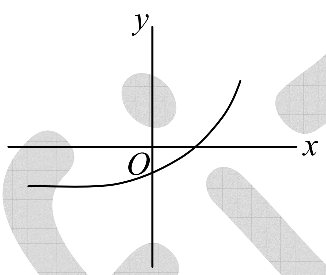
Ans. : (b)

Solution: $u_x' = v$ $V = v$

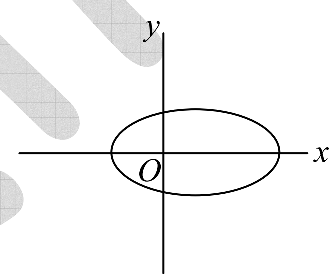
$$u_x = \frac{u_x' + V}{1+\dfrac{u_x' V}{c^2}} \quad u_x = \frac{v+v}{1+\dfrac{v^2}{c^2}} = \frac{2v}{1+\dfrac{v^2}{c^2}}$$

Q93. Which of the following figures best describes the trajectory of a particle moving in a repulsive central potential $V(r) = \dfrac{a}{r}$ ($a > 0$ is a constant)?

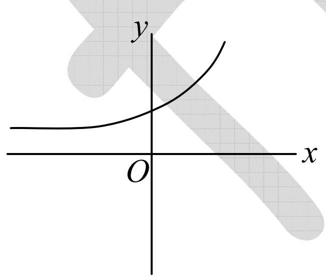
(a)



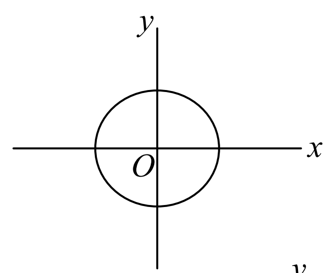
(b)



(c)

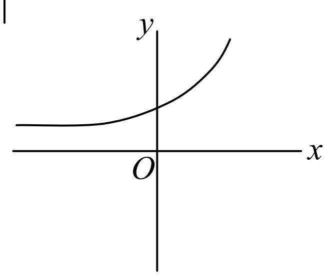


(d)



Ans. : (c)

Solution: The potential is $V(r) = \dfrac{a}{r}$ which is repulsive. So there is unbounded motion and mainly represent by scattering project



Q94. A particle moves in the one-dimensional potential $V(x)=\alpha x^6$, where $a>0$ is a constant. If the total energy of the particle is E, its time period in a periodic motion is proportional to

(a) $E^{-1/3}$ (b) $E^{-1/2}$ (c) $E^{1/3}$ (d) $E^{1/2}$

Ans. : (a)

Solution: $J=\oint Pdx$

$$J\propto (2mE)^{1/2}\left(\frac{E}{a}\right)^{1/6} \quad \Rightarrow E\propto J^{3/2} \quad \nu=\frac{dE}{dJ}\propto J^{1/2} \Rightarrow \nu\propto E^{1/3} \quad \Rightarrow T\propto E^{-1/3}$$

Q95. A particle of mass m, kept in potential $V(x)=-\frac{1}{2}kx^2+\frac{1}{4}\lambda x^4$ (where k and λ are positive constants), undergoes small oscillations about an equilibrium point. The frequency of oscillations is

(a) $\frac{1}{2\pi}\sqrt{\frac{2\lambda}{m}}$ (b) $\frac{1}{2\pi}\sqrt{\frac{k}{m}}$ (c) $\frac{1}{2\pi}\sqrt{\frac{2k}{m}}$ (d) $\frac{1}{2\pi}\sqrt{\frac{\lambda}{m}}$

Ans. : (c)

Solution: $V=-\frac{1}{2}kx^2+\frac{1}{4}\lambda x^4$

$$\frac{dV}{dx}=0 \qquad -kx+\lambda x^3=0$$

$$x=0,\quad x^2=\frac{k}{\lambda}\Rightarrow x=x_0=\sqrt{\frac{k}{\lambda}}$$

$$\frac{d^2V}{dx^2}=-k \quad \text{at } x=0 \quad \text{so } x=0 \text{ is unstable part}$$

$$\frac{d^2V}{dx^2}=2k \text{ at } x_0=\sqrt{\frac{k}{\lambda}} \quad \text{so } x_0=\sqrt{\frac{k}{x}} \text{ is stable equation point}$$

$$\omega=\sqrt{\frac{\left.\frac{d^2V}{dx^2}\right|_{x=x_0}}{m}}=\sqrt{\frac{2k}{m}} \qquad f=\frac{1}{2\pi}\sqrt{\frac{2k}{m}}$$

Q96. A particle of mass m moves in a central potential $V(r)=-\frac{k}{r}$ in an elliptic orbit $r(\theta)=\frac{a(1-e^2)}{1+e\cos\theta}$, where $0\le\theta<2\pi$ and a and e denote the semi-major axis and eccentricity, respectively. If its total energy is $E=-\frac{k}{2a}$, the maximum kinetic energy is

(a) $E(1-e^2)$ (b) $E\frac{(e+1)}{(e-1)}$ (c) $E/(1-e^2)$ (d) $E\frac{(e-1)}{(e+1)}$

Ans. : (b)

Solution: $E=T+V \quad T=E-V$

$$T=-\frac{k}{2a}+\frac{k}{r}\; T=-\frac{k}{2a}+\frac{k}{a(1-e^2)}(1+\cos\theta)$$

T . maximum $\cos\theta=1$

$$T_{\max}=-\frac{k}{2a}+\frac{k(1+e)}{a(1-e^2)}=-\frac{k}{2a}+\frac{k}{a}\frac{(1+e)}{(1-e)(1+e)}$$

$$=-\frac{k}{a}\left[\frac{1}{2}+\frac{1}{(1-e)}\right]=-\frac{k}{2a}\left(\frac{1+e}{1-e}\right)=E\left(\frac{1+e}{1-e}\right)$$

Q97. The Hamiltonian of a one-dimensional system is $H=\frac{xp^2}{2m}+\frac{1}{2}kx$, where m and k are positive constants. The corresponding Euler-Lagrange equation for the system is

(a) $m\ddot{x}+k=0$ (b) $m\ddot{x}+2\dot{x}+kx^2=0$

(c) $2mx\ddot{x}-m\dot{x}^2+kx^2=0$ (d) $mx\ddot{x}+2m\dot{x}^2+kx^2=0$

Ans. : (c)

Solution: $H=\frac{xp^2}{2m}+\frac{1}{2}kx$

$$\frac{\partial H}{\partial p}=\dot{x}\Rightarrow\frac{xp}{m}=\dot{x} \qquad p=\frac{m\dot{x}}{x}$$

$$L=\dot{x}p-H\Rightarrow L=\dot{x}p-\frac{xp^2}{2m}-\frac{1}{2}kx=\frac{m\dot{x}^2}{x}-\frac{m\dot{x}^2}{2x}-\frac{1}{2}kx=\frac{m\dot{x}^2}{2x}-\frac{1}{2}kx$$

Eular Lagrangas equation is given by

$$\frac{d}{dr}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x}=0$$

$$\frac{m\ddot{x}}{x}-\frac{m\dot{x}\dot{x}}{2x^2}+\frac{1}{2}k=0$$

$$2xm\ddot{x}-m\dot{x}^2+kx^2=0$$

Q98. The energy of a free relativistic particle is $E=\sqrt{|\vec{p}|^2 c^2+m^2c^4}$, where m is its rest mass, $\vec{p}$ is its momentum and c is the speed of light in vacuum. The ratio v_g / v_p of the group velocity v_g of a quantum mechanical wave packet (describing this particle) to the phase velocity v_p is

(a) $|\vec{p}|c/E$ (b) $|\vec{p}|mc^3/E^2$ (c) $|\vec{p}|^2 c^3/E^2$ (d) $|\vec{p}|c/2E$

Ans. : (c)

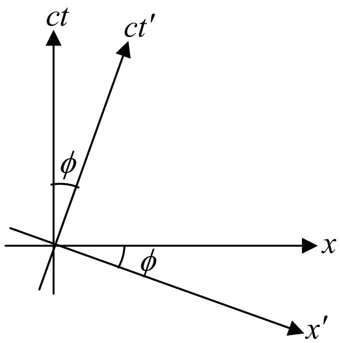
Solution: $E^2=p^2c^2+m^2c^4$ and $v_g=\frac{dE}{dp}, v_p=\frac{E}{p}$

$$2E\frac{dE}{dp}=2pc^2 \quad \Rightarrow \frac{E}{p}\frac{dE}{dp}=c^2$$

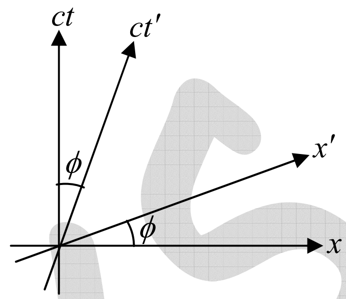
$$\frac{v_g}{v_p}=\frac{c^2}{v_p^2} \qquad \frac{v_g}{v_p}=\frac{c^3p^2}{E^2}$$

Q99. An inertial frame K' moves with a constant speed v with respect to another inertial frame K along their common x- direction. Let (x, ct) and (x', ct') denote the space-time coordinates in the frames K and K', respectively. Which of the following space-time diagrams correctly describes the t'- axis $(x' = 0 \text{ line})$ and the x'- axis $(t' = 0 \text{ line})$ in the x-ct plane? (In the following figures $\tan\phi = v/c$)

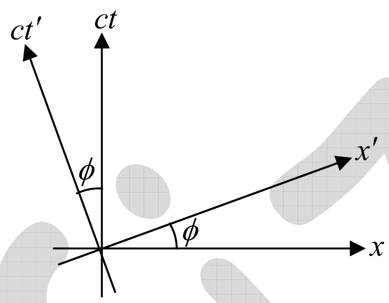
(a)



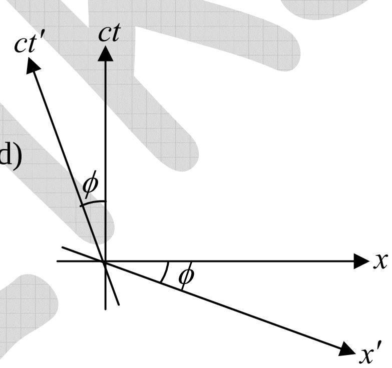
(c)



(c)

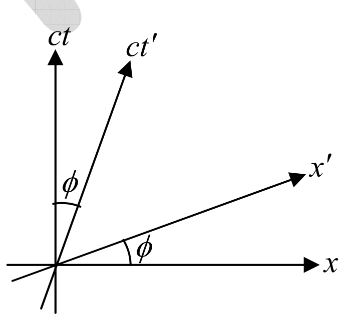


(d)



Ans. : (b)

Solution: $\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$



Where $v = \cosh\phi, \beta v = \sinh\phi \quad \beta = \tanh\phi$

NET/JRF (DEC - 2018)

Q100. A particle of mass m, moving along the x - direction, experiences a damping force $-\gamma v^2$, where γ is a constant and v is its instantaneous speed. If the speed at $t=0$ is v_0, the speed at time t is

(a) $v_0 e^{-\frac{\gamma v_0 t}{m}}$ (b) $\dfrac{v_0}{1+\ln\left(1+\frac{\gamma v_0 t}{m}\right)}$ (c) $\dfrac{m v_0}{m+\gamma v_0 t}$ (d) $\dfrac{2v_0}{1+e^{\frac{\gamma v_0 t}{m}}}$

Ans. : (c)

Solution: From Newton's second law, $m\dfrac{dv}{dt}=-\gamma v^2 \Rightarrow \dfrac{dv}{v^2}=-\dfrac{\gamma}{m}dt$

Integrating both sides gives $-\dfrac{1}{v}=-\dfrac{\gamma}{m}t+c$

where c is a constant of integration

Since $v=v_0$ at $t=0$, we obtain

$$-\frac{1}{v_0}=-\frac{\gamma}{m}.0+c \Rightarrow c=-\frac{1}{v_0}$$

Hence, $-\dfrac{1}{v}=-\dfrac{\gamma}{m}t-\dfrac{1}{v_0}$

$$\Rightarrow \frac{1}{v}=\frac{\gamma t}{m}+\frac{1}{v_0}=\frac{\gamma v_0 t+m}{m v_0} \Rightarrow v=\frac{m v_0}{\gamma v_0 t+m}=\frac{m v_0}{m+\gamma v_0 t}$$

Q101. In the attractive Kepler problem described by the central potential $V(r)=\dfrac{-k}{r}$ (where k is a positive constant), a particle of mass m with a non-zero angular momentum can never reach the centre due to the centrifugal barrier. If we modify the potential to

$$V(r)=-\frac{k}{r}-\frac{\beta}{r^3}$$

one finds that there is a critical value of the angular momentum ℓ_c below which there is no centrifugal barrier. This value of ℓ_c is

(a) $\left[12km^2\beta\right]^{1/2}$ (b) $\left[12km^2\beta\right]^{-1/2}$ (c) $\left[12km^2\beta\right]^{1/4}$ (d) $\left[12km^2\beta\right]^{-1/4}$

Ans. : (c)

Solution: $V_{eff} = \frac{L^2}{2mr^2} - \frac{k}{r} = 0 \Rightarrow -\frac{L^2}{mr^3} + \frac{k}{r^2} = 0 \Rightarrow r_0 = \frac{L^2}{mk}$

when introduce new potential

$$V_{eff} = \frac{L^2}{2mr^2} - \frac{k}{r} - \frac{\beta}{r^3}$$

For critical value

$$\frac{\partial V_{eff}}{\partial r} = \frac{-L^2}{mr^3} + \frac{k}{r^2} + \frac{3\beta}{r^4}$$

$$\frac{\partial^2 V_{eff}}{\partial r^2} = \frac{+3L^2}{mr^4} \frac{-2k}{r^3} - \frac{12\beta}{r^5} \geq 0$$

For critical value

$$= \frac{3L^2}{m\left(\frac{L^2}{mk}\right)^4} - \frac{2k}{\left(\frac{L^2}{mk}\right)^3} - \frac{12\beta}{\left(\frac{L^2}{mk}\right)^5} = 0 = \frac{3m^3k^4}{L^6} - \frac{2m^3x^4}{L^6} - \frac{12m^5x^5\beta}{L^{10}} = 0$$

$$L_C = \left(12m^2k\beta\right)^{1/4} \frac{m^3k^4}{L^6}\left(3 - 2 - 12\frac{m^2k\beta}{L^4}\right) = 0 \Rightarrow L_c = \left(12m^2k\beta\right)^{1/4}$$

Q102. The time period of a particle of mass m, undergoing small oscillations around $x = 0$, in the potential $V = V_0 \cosh\left(\frac{x}{L}\right)$, is

(a) $\pi\sqrt{\frac{mL^2}{V_0}}$ (b) $2\pi\sqrt{\frac{mL^2}{2V_0}}$ (c) $2\pi\sqrt{\frac{mL^2}{V_0}}$ (d) $2\pi\sqrt{\frac{2mL^2}{V_0}}$

Ans. : (c)

Solution: $V = V_0 \cosh\left(\frac{x}{L}\right)$

For equilibrium point $\frac{\partial V}{\partial x} = 0 \Rightarrow \frac{V_0}{L}\sinh\left(\frac{x}{L}\right) = 0$

$$k = \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=0} = \frac{V_0}{L^2}$$

$$\omega = \sqrt{\frac{k}{m}} \Rightarrow \frac{2\pi}{T} = \sqrt{\frac{V_0}{mL^2}} \Rightarrow T = 2\pi\sqrt{\frac{mL^2}{V_0}}$$

Q103. Consider the decay $A \rightarrow B + C$ of a relativistic spin - $\frac{1}{2}$ particle A. Which of the following statements is true in the rest frame of the particle A?

(a) The spin of both B and C may be $\frac{1}{2}$

(b) The sum of the masses of B and C is greater than the mass of A

(c) The energy of B is uniquely determined by the masses of the particles

(d) The spin of both B and C may be integral

Ans. : (c)

Q104. The motion of a particle in one dimension is described by the Langrangian $L = \frac{1}{2}\left(\left(\frac{dx}{dt}\right)^2 - x^2\right)$ in suitable units. The value of the action along the classical path from $x = 0$ at $t = 0$ to $x = x_0$ at $t = t_0$, is

(a) $\frac{x_0^2}{2\sin^2 t_0}$ (b) $\frac{1}{2}x_0^2 \tan t_0$ (c) $\frac{1}{2}x_0^2 \cot t_0$ (d) $\frac{x_0^2}{2\cos^2 t_0}$

Ans. : (c)

Solution: $L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2$

From Lagrangian equation of motion, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\ddot{x} + x = 0$

The solution is $x = A\sin t + B\cos t$

$t = 0, \quad x = 0, \quad B = 0$

$x = A\sin t$

$t = t_0, \quad x = x_0, \quad A = \frac{x_0}{\sin t_0}$

$x = \frac{x_0}{\sin t_0}\sin t, \quad \dot{x} = \frac{x_0}{\sin t_0}\cos t$

$$A = \int_0^{t_0} L\,dt = \int_0^{t_0} \frac{1}{2}\dot{x}^2 dt - \int_0^{t_0} \frac{1}{2}x^2 dt = \frac{1}{2}\frac{x_0^2}{\sin^2 t_0}\int_0^{t_0} \cos^2 t\, dt - \frac{1}{2}\frac{x_0^2}{\sin^2 t_0}\int_0^{t_0} \sin^2 t dt$$

$$= \frac{1}{2}\frac{x_0^2}{\sin^2 t_0}\left[\int_0^{t_0} \cos^2 t\, dt - \int_0^{t} \sin^2 t dt\right] = \frac{1}{2}\frac{x_0^2}{\sin^2 t_0}\int_t^{t_0} \cos 2t\, dt = \frac{1}{2}\frac{x_0^2}{\sin^2 t_0}\left.\frac{\sin 2t_0}{2}\right|_0^{t_0} = \frac{x_0^2}{2}\cot t_0$$

Q105. The Hamiltonian of a classical one-dimensional harmonic oscillator is $H = \frac{1}{2}\left(p^2 + x^2\right)$, in suitable units. The total time derivative of the dynamical variable $\left(p + \sqrt{2}x\right)$ is

(a) $\sqrt{2}p - x$ (b) $p - \sqrt{2}x$ (c) $p + \sqrt{2}x$ (d) $x + \sqrt{2}p$

Ans. : (a)

Solution: $H = \frac{p^2}{2} + \frac{x^2}{2}$ Let say dynamical variable $A = \left(p + \sqrt{2}x\right)$

$$\frac{dA}{dt} = [A, H] + \frac{\partial A}{\partial t}$$

It is given $\frac{\partial A}{\partial t} = 0 \Rightarrow \frac{dA}{dt} = [A, H]$

$$\frac{dA}{dt} = \left[p + \sqrt{2}x, \frac{p^2}{2} + \frac{x^2}{2}\right] = \left[p, \frac{x^2}{2}\right] + \left[\sqrt{2}x, \frac{p^2}{2}\right]$$

$$= \frac{-2x}{2} + \frac{\sqrt{2}\,2p}{2} = -x + \sqrt{2}p = \sqrt{2}p - x$$

Q106. A relativistic particle of mass m and charge e is moving in a uniform electric field of strength ε. Starting from rest at $t = 0$, how much time will it take to reach the speed $\frac{c}{2}$?

(a) $\frac{1}{\sqrt{3}}\frac{mc}{e\varepsilon}$ (b) $\frac{mc}{e\varepsilon}$ (c) $\sqrt{2}\frac{mc}{e\varepsilon}$ (d) $\sqrt{\frac{3}{2}}\frac{mc}{e\varepsilon}$

Ans. : (a)

Solution: $\frac{dp}{dt} = e\varepsilon$

$$p = e\varepsilon t + c$$

At $t = 0$, $p = 0$, $c = 0$

$$\frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} = e\varepsilon t$$

$$t = \frac{m}{e\varepsilon}\frac{v}{\sqrt{1 - \frac{v^2}{c^2}}}$$ Put $v = \frac{c}{2}$, $t = \frac{m}{e\varepsilon}\frac{c/2}{\sqrt{1 - \frac{1}{4}}} = \frac{mc}{\sqrt{3}e\varepsilon}$ $\Rightarrow t = \frac{mc}{\sqrt{3}eE}$