# QUANTUM MECHANICS SOLUTIONS

## NET/JRF (JUNE-2011)

Q1. The wavefunction of a particle is given by $\psi = \left(\frac{1}{\sqrt{2}}\phi_0 + i\phi_1\right)$ where $\phi_0$ and $\phi_1$ are the normalized eigenfunctions with energies $E_0$ and $E_1$ corresponding to the ground state and first excited state, respectively. The expectation value of the Hamiltonian in the state $\psi$ is

(a) $\frac{E_0}{2} + E_1$ (b) $\frac{E_0}{2} - E_1$ (c) $\frac{E_0 - 2E_1}{3}$ (d) $\frac{E_0 + 2E_1}{3}$

Ans. : (d)

Solution: $\psi = \frac{1}{\sqrt{2}}\phi_0 + i\phi_1$ and $\langle H \rangle = \frac{\langle \psi | H | \psi \rangle}{\langle \psi | \psi \rangle} = \frac{E_0 + 2E_1}{3}$

Q2. The energy levels of the non-relativistic electron in a hydrogen atom (i.e. in a Coulomb potential $V(r) \propto -1/r$) are given by $E_{nlm} \propto -1/n^2$, where $n$ is the principal quantum number, and the corresponding wave functions are given by $\psi_{nlm}$, where $l$ is the orbital angular momentum quantum number and $m$ is the magnetic quantum number. The spin of the electron is not considered. Which of the following is a correct statement?

(a) There are exactly $(2l+1)$ different wave functions $\psi_{nlm}$, for each $E_{nlm}$.

(b) There are $l(l+1)$ different wave functions $\psi_{nlm}$, for each $E_{nlm}$.

(c) $E_{nlm}$ does not depend on $l$ and $m$ for the Coulomb potential.

(d) There is a unique wave function $\psi_{nlm}$ for each $E_{nlm}$.

Ans. : (c)

Q3. The Hamiltonian of an electron in a constant magnetic field $\vec{B}$ is given by $H = \mu\vec{\sigma}\cdot\vec{B}$. where $\mu$ is a positive constant and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli matrices. Let $\omega = \mu B/\hbar$ and $I$ be the $2\times 2$ unit matrix. Then the operator $e^{iHt/\hbar}$ simplifies to

(a) $I\cos\frac{\omega t}{2} + \frac{i\vec{\sigma}\cdot\vec{B}}{B}\sin\frac{\omega t}{2}$

(b) $I\cos\omega t + \frac{i\vec{\sigma}\cdot\vec{B}}{B}\sin\omega t$

(c) $I\sin\omega t + \frac{i\vec{\sigma}\cdot\vec{B}}{B}\cos\omega t$

(d) $I\sin 2\omega t + \frac{i\vec{\sigma}\cdot\vec{B}}{B}\cos 2\omega t$

Ans. : (b)

Solution: $H = \mu\vec{\sigma}\vec{B}$ where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are pauli spin matrices and $\vec{B}$ are constant magnetic field. $\vec{\sigma} = (\sigma_1\hat{i}, \sigma_2\hat{j}, \sigma_3\hat{k})$, $\vec{B} = (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$ and Hamiltonion $H = \mu\vec{\sigma}\cdot\vec{B}$ in matrices form is given by

$$H = \mu\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}.$$

Eigenvalue of given matrices are given by $+\mu B$ and $-\mu B$. $H$ matrices are not diagonals so $e^{iHt/\hbar}$ is equivalent to

$$S^{-1}\begin{pmatrix} e^{\frac{i\mu Bt}{\hbar}} & 0 \\ 0 & e^{\frac{-i\mu Bt}{\hbar}} \end{pmatrix}S$$

where $S$ is unitary matrices

and $$S^{-1} = S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}.$$

$$S^{-1}\begin{pmatrix} e^{\frac{i\mu Bt}{\hbar}} & 0 \\ 0 & e^{\frac{-i\mu Bt}{\hbar}} \end{pmatrix}S = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} e^{\frac{i\mu Bt}{\hbar}} & 0 \\ 0 & e^{\frac{-i\mu Bt}{\hbar}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$, where $\omega = \mu B/\hbar$.

$e^{iHt/\hbar} = \begin{pmatrix} \cos\omega t & i\sin\omega t \\ i\sin\omega t & \cos\omega t \end{pmatrix}$, which is equivalent to $I\cos\omega t + i\sigma_x \sin\omega t$ can be written as $I\cos\omega t + \frac{i\vec{\sigma}\cdot\vec{B}}{B}\sin\omega t$, where $\sigma_x = \frac{i\vec{\sigma}\cdot B}{B}$

Q4. If the perturbation $H' = ax$, where $a$ is a constant, is added to the infinite square well potential

$$V(x) = \begin{cases} 0 & \text{for } 0 \le x \le \pi \\ \infty & \text{otherwise.} \end{cases}$$

The correction to the ground state energy, to first order in $a$, is

(a) $\frac{a\pi}{2}$ (b) $a\pi$ (c) $\frac{a\pi}{4}$ (d) $\frac{a\pi}{\sqrt{2}}$

Ans. : (a)

Solution: $E_0^1 = \int_0^{\pi} \psi_0^* H' \psi_0 \, dx = \frac{a \cdot 2}{\pi} \int_0^{\pi} x \sin^2 \frac{\pi x}{\pi} dx = \frac{a\pi}{2} \quad \because \psi_0 = \sqrt{\frac{2}{\pi}} \sin \frac{\pi x}{\pi}$.

Q5. A particle in one dimension moves under the influence of a potential $V(x) = ax^6$, where $a$ is a real constant. For large $n$ the quantized energy level $E_n$ depends on $n$ as:

(a) $E_n \sim n^3$ (b) $E_n \sim n^{4/3}$ (c) $E_n \sim n^{6/5}$ (d) $E_n \sim n^{3/2}$

Ans. : (d)

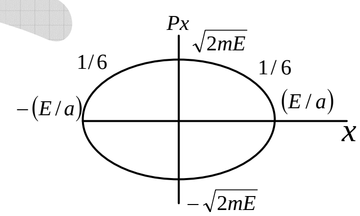
Solution: $V(x) = ax^6$, $H = \frac{p_x^2}{2m} + ax^6$, $E = \frac{p_x^2}{2m} + ax^6$ and $p_x = \left[2m\left(E - ax^6\right)\right]^{\frac{1}{2}}$.

According to W.K.B approximation $pdx \cong nh$

$$\int \left(2m\left(E - ax^6\right)\right)^{1/2} dx \propto n$$

We can find this integration without solving the integration

$$E = \frac{p_x^2}{2m} + ax^6 \Rightarrow \frac{p_x^2}{2mE} + \frac{x^6}{E/a} = 1 \Rightarrow x = \left(\frac{E}{a}\right)^{\frac{1}{6}} \text{ at } p_x = 0.$$




Area of Ellipse = $\pi$ (semi major axis $\times$ semiminor axis)

$$= \pi\sqrt{2mE} \times \left(\frac{E}{a}\right)^{\frac{1}{6}} \propto n \Rightarrow E \propto n^{\frac{3}{2}}.$$

Q6. **(A)** In a system consisting of two spin $\frac{1}{2}$ particles labeled 1 and 2, let $\vec{S}^{(1)} = \frac{\hbar}{2}\vec{\sigma}^{(1)}$ and $\vec{S}^{(2)} = \frac{\hbar}{2}\vec{\sigma}^{(2)}$ denote the corresponding spin operators. Here $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ and $\sigma_x, \sigma_y, \sigma_z$ are the three Pauli matrices.

In the standard basis the matrices for the operators $S_x^{(1)} S_y^{(2)}$ and $S_y^{(1)} S_x^{(2)}$ are respectively,

(a) $\frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $\frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

(c) $\frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$

(d) $\frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Ans. : (c)

Solution: $S_x^{(1)}S_y^{(2)} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \Rightarrow \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$

$$S_y^{(1)}S_x^{(2)} = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \frac{\hbar^2}{4}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

**(B)** These two operators satisfy the relation

(a) $\left\{S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right\} = S_z^{(1)}S_z^{(2)}$

(b) $\left\{S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right\} = 0$

(c) $\left[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right] = iS_z^{(1)}S_z^{(2)}$

(d) $\left[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right] = 0$

Ans. : (d)

Solution: We have matrix $S_x^{(1)}S_y^{(2)}$ and $S_y^{(1)}S_x^{(2)}$ from question 6(A) so commutation is given by $\left[S_x^{(1)}S_y^{(2)}, S_y^{(1)}S_x^{(2)}\right] = 0$.

## NET/JRF (DEC-2011)

Q7. The energy of the first excited quantum state of a particle in the two-dimensional potential $V(x,y) = \frac{1}{2}m\omega^2\left(x^2 + 4y^2\right)$ is

(a) $2\hbar\omega$

(b) $3\hbar\omega$

(c) $\frac{3}{2}\hbar\omega$

(d) $\frac{5}{2}\hbar\omega$

Ans. : (d)

Solution: $V(x,y) = \frac{1}{2}m\omega^2\left(x^2+4y^2\right) = \frac{1}{2}m\omega^2x^2 + \frac{1}{2}m4\omega^2y^2$, $E = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)2\hbar\omega$

For ground state energy $n_x = 0, n_y = 0 \Rightarrow E = \frac{\hbar\omega}{2} + \frac{1}{2}2\hbar\omega = \frac{3\hbar\omega}{2}$

First exited state energy $n_x = 1, n_y = 0 \Rightarrow \frac{3\hbar\omega}{2} + \hbar\omega = \frac{5\hbar\omega}{2}$

Q8. Consider a particle in a one dimensional potential that satisfies $V(x)=V(-x)$. Let $|\psi_0\rangle$ and $|\psi_1\rangle$ denote the ground and the first excited states, respectively, and let $|\psi\rangle=\alpha_0|\psi_0\rangle+\alpha_1|\psi_1\rangle$ be a normalized state with $\alpha_0$ and $\alpha_1$ being real constants. The expectation value $\langle x\rangle$ of the position operator $x$ in the state $|\psi\rangle$ is given by

(a) $\alpha_0^2\langle\psi_0|x|\psi_0\rangle+\alpha_1^2\langle\psi_1|x|\psi_1\rangle$ (b) $\alpha_0\alpha_1[\langle\psi_0|x|\psi_1\rangle+\langle\psi_1|x|\psi_0\rangle]$

(c) $\alpha_0^2+\alpha_1^2$ (d) $2\alpha_0\alpha_1$

Ans. : (b)

Solution: Since $V(x)=V(-x)$ so potential is symmetric.

$\langle\psi_0|x|\psi_0\rangle=0$, $\langle\psi_1|x|\psi_1\rangle=0$

$$\langle\psi|x|\psi\rangle=(\alpha_0\langle\psi_0|+\alpha_1\langle\psi_1|)\times(\alpha_0|\psi_0\rangle+\alpha_1|\psi_1\rangle)=\alpha_0\alpha_1[\langle\psi_0|x|\psi_1\rangle+\langle\psi_1|x|\psi_0\rangle]$$

Q9. The perturbation $H'=bx^4$, where $b$ is a constant, is added to the one dimensional harmonic oscillator potential $V(x)=\frac{1}{2}m\omega^2x^2$. Which of the following denotes the correction to the ground state energy to first order in $b$?

[Hint: The normalized ground state wave function of the one dimensional harmonic oscillator potential is $\psi_0=\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}e^{-m\omega x^2/2\hbar}$. You may use the following integral $\int_{-\infty}^{\infty}x^{2n}e^{-ax^2}dx=a^{-n-\frac{1}{2}}\Gamma\left(n+\frac{1}{2}\right)$].

(a) $\frac{3b\hbar^2}{4m^2\omega^2}$ (b) $\frac{3b\hbar^2}{2m^2\omega^2}$ (c) $\frac{3b\hbar^2}{2\pi m^2\omega^2}$ (d) $\frac{15b\hbar^2}{4m^2\omega^2}$

Ans. : (a)

Solution: $H'=bx^4, V(x)=\frac{1}{2}m\omega^2x^2$.

Correction in ground state is given by $E_0^1=\langle\psi_0|H'|\psi_0\rangle$ where $\psi_0=\left(\frac{m\omega}{\hbar\pi}\right)^{1/4}e^{-\frac{m\omega x^2}{2\hbar}}$.

$$E_0^1=\int_{-\infty}^{\infty}\psi_0^* bx^4\psi_0 dx \quad =\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{2}}\cdot b\int x^4 e^{-\frac{m\omega x^2}{\hbar}}dx=\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{2}}\cdot b\int_{-\infty}^{\infty}\left(x^2\right)^2 e\frac{-m\omega x^2}{\hbar}dx$$

It is given in the equation $\int_{-\infty}^{\infty} x^{2n}e^{-\alpha x^2}dx=\alpha^{-n-1/2}\left|\left(n+\frac{1}{2}\right)\right.$

Thus $n=2$ and $\alpha=\frac{m\omega}{\hbar}$

$$\Rightarrow E_0^1=\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{2}}\cdot b\int_{-\infty}^{\infty}\left(x^2\right)^2 e\frac{-m\omega x^2}{\hbar}dx=b\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{2}}\left(\frac{m\omega}{\hbar}\right)^{-2-\frac{1}{2}}\left|\left(2+\frac{1}{2}\right)\right.$$

$$\Rightarrow E_0^1=b\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{2}}\left(\frac{m\omega}{\hbar}\right)^{\frac{-5}{2}}\left|\frac{5}{2}\right.=\frac{3}{4}\frac{b\hbar^2}{m^2\omega^2}.$$

Q10. Let $|0\rangle$ and $|1\rangle$ denote the normalized eigenstates corresponding to the ground and first excited states of a one dimensional harmonic oscillator. The uncertainty $\Delta p$ in the state $\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$, is

(a) $\Delta p=\sqrt{\hbar m\omega}/2$

(b) $\Delta p=\sqrt{\hbar m\omega/2}$

(c) $\Delta p=\sqrt{\hbar m\omega}$

(d) $\Delta p=\sqrt{2\hbar m\omega}$

Ans. : (c)

Solution: $|\psi\rangle=\frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right)$, $p=i\sqrt{\frac{m\omega\hbar}{2}}\left(a^\dagger-a\right)$

$$a^\dagger|\psi\rangle=\frac{1}{\sqrt{2}}\left(\sqrt{1}\ \ |1\rangle+\sqrt{2}|2\rangle\ \right) \text{ and } a|\psi\rangle=\frac{1}{\sqrt{2}}\left(0+\sqrt{1}|0\rangle\right)$$

$$\langle p\rangle=i\sqrt{\frac{m\omega\hbar}{2}}\left(\langle\psi|a^\dagger-a|\psi\rangle\right)=0\ ,\ p^2=-\frac{m\omega\hbar}{2}\left(a^{\dagger^2}+a^2-(2N+1)\right)$$

$$\langle p^2\rangle=\frac{-m\omega\hbar}{2}\left[\langle a^{\dagger^2}\rangle+\langle a^2\rangle-\langle 2N+1\rangle\right]=\frac{m\omega\hbar}{2}\langle 2N+1\rangle=\frac{m\omega\hbar}{2}\left(2\cdot\frac{1}{2}+1\right)\ =m\omega\hbar$$

$$\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{m\omega\hbar}$$

Q11. The wave function of a particle at time $t=0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle)$, where $|u_1\rangle$ and $|u_2\rangle$ are the normalized eigenstates with eigenvalues $E_1$ and $E_2$ respectively, $(E_2 > E_1)$. The shortest time after which $|\psi(t)\rangle$ will become orthogonal to $|\psi(0)\rangle$ is

(a) $\frac{-\hbar\pi}{2(E_2 - E_1)}$ (b) $\frac{\hbar\pi}{E_2 - E_1}$ (c) $\frac{\sqrt{2}\hbar\pi}{E_2 - E_1}$ (d) $\frac{2\hbar\pi}{E_2 - E_1}$

Ans. : (b)

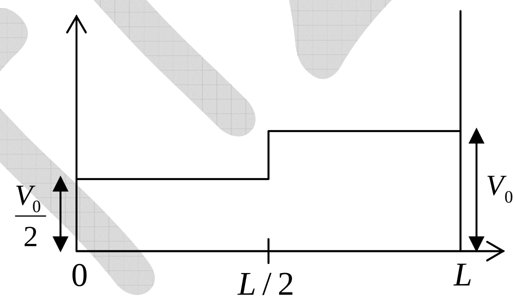
Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}(|u_1\rangle + |u_2\rangle) \Rightarrow |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left(|u_1\rangle e^{\frac{-iE_1 t}{\hbar}} + |u_2\rangle e^{\frac{-iE_2 t}{\hbar}}\right)$

$|\psi(t)\rangle$ is orthogonal to $|\psi(0)\rangle \Rightarrow \langle\psi(0)|\psi(t)\rangle = 0 \Rightarrow \frac{1}{2}e^{\frac{-iE_1 t}{\hbar}} + \frac{1}{2}e^{\frac{-iE_2 t}{\hbar}} = 0$

$\Rightarrow e^{\frac{-iE_1 t}{\hbar}} + e^{\frac{-iE_2 t}{\hbar}} = 0 \Rightarrow e^{\frac{-iE_1 t}{\hbar}} = -e^{\frac{-iE_2 t}{\hbar}} \Rightarrow e^{i\frac{(E_2-E_1)}{\hbar}} = -1$

$\Rightarrow \cos\frac{(E_2 - E_1)t}{\hbar} = \cos\pi \Rightarrow t = \frac{\pi\hbar}{E_2 - E_1}$

Q12. A constant perturbation as shown in the figure below acts on a particle of mass $m$ confined in an infinite potential well between $0$ and $L$.




The first-order correction to the ground state energy of the particle is

(a) $\frac{V_0}{2}$ (b) $\frac{3V_0}{4}$ (c) $\frac{V_0}{4}$ (d) $\frac{3V_0}{2}$

Ans. : (b)

Solution: $E_1^1 = \langle \psi_1 | V_p | \psi_1 \rangle = \int_0^{\frac{L}{2}} \frac{V_0}{2} \frac{2}{L} \sin^2 \frac{\pi x}{L} dx + \int_{\frac{L}{2}}^{L} V_0 \frac{2}{L} \sin^2 \frac{\pi x}{L} dx$

$$E_1^1 = \frac{V_0}{L} \int_0^{\frac{L}{2}} \frac{1}{2}\left(1-\cos\frac{2\pi x}{L}\right) dx + \frac{2V_0}{L} \int_{\frac{L}{2}}^{L} \frac{1}{2}\left(1-\cos\frac{2\pi x}{L}\right) dx$$

$$\Rightarrow E_1^1 = \frac{V_0}{2L}\left(\frac{L}{2}\right) + \frac{2V_0}{2L}\left(L - \frac{L}{2}\right) = \frac{V_0}{4} + \frac{2V_0}{4} = \frac{3V_0}{4}$$

**NET/JRF (JUNE-2012)**

Q13. The component along an arbitrary direction $\hat{n}$, with direction cosines $(n_x, n_y, n_z)$, of the spin of a spin $-\frac{1}{2}$ particle is measured. The result is

(a) 0 (b) $\pm\frac{\hbar}{2} n_z$ (c) $\pm\frac{\hbar}{2}(n_x + n_y + n_z)$ (d) $\pm\frac{\hbar}{2}$

Ans. : (d)

Solution: $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$ and $n_x^2 + n_y^2 + n_z^2 = 1$, $\vec{S} = S_x \hat{i} + S_y \hat{j} + S_z \hat{k}$

$$\vec{n}\cdot\vec{S} = n_x \begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix} + n_y \begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix} + n_z \begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & \frac{-\hbar}{2} \end{pmatrix}$$

$$\vec{n}\cdot\vec{S} = \begin{pmatrix} n_z \frac{\hbar}{2} & \frac{\hbar}{2}(n_x - i n_y) \\ \frac{\hbar}{2}(n_x + i n_y) & -n_z \frac{\hbar}{2} \end{pmatrix}$$

Let $\lambda$ is eigen value of $\vec{n}\cdot\vec{S}$

$$\begin{vmatrix} n_z \frac{\hbar}{2} - \lambda & \frac{\hbar}{2}(n_x - i n_y) \\ \frac{\hbar}{2}(n_x + i n_y) & -n_z \frac{\hbar}{2} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow -\left(\frac{n_z\hbar}{2}-\lambda\right)\left(\frac{n_z\hbar}{2}+\lambda\right)-\frac{\hbar^2}{4}\left(n_x^2+n_y^2\right)=0 \Rightarrow -\left(\frac{n_z^2\hbar^2}{4}-\lambda^2\right)-\frac{\hbar^2}{4}\left(n_x^2+n_y^2\right)=0.$$

$$\Rightarrow -\frac{\hbar^2}{4}\left(n_x^2+n_y^2+n_z^2\right)+\lambda^2=0 \Rightarrow \lambda=\pm\frac{\hbar}{2}.$$

Q14. A particle of mass $m$ is in a cubic box of size $a$. The potential inside the box $(0 \le x < a, 0 \le y < a, 0 \le z < a)$ is zero and infinite outside. If the particle is in an eigenstate of energy $E=\frac{14\pi^2\hbar^2}{2ma^2}$, its wavefunction is

(a) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{3\pi x}{a}\sin\frac{5\pi y}{a}\sin\frac{6\pi z}{a}$
(b) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{7\pi x}{a}\sin\frac{4\pi y}{a}\sin\frac{3\pi z}{a}$

(c) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{4\pi x}{a}\sin\frac{8\pi y}{a}\sin\frac{2\pi z}{a}$
(d) $\psi=\left(\frac{2}{a}\right)^{3/2}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}\sin\frac{3\pi z}{a}$

Ans. : (d)

Solution: $E_{n_x,n_y,n_z}=\left(n_x^2+n_y^2+n_z^2\right)\frac{\pi^2\hbar^2}{2ma^2}=\frac{14\pi^2\hbar^2}{2ma^2}$

$$\Rightarrow n_x^2+n_y^2+n_z^2=14 \Rightarrow n_x=1, n_y=2, n_z=3.$$

Q15. Let $\psi_{nlm_l}$ denote the eigenfunctions of a Hamiltonian for a spherically symmetric potential $V(r)$. The wavefunction $\psi=\frac{1}{4}\left[\psi_{210}+\sqrt{5}\psi_{21-1}+\sqrt{10}\psi_{211}\right]$ is an eigenfunction only of

(a) $H$, $L^2$ and $L_z$ (b) $H$ and $L_z$ (c) $H$ and $L^2$ (d) $L^2$ and $L_z$

Ans. : (c)

Solution: $H\psi=E_n\psi$

$L^2\psi=l(l+1)\hbar^2\psi$ and $L_z\psi\neq m\hbar\psi$.

Q16. The commutator $\left[x^2, p^2\right]$ is

(a) $2i\hbar xp$ (b) $2i\hbar(xp+px)$ (c) $2i\hbar px$ (d) $2i\hbar(xp-px)$

Ans. : (b)

Solution: $\left[x^2,p^2\right]=x\left[x,p^2\right]+\left[x,p^2\right]x=xp\left[x,p\right]+x\left[x,p\right]p+p\left[x,p\right]x+\left[x,p\right]px$

$\left[x^2, p^2\right] = xp(i\hbar) + x(i\hbar)p + p(i\hbar)x + (i\hbar)px = 2i\hbar(xp + px)$.

Q17. A free particle described by a plane wave and moving in the positive $z$ -direction undergoes scattering by a potential

$$V(r) = \begin{cases} V_0, & if \quad r \le R \\ 0\ , & if \quad r > R \end{cases}$$

If $V_0$ is changed to $2V_0$, keeping $R$ fixed, then the differential scattering cross-section, in the Born approximation.

(a) increases to four times the original value

(b) increases to twice the original value

(c) decreases to half the original value

(d) decreases to one fourth the original value

Ans. : (a)

Solution: $V(r) = \begin{cases} V_0, & r \le R \\ 0, & r > R \end{cases}$

Low energy scattering amplitude $f(\theta,\phi) = -\frac{m}{2\pi\hbar^2} V_0 \frac{4}{3}\pi R^3$

And differential scattering is given by $\frac{d\sigma_1}{d\Omega} = |f|^2 = \left(\frac{2mV_0R^3}{3\hbar^2}\right)^2$

Now $V(r) = 2V_0$ for $r < R \Rightarrow \frac{d\sigma_2}{d\Omega} = \left(\frac{2m(2V_0)R^3}{3\hbar^2}\right)^2 = 4\left(\frac{2mV_0R^3}{3\hbar^2}\right)^2 = 4\frac{d\sigma_1}{d\Omega}$

Q18. A variational calculation is done with the normalized trial wavefunction $\psi(x) = \frac{\sqrt{15}}{4a^{5/2}}(a^2 - x^2)$ for the one-dimensional potential well

$$V(x) = \begin{cases} 0 & if\ |x| \le a \\ \infty & if\ |x| > a \end{cases}$$

The ground state energy is estimated to be

(a) $\frac{5\hbar^2}{3ma^2}$ (b) $\frac{3\hbar^2}{2ma^2}$ (c) $\frac{3\hbar^2}{5ma^2}$ (d) $\frac{5\hbar^2}{4ma^2}$

Ans. : (d)

Solution: $\psi(x) = \frac{\sqrt{15}}{4a^{\frac{5}{2}}}(a^2 - x^2)$, $\quad V(x) = 0, |x| \le a$ and $\mathrm{V}(\mathrm{x}) = \infty$, $|x| > a$

$$\langle E\rangle=\int_{-a}^{a}\psi H\psi dx \text{ where } H=\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

$$\langle E\rangle=\int_{-a}^{a}\left[\frac{\sqrt{15}}{4a^{5/2}}\left(a^2-x^2\right)\right]\left[\frac{-\hbar^2}{2m}\frac{d^2}{dx^2}\left\{\frac{\sqrt{15}}{4a^{5/2}}\left(a^2-x^2\right)\right\}\right]dx \quad =\frac{15}{16a^5}\frac{-\hbar^2}{2m}\int_{-a}^{a}\left(a^2-x^2\right)(-2)dx$$

$$\Rightarrow\langle \mathrm{E}\rangle=\frac{15}{16\mathrm{a}^5}\frac{2\hbar^2}{2\mathrm{m}}\int_{-\mathrm{a}}^{\mathrm{a}}\left(\mathrm{a}^2-\mathrm{x}^2\right)\mathrm{dx}=\frac{15}{16a^5}\frac{\hbar^2}{m}\frac{4a^3}{3}=\frac{5\hbar^2}{4ma^2}$$

Q19. A particle in one-dimension is in the potential

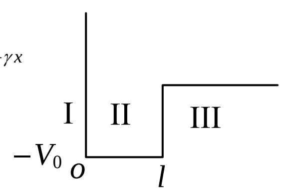
$$V(x)=\begin{cases}\infty & , \text{ if } \quad x<0\\ -V_0 & , \text{ if } \quad 0\le x\le l\\ 0 & , \text{ if } \quad x>l\end{cases}$$

If there is at least one bound state, the minimum depth of potential is

(a) $\frac{\hbar^2\pi^2}{8ml^2}$ (b) $\frac{\hbar^2\pi^2}{2ml^2}$ (c) $\frac{2\hbar^2\pi^2}{ml^2}$ (d) $\frac{\hbar^2\pi^2}{ml^2}$

Ans. : (a)

Solution: For bound state, $-V_0<E<0$




Wave function in region I, $\psi_I=0$, $\psi_{II}=A\sin kx+B\cos kx$, $\psi_{III}=ce^{-\gamma x}$

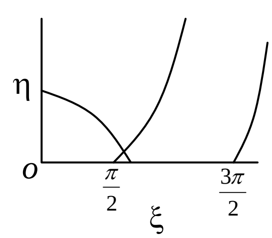
where $k=\frac{\sqrt{2m(V_0+E)}}{\hbar^2}$, $\gamma=\frac{\sqrt{2m(-E)}}{\hbar^2}$.

Use Boundary condition at $x=0$ and $x=l$

(wave function is continuous and differential at $x=0$ and $x=l$), one will get

$k\cot kl=-\gamma \Rightarrow kl\cot kl=-\gamma l \Rightarrow \eta=-\xi\cot\xi$ where $\gamma l=\eta$, $kl=\xi$.

$$\Rightarrow \eta^2+\xi^2=\frac{2mV_0l^2}{\hbar^2}$$




For one bound state $\left(\frac{2mV_0l^2}{\hbar^2}\right)^{1/2}=\frac{\pi}{2} \Rightarrow V_0=\frac{\pi^2\hbar^2}{8ml^2}$.

Q20. Which of the following is a self-adjoint operator in the spherical polar coordinate system $(r,\theta,\phi)$?

(a) $-\frac{i\hbar}{\sin^2\theta}\frac{\partial}{\partial\theta}$ (b) $-i\hbar\frac{\partial}{\partial\theta}$ (c) $-\frac{i\hbar}{\sin\theta}\frac{\partial}{\partial\theta}$ (d) $-i\hbar\sin\theta\frac{\partial}{\partial\theta}$

Ans. : (c)

Solution: $\frac{-i\hbar}{\sin\theta}\frac{\partial}{\partial\theta}$ is Hermitian.

## NET/JRF (DEC-2012)

Q21. Let $v$, $p$ and $E$ denote the speed, the magnitude of the momentum, and the energy of a free particle of rest mass $m$. Then

(a) $dE/dp =$ constant

(b) $p = mv$

(c) $v = cp/\sqrt{p^2+m^2c^2}$

(d) $E = mc^2$

Ans. : (c)

Solution: $p = m'v = \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow p^2 = \frac{m^2v^2}{1-\frac{v^2}{c^2}} \Rightarrow m^2v^2 = p^2 - \frac{p^2v^2}{c^2}$, $m \to$ rest mass energy

$$\Rightarrow v^2\left(m^2+\frac{p^2}{c^2}\right) = p^2 \Rightarrow v^2 = \frac{p^2}{\frac{m^2c^2+p^2}{c^2}} \Rightarrow v = \frac{pc}{\sqrt{p^2+m^2c^2}}$$

Q22. The wave function of a state of the Hydrogen atom is given by,

$$\psi = \psi_{200} + 2\psi_{211} + 3\psi_{210} + \sqrt{2}\psi_{21-1}$$

where $\psi_{nlm}$ is the normalized eigen function of the state with quantum numbers $n, l, m$ in the usual notation. The expectation value of $L_z$ in the state $\psi$ is

(a) $\frac{15\hbar}{6}$

(b) $\frac{11\hbar}{6}$

(c) $\frac{3\hbar}{8}$

(d) $\frac{\hbar}{8}$

Ans. : (d)

Solution: Firstly normalize $\psi$, $\psi = \frac{1}{\sqrt{16}}\psi_{200} + \frac{2}{\sqrt{16}}\psi_{211} + \frac{3}{\sqrt{16}}\psi_{210} + \frac{\sqrt{2}}{\sqrt{16}}\psi_{21-1}$

$$P(0\hbar) = \frac{1}{16} + \frac{9}{16} = \frac{10}{16}.$$

Probability of getting $(1\hbar)$ i.e. $P(\hbar) = \frac{4}{16}$ and $P(-\hbar) = \frac{2}{16}$.

Now, $\langle L_z\rangle = \frac{\langle\psi|L_z|\psi\rangle}{\langle\psi|\psi\rangle} = 0\hbar\times\frac{10}{16} + 1\hbar\times\frac{4}{16} + (-1\hbar)\times\frac{2}{16} = \frac{4}{16}\hbar - \frac{2}{16}\hbar = \frac{2}{16}\hbar = \frac{\hbar}{8}$

Q23. The energy eigenvalues of a particle in the potential $V(x)=\frac{1}{2}m\omega^2x^2-ax$ are

(a) $En=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{a^2}{2m\omega^2}$ (b) $En=\left(n+\frac{1}{2}\right)\hbar\omega+\frac{a^2}{2m\omega^2}$

(c) $En=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{a^2}{m\omega^2}$ (d) $En=\left(n+\frac{1}{2}\right)\hbar\omega$

Ans. : (a)

Solution: Hamiltonian $(H)$ of Harmonic oscillator, $H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2x^2$

Eigenvalue of this, $E_n=\left(n+\frac{1}{2}\right)\hbar\omega$

But here, $H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2x^2-ax\Rightarrow H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2\left[x^2-\frac{2ax}{m\omega^2}+\frac{a^2}{m^2\omega^4}\right]-\frac{a^2}{2m\omega^2}$

$$H=\frac{p_x^2}{2m}+\frac{1}{2}m\omega^2\left[x-\frac{a}{m\omega^2}\right]^2-\frac{a^2}{2m\omega^2}$$

Energy eigenvalue, $E_n=\left(n+\frac{1}{2}\right)\hbar\omega-\frac{a^2}{2m\omega^2}$

Q24. If a particle is represented by the normalized wave function

$$\psi(x)=\begin{cases}\frac{\sqrt{15}\left(a^2-x^2\right)}{4a^{5/2}}, & \text{for } -a<x<a\\ 0, & \text{otherwise}\end{cases}$$

the uncertainty $\Delta p$ in its momentum is

(a) $2\hbar/5a$ (b) $5\hbar/2a$ (c) $\sqrt{10}\hbar/a$ (d) $\sqrt{5}\hbar/\sqrt{2}a$

Ans. : (d)

Solution: $\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}$ and $\langle p\rangle=\frac{\langle\psi|-i\hbar\frac{\partial}{\partial x}|\psi\rangle}{\langle\psi|\psi\rangle}$

$$\Rightarrow\langle p\rangle=\int_{-a}^{a}\frac{\sqrt{15}\left(a^2-x^2\right)}{4a^{5/2}}(-i\hbar)\frac{\sqrt{15}}{4a^{5/2}}\frac{\partial}{\partial x}\left(a^2-x^2\right)dx$$

$$=\int_{-a}^{a}\frac{15}{16a^5}(-i\hbar)\left(a^2-x^2\right)(-2x)dx=+i\hbar\frac{2\times15}{16\times a^5}\int_{-a}^{a}\left(a^2x-x^3\right)dx=0,\ (\because\text{ odd function})$$

$$\langle p^2\rangle = -\hbar^2\times\frac{15}{16a^5}\int_{-a}^{a}(a^2-x^2)\frac{\partial^2}{\partial x^2}(a^2-x^2)dx$$

$$=-\hbar^2\times\frac{15}{16a^5}\times(-2)\int_{-a}^{a}(a^2-x^2)dx \;=\hbar^2\times\frac{15}{16a^5}\times 2\left\{a^2\cdot x-\frac{x^3}{3}\right\}_{-a}^{a}$$

$$=\hbar^2\times\frac{15}{16a^5}\times 2\left[2a^3-\frac{2a^3}{3}\right]=\hbar^2\times\frac{15}{16}\times\frac{2}{a^5}\times 2a^3\left[1-\frac{1}{3}\right]=\frac{15\hbar^2}{4a^2}\times\frac{2}{3}=\frac{5\hbar^2}{2a^2}$$

Now, $\Delta p=\sqrt{\langle p^2\rangle-\langle p\rangle^2}=\sqrt{\frac{5\hbar^2}{2a^2}-0}=\frac{\sqrt{5}\hbar}{\sqrt{2}a}$

Q25. Given the usual canonical commutation relations, the commutator $[A,B]$ of $A=i(xp_y-yp_x)$ and $B=(yp_z+zp_y)$ is

(a) $\hbar(xp_z-p_xz)$ (b) $-\hbar(xp_z-p_xz)$ (c) $\hbar(xp_z+p_xz)$ (d) $-\hbar(xp_z+p_xz)$

Ans. : (c)

Solution: $[A,B]=[(ixp_y-iyp_x),(yp_z+zp_y)]$

$$[A,B]=i[xp_y,yp_z]-i[yp_x,yp_z]+i[xp_y,zp_y]-i[yp_x,zp_y]$$

$$[A,B]=i[xp_y,yp_z]-0+0-i[yp_x,zp_y]=i[xp_y,yp_z]-i[yp_x,zp_y]$$

$$[A,B]=ix[p_y,yp_z]+i[x,yp_z]p_y-iy[p_x,zp_y]-i[y,zp_y]p_x$$

$$[A,B]=ix[p_y,yp_z]+0-0-i[y,zp_y]p_x=ix[p_y,yp_z]-i[y,zp_y]p_x$$

$$[A,B]=ix\times(-i\hbar)\,p_z-izi\hbar\times p_x=\hbar[xp_z+zp_x]$$

$$[A,B]=\hbar(xp_z+p_xz)$$

Q26. The energies in the ground state and first excited state of a particle of mass $m=\frac{1}{2}$ in a potential $V(x)$ are $-4$ and $-1$, respectively, (in units in which $\hbar=1$). If the corresponding wavefunctions are related by $\psi_1(x)=\psi_0(x)\sinh x$, then the ground state eigenfunction is

(a) $\psi_0(x)=\sqrt{\sec hx}$ (b) $\psi_0(x)=\sec hx$

(c) $\psi_0(x)=\sec h^2x$ (d) $\psi_0(x)=\sec h^3x$

Ans. : (c)

Solution: Given that ground state energy $E_0 = -4$, first excited state energy $E_1 = -1$ and $\psi_0$, $\psi_1$ are corresponding wave functions.

Solving Schrödinger equation (use $m = \frac{1}{2}$ and $\hbar = 1$)

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi_0}{\partial x^2} + V\psi_0 = E_0\psi_0 \quad \Rightarrow -\frac{\partial^2 \psi_0}{\partial x^2} + V\psi_0 = -4\psi_0 \ldots\ldots(1)$$

$$\frac{-\hbar^2}{2m}\frac{\partial^2 \psi_1}{\partial x^2} + V\psi_1 = E_1\psi_1 \quad \Rightarrow -\frac{\partial^2 \psi_1}{\partial x^2} + V\psi_1 = -1\psi_1 \ldots\ldots..(2)$$

Put $\psi_1 = \psi_0 \sinh x$ in equation (2) one will get

$$-\left[\frac{\partial^2 \psi_0}{\partial x^2}.\sinh x + 2\frac{\partial \psi_0}{\partial x}\cosh x + \psi_0 \sinh x\right] + V\psi_0 \sinh x = -\psi_0 \sinh x$$

$$-\left[\frac{\partial^2 \psi_0}{\partial x^2} + 2\frac{\partial \psi_0}{\partial x}\coth x + \psi_0\right] + V\psi_0 = -\psi_0$$

$$\left[-\frac{\partial^2 \psi_0}{\partial x^2} + V\psi_0\right] - 2\frac{\partial \psi_0}{\partial x}\coth x - \psi_0 = -\psi_0 \text{ using relation } -\frac{\partial^2 \psi_0}{\partial x^2} + V\psi_0 = -4\psi_0$$

$$-4\psi_0 - 2\frac{\partial \psi_0}{\partial x}\coth x - \psi_0 = -\psi_0 \quad \Rightarrow \frac{d\psi_0}{\psi_0} = -2\tanh x dx \Rightarrow \psi_0 = \sec h^2 x.$$

**NET/JRF (JUNE-2013)**

Q27. In a basis in which the $z$ - component $S_z$ of the spin is diagonal, an electron is in a spin state $\psi = \begin{pmatrix} (1+i)/\sqrt{6} \\ \sqrt{2/3} \end{pmatrix}$. The probabilities that a measurement of $S_z$ will yield the values $\hbar/2$ and $-\hbar/2$ are, respectively,

(a) $1/2$ and $1/2$ (b) $2/3$ and $1/3$ (c) $1/4$ and $3/4$ (d) $1/3$ and $2/3$

Ans. : (d)

Solution: Eigen state of $S_z$ is $|\phi_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|\phi_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$ corresponds to Eigen value $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively.

$$P\left(\frac{\hbar}{2}\right) = \frac{|\langle \phi_1|\psi\rangle|^2}{\langle \psi|\psi\rangle} = \left|\frac{1+i}{\sqrt{6}}\right|^2 = \frac{2}{6} = \frac{1}{3}, \quad P\left(-\frac{\hbar}{2}\right) = \frac{|\langle \phi_2|\psi\rangle|^2}{\langle \psi|\psi\rangle} = \frac{2}{3}$$

Q28. Consider the normalized state $|\psi\rangle$ of a particle in a one-dimensional harmonic oscillator:

$$|\psi\rangle = b_1|0\rangle + b_2|1\rangle$$

where $|0\rangle$ and $|1\rangle$ denote the ground and first excited states respectively, and $b_1$ and $b_2$ are real constants. The expectation value of the displacement $x$ in the state $|\psi\rangle$ will be a minimum when

(a) $b_2 = 0, b_1 = 1$ (b) $b_2 = \frac{1}{\sqrt{2}} b_1$ (c) $b_2 = \frac{1}{2} b_1$ (d) $b_2 = b_1$

Ans. : (d)

Solution: $\langle x\rangle = b_1^2\langle 0|x|0\rangle + b_2^2\langle 1|x|1\rangle + 2b_1 b_2\langle 0|x|1\rangle$

Since $\langle 0|x|0\rangle = 0$ and $\langle 1|x|1\rangle = 0 \Rightarrow \langle x\rangle = 2b_1 b_2\langle 0|x|1\rangle$.

Min of $\langle x\rangle$ means min $2b_1 b_2$. We know that $b_1^2 + b_2^2 = 1$.

$\langle x\rangle_{\min} = \left[(b_1+b_2)^2 - (b_1^2+b_2^2)\right]\langle 0|x|1\rangle = \left[(b_1+b_2)^2 - 1\right]\langle 0|x|1\rangle \Rightarrow \left[1-(b_1-b_2)^2\right]\langle 0|x|1\rangle$ will be minimum and minimum value of $\left[1-(b_1-b_2)^2\right]$, there must be maximum of $(b_1-b_2)^2$, so $\Rightarrow b_1 = b_2$

Q29. The un-normalized wavefunction of a particle in a spherically symmetric potential is given by

$$\psi(\vec{r}) = zf(r)$$

where $f(r)$ is a function of the radial variable $r$. The eigenvalue of the operator $\vec{L}^2$ (namely the square of the orbital angular momentum) is

(a) $\hbar^2/4$ (b) $\hbar^2/2$ (c) $\hbar^2$ (d) $2\hbar^2$

Ans. : (d)

Solution: $\psi(r) = zf(r) = r\cos\theta\, f(r)$

$\psi\left(r = Y_1^0(\theta, \phi)\right)$, $L^2\psi(r) = L^2 Y_1^0(\theta, \phi)$, where $l = 1$

$L^2 = l(l+1)\hbar^2 = 1(1+1)\hbar^2 = 2\hbar^2$

Q30. If $\psi_{nlm}$ denotes the eigenfunciton of the Hamiltonian with a potential $V = V(r)$ then the expectation value of the operator $L_x^2 + L_y^2$ in the state

$$\psi = \frac{1}{5}\left[3\psi_{211} + \psi_{210} - \sqrt{15}\psi_{21-1}\right]$$

is

(a) $39\hbar^2/25$ (b) $13\hbar^2/25$ (c) $2\hbar^2$ (d) $26\hbar^2/25$

Ans. : (d)

Solution: $L_x^2 + L_y^2 = L^2 - L_z^2 \quad \Rightarrow \langle L_x^2 + L_y^2\rangle = \langle L^2 - L_z^2\rangle = \langle L^2\rangle - \langle L_z^2\rangle$

$$\langle L^2\rangle - \langle L_z^2\rangle = 2\hbar^2 - \left(\frac{9}{25}\times 1\hbar^2 + \frac{1}{25}\times 0\hbar^2 + \frac{15}{25}\times 1\hbar^2\right)$$

$$\langle L^2\rangle - \langle L_z^2\rangle = 2\hbar^2 - \frac{24}{25}\hbar^2 \quad = \frac{50-24}{25}\hbar^2 = \frac{26}{25}\hbar^2$$

Q31. Consider a two-dimensional infinite square well

$$V(x,y) = \begin{cases} 0, & 0 < x < a, \quad 0 < y < a \\ \infty, & \text{otherwise} \end{cases}$$

Its normalized Eigenfunctions are $\psi_{n_x,n_y}(x,y) = \frac{2}{a}\sin\left(\frac{n_x \pi x}{a}\right)\sin\left(\frac{n_y \pi y}{a}\right)$,

where $n_x, n_y = 1, 2, 3, ..$

If a perturbation $H' = \begin{cases} V_0 & 0 < x < \frac{a}{2}, \quad 0 < y < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases}$ is applied, then the correction to the energy of the first excited state to order $V_0$ is

(a) $\frac{V_0}{4}$

(b) $\frac{V_0}{4}\left[1 \pm \frac{64}{9\pi^2}\right]$

(c) $\frac{V_0}{4}\left[1 \pm \frac{16}{9\pi^2}\right]$

(d) $\frac{V_0}{4}\left[1 \pm \frac{32}{9\pi^2}\right]$

Ans. : (b)

Solution: For first excited state, which is doubly degenerate

$$|\phi_1\rangle = \frac{2}{a}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{a}, |\phi_2\rangle = \frac{2}{a}\sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{\pi y}{a}\right)$$

$$H_{11} = \langle \phi_1 | H | \phi_1 \rangle = V_0 \frac{2}{a}\int_0^{a/2} \sin^2\left(\frac{\pi x}{a}\right) dx \frac{2}{a}\int_0^{a/2} \sin^2\left(\frac{2\pi y}{a}\right) dy = V_0 \cdot \frac{1}{2}\cdot\frac{1}{2} = \frac{V_0}{4}$$

$$H_{12} = \langle \phi_1 | H | \phi_2 \rangle \qquad = V_0 \frac{2}{a}\int_0^{a/2} \sin\frac{\pi x}{a} \sin\frac{2\pi x}{a} dx \frac{2}{a}\int_0^{a/2} \sin\frac{2\pi y}{a}\sin\frac{\pi y}{a} dy$$

$$H_{12} = V_0\left(\frac{4}{3\pi}\right)\left(\frac{4}{3\pi}\right) = V_0\frac{16}{9\pi^2}, \; H_{21} = \langle \phi_2 | H' | \phi_1 \rangle = V_0 \frac{16}{9\pi^2} \text{ and } H_{22} = \langle \phi_2 | H' | \phi_2 \rangle = \frac{V_0}{4}.$$

Thus $\begin{pmatrix} \frac{V_0}{4} - \lambda & \frac{16V_0}{9\pi^2} \\ \frac{16V_0}{9\pi^2} & \frac{V_0}{4} - \lambda \end{pmatrix} = 0 \Rightarrow \left(\frac{V_0}{4} - \lambda\right)^2 - \left(\frac{16V_0}{9\pi^2}\right)^2 = 0$

$$\Rightarrow \left(\frac{V_0}{4} - \lambda\right) = \pm \frac{16V_0}{9\pi^2} \Rightarrow \lambda = \frac{V_0}{4}\left(1 \pm \frac{64}{9\pi^2}\right)$$

Q32. The bound on the ground state energy of the Hamiltonian with an attractive delta-function potential, namely

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - a\delta(x)$$

using the variational principle with the trial wavefunction $\psi(x) = A\exp(-bx^2)$ is

$\left[\text{Note}: \int_0^\infty e^{-t} t^a dt = \Gamma(a+1)\right]$

(a) $-ma^2/4\pi\hbar^2$ (b) $-ma^2/2\pi\hbar^2$ (c) $-ma^2/\pi\hbar^2$ (d) $-ma^2/\sqrt{5}\pi\hbar^2$

Ans. : (c)

Solution: For given wavefunction $\langle T \rangle = \frac{\hbar^2 b}{2m}$ and $\langle V \rangle = -a\sqrt{\frac{2b}{\pi}} \Rightarrow \langle E \rangle = \frac{\hbar^2 b}{2m} - a\sqrt{\frac{2b}{\pi}}$

For variation of parameter $\frac{d\langle E \rangle}{db} = 0 \Rightarrow \frac{d\langle E \rangle}{db} = \frac{\hbar^2}{2m} - a\sqrt{\frac{2}{\pi}} \times \frac{1}{2} b^{-1/2} = 0 \Rightarrow b = \frac{2m^2a^2}{\pi\hbar^4}$.

$$\Rightarrow \langle E \rangle_{\min} = -\frac{ma^2}{\pi\hbar^2}.$$

Q33. If the operators $A$ and $B$ satisfy the commutation relation $[A, B] = I$, where $I$ is the identity operator, then

(a) $[e^A, B] = e^A$ (b) $[e^A, B] = [e^B, A]$

(c) $[e^A, B] = [e^{-B}, A]$ (d) $[e^A, B] = I$

Ans. : (a)

Solution: $[A,B]=I$ and $e^A=\left[1+\frac{A}{1}+\frac{A^2}{\lfloor 2}+.......\right]$

$$\left[e^A,B\right]=\left[1+\frac{A}{1}+\frac{A^2}{\lfloor 2}+.......,B\right]=[1,\ B]+[A,\ B]+\frac{\left[A^2,\ B\right]}{\lfloor 2}+\frac{\left[A^3,\ B\right]}{\lfloor 3}....$$

$$\left[e^A,B\right]=0+I+\frac{A[A,\ B]+[A,\ B]A}{2!}+\frac{A\left[A^2,B\right]+\left[A^2,B\right]A}{3!}+....$$

$\left[e^A,\ B\right]=1+A+\frac{A^2}{2!}+....=e^A$ where $[A,B]=I$, $\left[A^2,B\right]=2A$ and $\left[A^3,B\right]=3A^2$.

Q34. Two identical bosons of mass $m$ are placed in a one-dimensional potential $V(x)=\frac{1}{2}m\omega^2x^2$. The bosons interact via a weak potential,

$$V_{12}=V_0\exp\left[-m\Omega\left(x_1-x_2\right)^2/4\hbar\right]$$

where $x_1$ and $x_2$ denote coordinates of the particles. Given that the ground state wavefunction of the harmonic oscillator is $\psi_0(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{-\frac{m\omega x^2}{2\hbar}}$. The ground state energy of the two-boson system, to the first order in $V_0$, is

(a) $\hbar\omega+2V_0$

(b) $\hbar\omega+\frac{V_0\Omega}{\omega}$

(c) $\hbar\omega+V_0\left(1+\frac{\Omega}{2\omega}\right)^{-\frac{1}{2}}$

(d) $\hbar\omega+V_0\left(1+\frac{\omega}{\Omega}\right)$

Ans. : (c)

Solution: There are two bosons trapped in harmonic oscillator.

So, energy for ground state without perturbation is, $2\cdot\frac{\hbar\omega}{2}=\hbar\omega$.

If perturbation is introduced, we have to calculate $\langle V_{1,2}\rangle$

where $V_{1,2}=V_0\exp\left[-m\Omega\left(x_1-x_2\right)^2/4\hbar\right]$.

But calculating $\langle V_{1,2}\rangle$ on state $\psi_0(x)=\left(\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}e^{-\frac{m\omega x_1^2}{2\hbar}}e^{-\frac{m\omega x_2^2}{2\hbar}}\right)$ is very tedious task.

So lets use a trick i.e perturbation is nothing but approximation used in Taylor series. So just expand $V_{1,2}=V_0\exp\left[-m\Omega\left(x_1-x_2\right)^2/4\hbar\right]$ and take average value of first term

$$V_{1,2}=V_0\exp\left[-m\Omega\left(x_1-x_2\right)^2/4\hbar\right]=V_0\left(1-\frac{m\Omega\left(x_1-x_2\right)^2}{4\hbar}+\ldots\right)$$

$$=V_0\left(1-\frac{m\Omega\left(x_1^2+x_2^2-2x_1.x_2\right)}{4\hbar}+\ldots\right)$$

$$\langle V_{1,2}\rangle=V_0\left(1-\frac{m\Omega\left(\langle x_1^2\rangle+\langle x_2^2\rangle-2\langle x_1\rangle.\langle x_2\rangle\right)}{4\hbar}+\ldots\right)=V_o\left(1-\frac{m\Omega\left(\frac{\hbar}{2m\omega}+\frac{\hbar}{2m\omega}-0\right)}{4\hbar}\right)\ldots$$

$$\Rightarrow\langle V_{12}\rangle=V_o(1-\frac{\Omega}{4\omega})\;\approx V_0\left(1+\frac{\Omega}{2\omega}\right)^{\frac{-1}{2}}\text{, so }E=\hbar\omega+V_0\left(1+\frac{\Omega}{2\omega}\right)^{\frac{-1}{2}}.$$

## NET/JRF (DEC-2013)

Q35. A spin -$\frac{1}{2}$ particle is in the state $\chi=\frac{1}{\sqrt{11}}\begin{pmatrix}1+i\\3\end{pmatrix}$ in the eigenbasis of $S^2$ and $S_z$. If we measure $S_z$, the probabilities of getting $+\frac{h}{2}$ and $-\frac{h}{2}$, respectively are

(a) $\frac{1}{2}$ and $\frac{1}{2}$ (b) $\frac{2}{11}$ and $\frac{9}{11}$ (c) 0 and 1 (d) $\frac{1}{11}$ and $\frac{3}{11}$

Ans. : (b)

Solution: $P\left(\frac{\hbar}{2}\right)=\left|\frac{1}{\sqrt{11}}(10)\begin{pmatrix}1+i\\3\end{pmatrix}\right|^2=\frac{1}{11}\times 2=\frac{2}{11}$ $\qquad\because\langle\psi|\psi\rangle=1$

$$P\left(-\frac{\hbar}{2}\right)=\left|\frac{1}{\sqrt{11}}(01)\begin{pmatrix}1+i\\3\end{pmatrix}\right|^2=\frac{9}{11}$$

i.e. probability of $S_z$ getting $\left(\frac{\hbar}{2}\right)$ and $\left(-\frac{\hbar}{2}\right)$

Q36. The motion of a particle of mass $m$ in one dimension is described by the Hamiltonian $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x$. What is the difference between the (quantized) energies of the first two levels? (In the following, $\langle x \rangle$ is the expectation value of $x$ in the ground state)

(a) $\hbar\omega - \lambda\langle x \rangle$ (b) $\hbar\omega + \lambda\langle x \rangle$ (c) $\hbar\omega + \frac{\lambda^2}{2m\omega^2}$ (d) $\hbar\omega$

Ans. : (d)

Solution: $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 + \lambda x \Rightarrow V(x) = \frac{1}{2}m\omega^2 x^2 + \lambda x$

$$V(x) = \frac{1}{2}m\omega^2\left[x^2 + \frac{2}{m\omega^2}\lambda x\right] = \frac{1}{2}m\omega^2\left[x^2 + 2\cdot x\cdot\frac{\lambda}{m\omega^2} + \frac{\lambda^2}{m^2\omega^4} - \frac{\lambda^2}{m^2\omega^4}\right]$$

$$V(x) = \frac{1}{2}m\omega^2\left(x + \frac{\lambda}{m\omega^2}\right)^2 - \frac{\lambda^2}{2m\omega^2}$$

$$\therefore\ E_n = \left(n + \frac{1}{2}\right)\hbar\omega - \frac{\lambda^2}{2m\omega^2} \Rightarrow E_1 - E_0 = \frac{3}{2}\hbar\omega - \frac{1}{2}\hbar\omega = \hbar\omega$$

Q37. Let $\psi_{nlm}$ denote the eigenfunctions of a Hamiltonian for a spherically symmetric potential $V(r)$. The expectation value of $L_z$ in the state

$$\psi = \frac{1}{6}\left[\psi_{200} + \sqrt{5}\psi_{210} + \sqrt{10}\psi_{21-1} + \sqrt{20}\psi_{211}\right] \text{ is}$$

(a) $-\frac{5}{18}\hbar$ (b) $\frac{5}{6}\hbar$ (c) $\hbar$ (d) $\frac{5}{18}\hbar$

Ans. : (d)

Solution: $\langle L_z \rangle = \langle \psi | L_z | \psi \rangle = \frac{1}{36}\times 0\hbar + \frac{5}{36}\times 0\hbar + \frac{10}{36}\times(-1\hbar) + \frac{20}{36}(1\hbar) = \frac{10}{36}\hbar = \frac{5}{18}\hbar \quad \because \langle \psi | \psi \rangle = 1$

Q38. If $\psi(x) = A\exp(-x^4)$ is the eigenfunction of a one dimensional Hamiltonian with eigen value $E = 0$, the potential $V(x)$ (in units where $\hbar = 2m = 1$) is

(a) $12x^2$ (b) $16x^6$ (c) $16x^6 + 12x^2$ (d) $16x^6 - 12x^2$

Ans. : (d)

Solution: Schrodinger equation

$$-\nabla^2\psi + V\psi = 0 \text{ (where } \hbar = 2m = 1 \text{ and } E = 0\text{)}$$

$$-\frac{\partial^2}{\partial x^2}\left(Ae^{-x^4}\right) + VAe^{-x^4} = 0 \Rightarrow -\frac{\partial}{\partial x}\left[e^{-x^4}\times -4x^3\right] + Ve^{-x^4} = 0$$

$$4\left[\left\{3x^2e^{-x^4}+x^3\left(-4x^3e^{-x^4}\right)\right\}\right]+Ve^{-x^4}=0\Rightarrow 12x^2e^{-x^4}-16x^6e^{-x^4}+Ve^{-x^4}=0$$

$$\Rightarrow V=16x^6-12x^2$$

Q39. A particle is in the ground state of an infinite square well potential is given by,

$$V(x)=\begin{cases}0 & \text{for } -a\le x\le a\\ \infty & \text{otherwise}\end{cases}$$

The probability to find the particle in the interval between $-\frac{a}{2}$ and $\frac{a}{2}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{2}+\frac{1}{\pi}$ (c) $\frac{1}{2}-\frac{1}{\pi}$ (d) $\frac{1}{\pi}$

Ans. : (b)

Solution: The probability to find the particle in the interval between $-\frac{a}{2}$ and $\frac{a}{2}$ is

$$=\int_{-a/2}^{a/2}\sqrt{\frac{2}{2a}}\cdot\sqrt{\frac{2}{2a}}\cos\frac{\pi x}{2a}\cdot\cos\frac{\pi x}{2a}dx=\int_{-a/2}^{a/2}\frac{1}{a}\cos^2\frac{\pi x}{2a}dx=\frac{1}{a}\times\frac{1}{2}\left[\int_{-a/2}^{a/2}\left(1+\cos\frac{2\pi x}{2a}\right)dx\right]$$

$$=\frac{1}{2a}\left[x+\frac{a}{\pi}\sin\frac{\pi x}{a}\right]_{-a/2}^{a/2}=\frac{1}{2a}\left[\frac{a}{2}+\frac{a}{2}+\frac{a}{\pi}(1+1)\right]=\frac{1}{2a}\left[a+\frac{2a}{\pi}\right]=\left(\frac{1}{2}+\frac{1}{\pi}\right)$$

Q40. The expectation value of the $x$- component of the orbital angular momentum $L_x$ in the state $\psi=\frac{1}{5}\left[3\psi_{2,1,-1}+\sqrt{5}\psi_{2,1,0}-\sqrt{11}\psi_{2,1,+1}\right]$

(where $\psi_{nlm}$ are the eigenfunctions in usual notation), is

(a) $-\frac{\hbar\sqrt{10}}{25}\left(\sqrt{11}-3\right)$ (b) 0 (c) $\frac{\hbar\sqrt{10}}{25}\left(\sqrt{11}+3\right)$ (d) $\hbar\sqrt{2}$

Ans. : (a)

Solution: $L_-|l,m\rangle=\sqrt{l(l+1)-m(m-1)}\hbar|l,m-1\rangle$ and $L_+|l,m\rangle=\sqrt{l(l+1)-m(m+1)}\hbar|l,m+1\rangle$

$$L_x=\frac{L_++L_-}{2}\quad\Rightarrow\langle L_x\rangle=\frac{\langle L_+\rangle+\langle L_-\rangle}{2}$$

$$L_+\psi=\frac{1}{5}\left[3\sqrt{2}\hbar\psi_{210}+\sqrt{5}\sqrt{2}\hbar\psi_{211}\right]$$

$$\langle\psi|L_+|\psi\rangle=\frac{1}{25}.3\sqrt{10}\hbar-\frac{1}{25}\sqrt{110}\hbar=\frac{1}{25}\sqrt{10}(3-\sqrt{11})\hbar$$

$$L_{-}\psi = \frac{1}{5}\left[\sqrt{2}\hbar\sqrt{5}\psi_{21-1} - \sqrt{2}\hbar\sqrt{11}\psi_{210}\right]$$

$$\langle\psi|L_{-}|\psi\rangle = \frac{1}{25}.3\sqrt{10}\hbar - \frac{1}{25}\sqrt{10}\sqrt{11}\hbar$$

$$\langle L_x\rangle = \frac{\langle L_{+}\rangle + \langle L_{-}\rangle}{2} = \frac{1}{25}\sqrt{10}(3-\sqrt{11})\hbar$$

$$\langle\psi|L_x|\psi\rangle = \frac{1}{25}.3\sqrt{10}\hbar - \frac{1}{25}\sqrt{10}\sqrt{11}\hbar = -\frac{\hbar\sqrt{10}}{25}\left(\sqrt{11}-3\right)$$

Q41. A particle is prepared in a simultaneous eigenstate of $L^2$ and $L_z$. If $l(l+1)\hbar^2$ and $m\hbar$ are respectively the eigenvalues of $L^2$ and $L_z$, then the expectation value $\langle L_x^2\rangle$ of the particle in this state satisfies

(a) $\langle L_x^2\rangle = 0$

(b) $0 \le \langle L_x^2\rangle \le \ell^2\hbar^2$

(c) $0 \le \langle L_x^2\rangle \le \frac{\ell(\ell+1)\hbar^2}{2}$

(d) $\frac{\ell\hbar^2}{2} \le \langle L_x^2\rangle \le \frac{\ell(\ell+1)\hbar^2}{2}$

Ans. : (d)

Solution: $\langle L_x^2\rangle = \frac{1}{2}\left(l(l+1)\hbar^2 - m^2\hbar^2\right)$

For max value $m=0$ and for min $m=l$

$$\frac{l\hbar^2}{2} \le \langle L_x^2\rangle \le \frac{l(l+1)\hbar^2}{2}$$

$A, B, C$ are Non zero Hermitian operator.

$[A,B] = C \Rightarrow AB - BA \Rightarrow AB - Ab = 0 = C$

but $C \neq 0$

if $AB = BA$ i.e. $[A,B] = C$ false (2)

**NET/JRF (JUNE-2014)**

Q42. Consider a system of two non-interacting identical fermions, each of mass $m$ in an infinite square well potential of width $a$. (Take the potential inside the well to be zero and ignore spin). The composite wavefunction for the system with total energy $E = \frac{5\pi^2\hbar^2}{2ma^2}$ is

(a) $\frac{2}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$

(b) $\frac{2}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$

(c) $\frac{2}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{2a}\right) - \sin\left(\frac{3\pi x_1}{2a}\right)\sin\left(\frac{\pi x_2}{a}\right)\right]$

(d) $\frac{2}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right) - \sin\left(\frac{\pi x_2}{a}\right)\cos\left(\frac{\pi x_2}{a}\right)\right]$

Ans. : (a)

Solution: Fermions have antisymmetric wave function

$$\psi(x_1x_2) = \frac{2}{a}\left[\sin\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \sin\left(\frac{2\pi x_1}{a}\right)\cdot\sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$\because E_n = \frac{5\pi^2\hbar^2}{2ma^2} \Rightarrow n_{x_1} = 1,\ n_{x_2} = 2$$

Q43. A particle of mass $m$ in the potential $V(x,y) = \frac{1}{2}m\omega^2\left(4x^2 + y^2\right)$, is in an eigenstate of energy $E = \frac{5}{2}\hbar\omega$. The corresponding un-normalized eigen function is

(a) $y\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2 + y^2\right)\right]$ (b) $x\exp\left[-\frac{m\omega}{2\hbar}\left(2x^2 + y^2\right)\right]$

(c) $y\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + y^2\right)\right]$ (d) $xy\exp\left[-\frac{m\omega}{2\hbar}\left(x^2 + y^2\right)\right]$

Ans. : (a)

Solution: $V(x,y) = \frac{1}{2}m\omega^2\left(4x^2 + y^2\right)$, $E = \frac{5}{2}\hbar\omega$

$\Rightarrow V(x,y)=\frac{1}{2}m(2\omega)^2x^2+\frac{1}{2}m\omega^2y^2$

Now, $E_n=\left(n_x+\frac{1}{2}\right)\hbar\omega_x+\left(n_y+\frac{1}{2}\right)\hbar\omega_y=\left(n_x+\frac{1}{2}\right)2\hbar\omega+\left(n_y+\frac{1}{2}\right)\hbar\omega$

$\Rightarrow E_n=\left(2n_x+n_y+\frac{3}{2}\right)\hbar\omega$

$\because E_n=\frac{5}{2}\hbar\omega$ when $n_x=0$ and $n_y=1$ .

Q44. A particle of mass $m$ in three dimensions is in the potential

$$V(r)=\begin{cases}0, & r<a\\ \infty, & r>a\end{cases}$$

Its ground state energy is

(a) $\frac{\pi^2\hbar^2}{2ma^2}$ (b) $\frac{\pi^2\hbar^2}{ma^2}$ (c) $\frac{3\pi^2\hbar^2}{2ma^2}$ (d) $\frac{9\pi^2\hbar^2}{2ma^2}$

Ans. : (a)

Solution: $\left(-\frac{\hbar^2}{2m}\right)\frac{d^2u(r)}{dr^2}+\frac{l(l+1)}{2mr^2}+V(r)u(r)=Eu(r)$

$\frac{d^2u(r)}{dr^2}=-K^2u(r)\quad \because K=\sqrt{\frac{2mE}{\hbar^2}},\ l=0,\ V(r)=0$

$u(r)=A\sin Kr+B\cos Kr$

Using boundary condition, $B=0,$

$u(r)=A\sin Kr,\ r=a,\ u(r)=0\Rightarrow \sin Ka=0\Rightarrow Ka=n\pi\Rightarrow E=\frac{\pi^2\hbar^2}{2ma^2}\qquad \because n=1$

Q45. Given that $\hat{p}_r=-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)$, the uncertainty $\Delta p_r$ in the ground state

$\psi_0(r)=\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$ of the hydrogen atom is

(a) $\frac{\hbar}{a_0}$ (b) $\frac{\sqrt{2}\hbar}{a_0}$ (c) $\frac{\hbar}{2a_0}$ (d) $\frac{2\hbar}{a_0}$

Ans. : (a)

Solution: $\hat{p}_r = -i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)$, $\psi_0(r)=\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}$, $\Delta P_r=\sqrt{\langle P_r^2\rangle-\langle P_r\rangle^2}$

$$\text{Now}\langle P_r\rangle=\int_0^\infty \frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\left\{\left[-i\hbar\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\right]\frac{e^{-r/a_0}}{\sqrt{\pi a_0^3}}\right\}4\pi r^2dr$$

$$=-\frac{4\pi i\hbar}{\pi a_0^3}\left[\int_0^\infty e^{-r/a_0}\left(e^{-r/a_0}\left(-\frac{1}{a_0}\right)+\frac{1}{r}e^{-r/a_0}\right)r^2dr\right]$$

$$=-\frac{4\pi i\hbar}{\pi a_0^3}\left[-\frac{1}{a_0}\int_0^\infty e^{-2r/a_0}r^2dr+\int_0^\infty re^{-2r/a_0}dr\right]$$

$$=-\frac{4\pi\ i\ \hbar}{\pi a_0^3}\left[-\frac{1}{a_0}\left(\frac{2!}{(2/a_0)^3}\right)+\left(\frac{1!}{(2/a_0)^2}\right)\right]$$

$$=-\frac{4\pi\ i\ \hbar}{\pi\ a_0^3}\left[-\frac{a_0^2}{4}+\frac{a_0^2}{4}\right]=0$$

$$\langle P_r^2\rangle=\frac{1}{\pi a_0^3}\int_0^\infty e^{-r/a_0}\left\{-\hbar^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)e^{-r/a_0}\right\}4\pi r^2dr$$

$$=-\frac{4\pi\hbar^2}{\pi a_0^3}\left[\int_0^\infty e^{-r/a_0}\left(e^{-r/a_0}\left(\frac{1}{a_0^2}\right)+\frac{2}{r}\cdot\left(-\frac{1}{a_0}\right)e^{-r/a_0}\right)\right]\ r^2dr$$

$$=-\frac{4\hbar^2}{a_0^3}\left[\int_0^\infty\frac{1}{a_0^2}r^2e^{-2r/a_0}dr-\frac{2}{a_0}\int_0^\infty re^{-2r/a_0}dr\right]=-\frac{4\hbar^2}{a_0^3}\left[\frac{1}{a_0^2}\frac{2\ !}{(2/a_0)^3}-\frac{2}{a_0}\frac{1\ !}{(2/a_0)^2}\right]$$

$$=-\frac{4\hbar^2}{a_0^3}\left[\frac{2\ !}{a_0^2}\times\frac{a_0^3}{8}-\frac{2}{a_0}\times\frac{a_0^2}{4}\right]=-\frac{4\hbar^2}{a_0^3}\left[\frac{a_0}{4}-\frac{a_0}{2}\right]=-\frac{4\hbar^2}{a_0^3}\times\left(-\frac{a_0}{4}\right)=\frac{\hbar^2}{a_0^2}$$

$$\therefore\ \Delta P=\sqrt{\langle P_r^2\rangle-\langle P_r\rangle^2}=\sqrt{\frac{\hbar^2}{a_0^2}-0}=\frac{\hbar}{a_0}$$

Q46. The ground state eigenfunction for the potential $V(x)=-\delta(x)$ where $\delta(x)$ is the delta function, is given by $\psi(x)=Ae^{-\alpha|x|}$, where $A$ and $\alpha>0$ are constants. If a perturbation $H'=bx^2$ is applied, the first order correction to the energy of the ground state will be

(a) $\frac{b}{\sqrt{2}\alpha^2}$ (b) $\frac{b}{\alpha^2}$ (c) $\frac{2b}{\alpha^2}$ (d) $\frac{b}{2\alpha^2}$

Ans. : (d)

Solution: $V(x) = -\delta(x)$, $\psi(x) = Ae^{-\alpha|x|}$

$$\langle\psi|\psi\rangle = 1 \Rightarrow \psi(x) = \sqrt{\alpha}e^{-\alpha|x|}$$

$$E_1^1 = \langle\phi_1|H'|\phi_1\rangle = \int_{-\infty}^{\infty}\sqrt{\alpha}e^{-\alpha|x|}bx^2\sqrt{\alpha}e^{-\alpha|x|}dx$$

$$\int_{-\infty}^{\infty}\alpha e^{-2\alpha|x|}bx^2dx = b\int_{-\infty}^{\infty}\alpha e^{-2\alpha|x|}x^2dx = b\alpha\left[\int_{-\infty}^{0}x^2e^{2\alpha x}dx + \int_0^{\infty}x^2e^{-2\alpha x}dx\right] = b\alpha\left[2\times\int_0^{\infty}x^2e^{-2\alpha x}dx\right]$$

$$\int_{-\infty}^{\infty}\alpha e^{-2\alpha|x|}bx^2dx = 2b\alpha\left[\frac{2!}{(2\alpha)^3}\right] = 2\times b\alpha\frac{2!}{8\alpha^3} = \frac{b}{2\alpha^2}$$

Q47. An electron is in the ground state of a hydrogen atom. The probability that it is within the Bohr radius is approximately equal to

(a) 0.60 (b) 0.90 (c) 0.16 (d) 0.32

Ans. : (d)

Solution: Probability: $\int_0^{a_0}\left|\frac{1}{\sqrt{\pi a_0^3}}e^{-r/a_0}\right|^2 4\pi r^2dr = \frac{4\pi}{\pi a_0^3}\int_0^{a_0}r^2e^{-2r/a_0}dr$

$$= \frac{4}{a_0^3}\left\{\left[r^2e^{-2r/a_0}\left(-\frac{a_0}{2}\right)\right]_0^{a_0} - \left[2r\left(e^{-2r/a_0}\right)\left(-\frac{a_0}{2}\right)\left(-\frac{a_0}{2}\right)\right]_0^{a_0} + \left[2e^{-2r/a_0}\left(-\frac{a_0}{2}\right)\left(-\frac{a_0}{2}\right)\left(-\frac{a_0}{2}\right)\right]_0^{a_0}\right\}$$

$$= \frac{4}{a_0^3}\left[a_0^2e^{-\frac{2a_0}{a_0}}\left(-\frac{a_0}{2}\right) - 2a_0\left(\frac{a_0^2}{4}\right)e^{-2a_0/a_0} - \frac{a_0^3}{4}e^{-2a_0/a_0} + 2e^{-0}\left(\frac{a_0^3}{8}\right)\right]$$

$$= \frac{4}{a_0^3}\left[-\frac{a_0^3}{2}\frac{1}{e^2} - \frac{a_0^3}{2}\frac{1}{e^2} - \frac{a_0^3}{4e^2} + \frac{a_0^3}{4}\right] = 4\left[-\frac{5}{4e^2} + \frac{1}{4}\right] = \left[-5\times\frac{1}{e^2} + 1\right]$$

$$= [-5\times 0.137 + 1] = [-0.685 + 1] = 0.32$$

Q48. A particle in the infinite square well potential

$$V(x)=\begin{cases}0 & , \quad 0<x<a \\ \infty & , \quad \text{otherwise}\end{cases}$$

is prepared in a state with the wavefunction

$$\psi(x)=\begin{cases}A\sin^3\left(\frac{\pi x}{a}\right), & 0<x<a \\ 0 & , \quad \text{otherwise}\end{cases}$$

The expectation value of the energy of the particle is

(a) $\frac{5\hbar^2\pi^2}{2ma^2}$ (b) $\frac{9\hbar^2\pi^2}{2ma^2}$ (c) $\frac{9\hbar^2\pi^2}{10ma^2}$ (d) $\frac{\hbar^2\pi^2}{2ma^2}$

Ans. : (c)

Solution: $V(x)=\begin{cases}0, & 0<x<a \\ \infty, & \text{otherwise}\end{cases}$ $\quad \psi(x)=\begin{cases}A\sin^3\left(\frac{\pi x}{a}\right), & 0<x<a \\ 0 & , \quad \text{otherwise}\end{cases}$

$$\psi(x)=A\sin^3\left(\frac{\pi x}{a}\right)=A\frac{3}{4}\sin\frac{\pi x}{a}-A\frac{1}{4}\sin\frac{3\pi x}{a} \qquad \left(\because \sin 3A=3\sin A-4\sin^3 A\right)$$

$$=\frac{A}{4}\left[\sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\times 3\sin\frac{\pi x}{a}-\sqrt{\frac{a}{2}}\sqrt{\frac{2}{a}}\sin\frac{3\pi x}{a}\right]\Rightarrow \psi(x)=\frac{A}{4}\left[3\sqrt{\frac{a}{2}}\phi_1(x)-\sqrt{\frac{a}{2}}\phi_3(x)\right]$$

$$\langle\psi|\psi\rangle=1\Rightarrow 9\frac{a}{32}A^2+\frac{a}{32}A^2=1\Rightarrow\frac{10a}{32}A^2=1 \quad \Rightarrow A=\sqrt{\frac{32}{10a}}$$

$$\psi(x)=\frac{1}{4}\left(3.\sqrt{\frac{a}{2}}\sqrt{\frac{32}{10a}}\phi_1(x)-\sqrt{\frac{a}{2}}\sqrt{\frac{32}{10a}}\phi_3(x)\right)=\frac{3}{\sqrt{10}}\phi_1(x)-\frac{1}{\sqrt{10}}\phi_3(x)$$

Now, $E_1=\frac{\pi^2\hbar^2}{2ma^2}$, $\quad E_3=\frac{9\pi^2\hbar^2}{2ma^2}\Rightarrow\langle E\rangle=a_nP(a_n)$

Probability $P(E_1)=\frac{|\langle\varphi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle}=\frac{9}{10}$, $\quad P(E_3)=\frac{|\langle\phi_2|\psi\rangle|^2}{\langle\psi|\psi\rangle}=\frac{1}{10}$

$$\langle E\rangle=\frac{9}{10}\times\frac{\pi^2\hbar^2}{2ma^2}+\frac{1}{10}\times\frac{9\pi^2\hbar^2}{2ma^2}\Rightarrow\langle E\rangle=\frac{9\pi^2\hbar^2}{10ma^2}$$

**NET/JRF (DEC-2014)**

Q49. Suppose Hamiltonian of a conservative system in classical mechanics is $H = \omega xp$, where $\omega$ is a constant and $x$ and $p$ are the position and momentum respectively. The corresponding Hamiltonian in quantum mechanics, in the coordinate representation, is

(a) $-i\hbar\omega\left(x\frac{\partial}{\partial x}-\frac{1}{2}\right)$ (b) $-i\hbar\omega\left(x\frac{\partial}{\partial x}+\frac{1}{2}\right)$

(c) $-i\hbar\omega x\frac{\partial}{\partial x}$ (d) $-\frac{i\hbar\omega}{2}x\frac{\partial}{\partial x}$

Ans. : (b)

Solution: Classically $H = \omega xp$, quantum mechanically $H$ must be Hermitian,

So, $H = \frac{\omega}{2}(xp + px)$ and $H\psi = \frac{\omega}{2}(xp\psi + px\psi)$

$$\Rightarrow H\psi = \frac{\omega}{2}\left(x(-i\hbar)\frac{\partial\psi}{\partial x} + \frac{-i\hbar\partial(x\psi)}{\partial x}\right) = \frac{\omega}{2}(-i\hbar)\left(x\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial x} + \psi\right)$$

$$\Rightarrow H\psi = \frac{-i\hbar\omega}{2}\left(2x\frac{\partial\psi}{\partial x} + \psi\right) = -i\hbar\omega\left(x\frac{\partial}{\partial x} + \frac{1}{2}\right)\psi$$

Q50. Let $\psi_1$ and $\psi_2$ denote the normalized eigenstates of a particle with energy eigenvalues $E_1$ and $E_2$ respectively, with $E_2 > E_1$. At time $t = 0$ the particle is prepared in a state

$$\Psi(t=0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$$

The shortest time $T$ at which $\Psi(t=T)$ will be orthogonal to $\Psi(t=0)$ is

(a) $\frac{2\hbar\pi}{(E_2 - E_1)}$ (b) $\frac{\hbar\pi}{(E_2 - E_1)}$ (c) $\frac{\hbar\pi}{2(E_2 - E_1)}$ (d) $\frac{\hbar\pi}{4(E_2 - E_1)}$

Ans. : (b)

Solution: $\psi(t=0) = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$ and $\psi(t=T) = \frac{1}{\sqrt{2}}e^{-\frac{iE_1T}{\hbar}}\psi_1 + \frac{1}{\sqrt{2}}e^{-\frac{iE_2T}{\hbar}}\psi_2$

$$\int \psi^*(0)\psi(T)dx = 0 \Rightarrow \frac{1}{2}e^{-\frac{iE_1}{\hbar}T} + \frac{1}{2}e^{-\frac{iE_2}{\hbar}T} = 0 \Rightarrow e^{-\frac{iE_1}{\hbar}T} = -e^{-\frac{iE_2}{\hbar}T} \Rightarrow e^{i\frac{T}{\hbar}(E_2 - E_1)} = -1$$

Equate real part $\Rightarrow \cos\left(\frac{T}{\hbar}(E_2 - E_1)\right) = -1 \Rightarrow T = \frac{\hbar}{(E_2 - E_1)}\cos^{-1}(-1) = \frac{\pi\hbar}{(E_2 - E_1)}$

Q51. Consider the normalized wavefunction

$$\phi = a_1\psi_{11} + a_2\psi_{10} + a_3\psi_{1-1}$$

where $\psi_{lm}$ is a simultaneous normalized eigenfunction of the angular momentum operators $L^2$ and $L_z$, with eigenvalues $l(l+1)\hbar^2$ and $m\hbar$ respectively. If $\phi$ is an eigenfunction of the operator $L_x$ with eigenvalue $\hbar$, then

(a) $a_1 = -a_3 = \frac{1}{2}, \; a_2 = \frac{1}{\sqrt{2}}$

(b) $a_1 = a_3 = \frac{1}{2}, \quad a_2 = \frac{1}{\sqrt{2}}$

(c) $a_1 = a_3 = \frac{1}{2}, \quad a_2 = -\frac{1}{\sqrt{2}}$

(d) $a_1 = a_2 = a_3 = \frac{1}{\sqrt{3}}$

Ans. : (b)

Solution: $L_x|\phi\rangle = \hbar|\phi\rangle \Rightarrow \frac{L_+ + L_-}{2}|\psi\rangle = \lambda|\psi\rangle$

For $L_+$, $\quad L_+\left[a_1\psi_{11} + a_2\psi_{10} + a_3\psi_{1-1}\right] = a_1 0\hbar\psi_{12} + a_2\sqrt{2}\hbar\psi_{11} + a_3\sqrt{2}\hbar\psi_{10}$

$$= a_2\sqrt{2}\hbar\psi_{11} + a_3\sqrt{2}\hbar\psi_{10}$$

For $L_-$, $\quad L_-\left[a_1\psi_{11} + a_2\psi_{10} + a_3\psi_{1-1}\right] = a_1\sqrt{2}\hbar\psi_{10} + a_2\sqrt{2}\hbar\psi_{1-1}$

Given $\frac{L_+ + L_-}{2}|\phi\rangle = \hbar|\phi\rangle$

$$\Rightarrow \frac{L_+ + L_-}{2}|\phi\rangle = \frac{1}{2}\left[a_2\sqrt{2}\hbar\psi_{11} + (a_1 + a_3)\sqrt{2}\hbar\psi_{10} + a_2\sqrt{2}\hbar\psi_{1-1}\right]$$

$$\because \frac{L_+ + L_-}{2}|\phi\rangle = \hbar\left[a_1\psi_{11} + a_2\psi_{10} + a_3\psi_{1-1}\right] \text{(Given)}$$

Thus $\frac{a_2}{\sqrt{2}} = a_1 \Rightarrow a_2 = \sqrt{2}a_1$

$\frac{a_1 + a_3}{\sqrt{2}} = a_2 \Rightarrow \frac{a_1 + a_3}{\sqrt{2}} = \sqrt{2}a_1 \Rightarrow a_1 = a_3 \qquad \because a_1^2 + a_2^2 + \frac{a_2^2}{2} = 1$

$a_1 = a_3 = \frac{1}{2}, \; a_2 = \frac{1}{\sqrt{2}}$

Q52. Let $x$ and $p$ denote, respectively, the coordinate and momentum operators satisfying the canonical commutation relation $[x, p] = i$ in natural units $(\hbar = 1)$. Then the commutator $\left[x,\ pe^{-p}\right]$ is

(a) $i(1-p)e^{-p}$ (b) $i(1-p^2)e^{-p}$ (c) $i(1-e^{-p})$ (d) $ipe^{-p}$

Ans. : (a)

Solution: $\because [x, p] = i$

$$\left[x,\ pe^{-p}\right] = [x, p]e^{-p} + p\left[x, e^{-p}\right] = ie^{-p} + p\left[x, 1 - p + \frac{p^2}{\lfloor 2} - \frac{p^3}{\lfloor 3}....\right]$$

$$= ie^{-p} + p\left[[x,1] - [x, p] + \left[x, \frac{p^2}{\lfloor 2}\right]....\right] = ie^{-p} + p\left[0 - i + \frac{2ip}{\lfloor 2} - \frac{3ip^2}{\lfloor 3}......\right]$$

$$\Rightarrow \left[x,\ pe^{-p}\right] = ie^{-p} - i\left[p - p^2 + \frac{p^3}{\lfloor 2}.....\right] = ie^{-p} - ipe^{-p} = i(1-p)e^{-p}$$

Q53. Let $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, where $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices. If $\vec{a}$ and $\vec{b}$ are two arbitrary constant vectors in three dimensions, the commutator $\left[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right]$ is equal to (in the following $I$ is the identity matrix)

(a) $(\vec{a}\cdot\vec{b})(\sigma_1 + \sigma_2 + \sigma_3)$ (b) $2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$

(c) $(\vec{a}\cdot\vec{b})I$ (d) $|\vec{a}||\vec{b}|I$

Ans. : (b)

Solution: $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\sigma = \sigma_x\hat{i} + \sigma_y\hat{j} + \sigma_z\hat{k}$

$$\left[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right] = \left[a_1\sigma_x + a_2\sigma_y + a_3\sigma_z,\ b_1\sigma_x + b_2\sigma_y + b_3\sigma_z\right]$$

$$\left[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right] = a_1b_1\left[\sigma_x, \sigma_x\right] + a_1b_2\left[\sigma_x, \sigma_y\right] + a_1b_3\left[\sigma_x, \sigma_z\right] + a_2b_1\left[\sigma_y, \sigma_x\right] + a_2b_2\left[\sigma_y, \sigma_y\right]$$

$$+ a_2b_3\left[\sigma_y, \sigma_z\right] + a_3b_1\left[\sigma_z, \sigma_x\right] + a_3b_2\left[\sigma_z, \sigma_y\right] + a_3b_3\left[\sigma_z, \sigma_z\right]$$

$$= a_1b_1\cdot 0 + a_1b_2\cdot 2i\sigma_z - 2ia_1b_3\sigma_y - a_2b_1\cdot 2i\sigma_z + 0 + a_2b_3\cdot 2i\sigma_x + a_3b_1\cdot 2i\sigma_y - a_3b_2\cdot 2i\sigma_x + 0$$

$$\Rightarrow \left[\vec{a}\cdot\vec{\sigma}, \vec{b}\cdot\vec{\sigma}\right] = = 2i(\vec{a}\times\vec{b})\cdot\vec{\sigma}$$

Q54. The ground state energy of the attractive delta function potential

$$V(x) = -b\delta(x),$$

where $b > 0$, is calculated with the variational trial function

$$\psi(x) = \begin{cases} A\cos\frac{\pi x}{2a}, & \text{for} \quad -a < x < a, \\ 0, & \text{otherwise,} \end{cases} \text{ is}$$

(a) $-\frac{mb^2}{\pi^2\hbar^2}$ (b) $-\frac{2mb^2}{\pi^2\hbar^2}$ (c) $-\frac{mb^2}{2\pi^2\hbar^2}$ (d) $-\frac{mb^2}{4\pi^2\hbar^2}$

Ans. : (b)

Solution: $V(x) = -b\delta(x)$; $b > 0$ and $\psi(x) = \left\{ A\cos\frac{\pi x}{2a}; \quad -a < x < a \right.$

Normalized $\psi = \sqrt{\frac{2}{2a}}\cos\frac{\pi x}{2a}$

$$\langle T \rangle = \int_{-a}^{a} \psi^* \left(\frac{-\hbar^2}{2m}\right)\frac{\partial^2}{\partial x^2}\psi\, dx = \frac{\pi^2\hbar^2}{8ma^2}$$

$$\langle V \rangle = \int_{-a}^{a} \psi^* - b\delta(x)\psi\, dx = \frac{2}{2a}(-b) = -\frac{b}{a}$$

$$\langle E \rangle = \frac{\pi^2\hbar^2}{8ma^2} - \frac{b}{a} \Rightarrow \frac{\partial\langle E \rangle}{\partial a} = \frac{-2\pi^2\hbar^2}{8ma^3} + \frac{b}{a^2} = 0 \Rightarrow \frac{-\pi^2\hbar^2}{4ma} + b = 0 \Rightarrow a = \frac{\pi^2\hbar^2}{4mb}$$

Put the value of $a$ in equation: $\langle E \rangle = \frac{\pi^2\hbar^2}{8ma^2} - \frac{b}{a} = \frac{\pi^2\hbar^2(4mb)^2}{8m(\pi^2\hbar^2)^2} - \frac{b(4mb)}{(\pi^2\hbar^2)} = -\frac{2mb^2}{\pi^2\hbar^2}$

Q55. Let $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ (where $c_0$ and $c_1$ are constants with $c_0^2 + c_1^2 = 1$) be a linear combination of the wavefunctions of the ground and first excited states of the one-dimensional harmonic oscillator. For what value of $c_0$ is the expectation value $\langle x \rangle$ a maximum?

(a) $\langle x \rangle = \sqrt{\frac{\hbar}{m\omega}}, \quad c_0 = \frac{1}{\sqrt{2}}$ (b) $\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}}, \quad c_0 = \frac{1}{2}$

(c) $\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}}, \quad c_0 = \frac{1}{\sqrt{2}}$ (d) $\langle x \rangle = \sqrt{\frac{\hbar}{m\omega}}, \quad c_0 = \frac{1}{2}$

Ans. : (c)

Solution: $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$

$$\langle X\rangle = \langle\psi|X|\psi\rangle$$

$$\Rightarrow \langle X\rangle = 2c_0c_1\langle 0|X|1\rangle = \left[\left(c_0^2+c_1^2\right)-\left(c_0-c_1\right)^2\right]\langle 0|X|1\rangle = \left[1-\left(c_0-c_1\right)^2\right]\langle 0|X|1\rangle$$

For max $\langle X\rangle = c_0 = c_1 \qquad \because c_0^2+c_1^2 = 1 \Rightarrow c_0 = \frac{1}{\sqrt{2}}$

$$\Rightarrow \langle X\rangle = 2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\langle 0|X|1\rangle = \langle 0|X|1\rangle$$

$$\sqrt{\frac{\hbar}{2m\omega}}\left(\langle 0|a+a^{+}|1\rangle\right) \Rightarrow \langle X\rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

Q56. Consider a particle of mass $m$ in the potential $V(x) = a|x|,\ a>0$. The energy eigen-values $E_n\,(n = 0, 1, 2, ....)$, in the WKB approximation, are

(a) $\left[\frac{3a\hbar\pi}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{1/3}$ (b) $\left[\frac{3a\hbar\pi}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{2/3}$

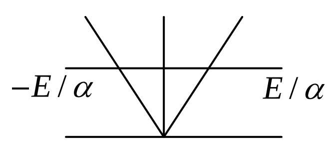
(c) $\frac{3a\hbar\pi}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)$ (d) $\left[\frac{3a\hbar\pi}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{4/3}$

Ans. : (b)

Solution: $V(x) = a|x|, \quad a>0$

According to W.K.B., $\int_{\alpha_1}^{\alpha_2} pdq = \left(n+\frac{1}{2}\right)\hbar$ where $a_1$ and $a_2$ are positive mid point

$$E = \frac{P^2}{2m} + a|x| \Rightarrow P = \sqrt{2m\left(E-a|x|\right)}$$




$$\int_{-E/a}^{E/a}\sqrt{2m\left(E-a|x|\right)}dx = \left(n+\frac{1}{2}\right)\hbar$$

$$\int_{-E/a}^{0}\sqrt{2m\left(E+ax\right)}dx + \int_{0}^{E/a}\sqrt{2m\left(E-ax\right)}dx = \left(n+\frac{1}{2}\right)\hbar$$

$$2\int_{0}^{E/a}\sqrt{2m\left(E-ax\right)}dx = \left(n+\frac{1}{2}\right)\hbar$$

$2m(E-ax)=t$, At $x=0$, $t=2mE$; $x=E/a,\ t=0$

$\Rightarrow -2madx=dt$

$$\Rightarrow 2ma\int_0^{2mE} t^{1/2}dt=\left(n+\frac{1}{2}\right)\hbar \Rightarrow 2ma\frac{2}{3}t^{3/2}\Big|_0^{2mE}=\left(n+\frac{1}{2}\right)\hbar$$

$$\Rightarrow \frac{4}{3}ma\,t^{3/2}\Big|_0^{2mE}=\left(n+\frac{1}{2}\right)\hbar \Rightarrow \frac{4}{3}ma(2mE)^{3/2}=\left(n+\frac{1}{2}\right)\hbar$$

$$\Rightarrow \frac{4}{3}\cdot 2^{3/2}am^{5/2}E^{3/2}=\left(n+\frac{1}{2}\right)\hbar \Rightarrow E=\left[\frac{3a\hbar\pi}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{2/3}$$

Q57. The Hamiltonian $H_0$ for a three-state quantum system is given by the matrix $H_0=\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$. When perturbed by $H'=\in\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$ where $\in<<1$, the resulting shift in the energy eigenvalue $E_0=2$ is

(a) $\in,-2\in$ (b) $-\in,\ 2\in$ (c) $\pm\in$ (d) $\pm 2\in$

Ans. : **None of the answer is correct.**

Solution: $H_0=\begin{pmatrix}1&0&0\\0&2&0\\0&0&2\end{pmatrix}$, $H'=\in_0\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$

$\begin{pmatrix}2&0\\0&2\end{pmatrix}$ in $H_0$ is not $\in_0\begin{pmatrix}0&1\\1&0\end{pmatrix}$ in $H'$ because $H'$ is not in block diagonal form. So we must diagonalised whole $H'$. The Eigen value at $H'=0,\ +\sqrt{2}\in_0,\ -\sqrt{2}\in_0$.

After diagonalisation $H'=\in_0\begin{pmatrix}0&0&0\\0&\sqrt{2}&0\\0&0&-\sqrt{2}\end{pmatrix}$, $\lambda=0$ is correction for Eigenvalue at $H_0$.

So $\pm\sqrt{2}\in_0$ is the correction for eigenvalue of $H_0=2$

Hence none of the options given is correct.

**NET/JRF (JUNE-2015)**

Q58. The ratio of the energy of the first excited state $E_1$, to that of the ground state $E_0$, to that of a particle in a three-dimensional rectangular box of side $L, L$ and $\frac{L}{2}$, is

(a) $3:2$ (b) $2:1$ (c) $4:1$ (d) $4:3$

Ans. (a)

Solution: $E = \frac{\pi^2\hbar^2}{2mL^2}\left[n_x^2 + n_y^2 + 4n_z^2\right]$, for ground state $n_x = 1, n_y = 1, n_z = 1 \Rightarrow E_0 = \frac{6\pi^2\hbar^2}{2mL^2}$

For first excited state $n_x = 1,\ n_y = 2,\ n_z = 1 \Rightarrow E = E_1 = \frac{\pi^2\hbar^2}{2mL^2}(1+4+4) = \frac{9\pi^2\hbar^2}{2mL^2}$

$\therefore \frac{E_1}{E_0} = \frac{9}{6} = \frac{3}{2}$

Q59. If $L_i$ are the components of the angular momentum operator $\vec{L}$, then the operator $\sum_{i=1,2,3}\left[\vec{L}, L_i\right]$ equals

(a) $\vec{L}$ (b) $2\vec{L}$ (c) $3\vec{L}$ (d) $-\vec{L}$

Ans. (b)

Solution: Let $\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$

$x = 1, y = 2, z = 3$

$\left[\vec{L}, L_x\right] = \left[L_y, L_x\right]j + \left[L_z, L_x\right]\hat{k} = -i\hbar L_z\hat{j} + L_y\hat{k}\,i\hbar$

$\left[\left[\vec{L}, L_x\right], L_x\right] = i\hbar\left[-L_z, L_x\right]\hat{j} + \left[L_y, L_x\right]i\hbar - i\hbar.i\hbar L_y\hat{j} - (i\hbar)L_z(i\hbar)L_z(i\hbar).\hat{k} = \hbar^2\left[L_y\hat{j} + L_z\hat{k}\right]$

similarly, $\left[\left[\vec{L}, L_y\right]L_y\right] = \hbar^2\left[L_x\hat{i} + L_z\hat{k}\right]$

$\left[\left[\vec{L}, L_z\right]L_z\right] = \hbar^2\left[L_x\hat{i} + L_y\hat{j}\right]$

$\sum_{i=1,2,3}\left[\left[L, Li\right]Li\right] = 2\hbar^2\left[L_x\hat{i} + L_y\hat{j} + L_z\hat{k}\right] = 2\vec{L}$ put $\hbar = 1$

Q60. The wavefunction of a particle in one-dimension is denoted by $\psi(x)$ in the coordinate representation and by $\phi(p)=\int \psi(x) e^{\frac{-ipx}{\hbar}} dx$ in the momentum representation. If the action of an operator $\hat{T}$ on $\psi(x)$ is given by $\hat{T}\psi(x)=\psi(x+a)$, where $a$ is a constant then $\hat{T}\phi(p)$ is given by

(a) $-\frac{i}{\hbar}ap\phi(p)$ (b) $e^{\frac{-iap}{\hbar}}\phi(p)$ (c) $e^{\frac{+iap}{\hbar}}\phi(p)$ (d) $\left(1+\frac{i}{\hbar}ap\right)\phi(p)$

Ans. (c)

Solution: $\phi(p)=\int \psi(x) e^{\frac{-ipx}{\hbar}} dx$

$T\psi(x)=\psi(x+a)$

$T\phi(p)=\int T\psi(x) e^{\frac{-ipx}{\hbar}} dx = \int \psi(x+a) e^{\frac{-ipx}{\hbar}} dx = e^{\frac{ipa}{\hbar}} \int \psi(x+a) e^{\frac{-ip(x+a)}{\hbar}} dx$

$\Rightarrow T\phi(p)=e^{\frac{ipa}{\hbar}}\phi(p)$

Q61. The differential cross-section for scattering by a target is given by

$$\frac{d\sigma}{d\Omega}(\theta,\phi)=a^2+b^2\cos^2\theta$$

If $N$ is the flux of the incoming particles, the number of particles scattered per unit time is

(a) $\frac{4\pi}{3}N\left(a^2+b^2\right)$ (b) $4\pi N\left(a^2+\frac{1}{6}b^2\right)$

(c) $4\pi N\left(\frac{1}{2}a^2+\frac{1}{3}b^2\right)$ (d) $4\pi N\left(a^2+\frac{1}{3}b^2\right)$

Ans. (d)

Solution: $\frac{d\sigma}{d\Omega}=a^2+b^2\cos^2\theta$

$$\sigma=a^2\int_0^{\pi}\int_0^{2\pi}\sin\theta d\theta d\phi+b^2\int_0^{\pi}\cos^2\theta\sin\theta d\theta\int_0^{2\pi}d\phi=a^2.4\pi+b^2.2\pi\times\frac{2}{3}=4\pi\left[a^2+\frac{b^2}{3}\right]$$

Number of particle scattered per unit time, $\sigma.N=4\pi N\left(a^2+\frac{b^2}{3}\right)$

Q62. A particle of mass $m$ is in a potential $V=\frac{1}{2}m\omega^2x^2$, where $\omega$ is a constant. Let $\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)$. In the Heisenberg picture $\frac{d\hat{a}}{dt}$ is given by

(a) $\omega\hat{a}$ (b) $-i\omega\hat{a}$ (c) $\omega\hat{a}^\dagger$ (d) $i\omega\hat{a}^\dagger$

Ans. : (b)

Solution: $V=\frac{1}{2}m\omega^2x^2$

$$\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i\hat{p}}{m\omega}\right)$$

$$\frac{d\hat{a}}{dt}=\frac{1}{i\hbar}\langle[a,H]\rangle+\left\langle\frac{\partial a}{\partial t}\right\rangle,\qquad \frac{\partial a}{\partial t}=0$$

$$\frac{d\hat{a}}{dt}=\frac{1}{i\hbar}\sqrt{\frac{m\omega}{2\hbar}}\left[\left[x,\frac{p^2}{2m}\right]+\frac{im\omega^2}{2m\omega}\left[\hat{p},x^2\right]\right]=\frac{1}{i\hbar}\sqrt{\frac{m\omega}{2\hbar}}\left(i\hbar\frac{2p}{2m}+\frac{i\omega}{2}(-2x)i\hbar\right)$$

$$=\sqrt{\frac{m\omega}{2\hbar}}\left(\frac{p}{m}-i\omega x\right)=-i\omega\sqrt{\frac{m\omega}{2\hbar}}\left(x+\frac{ip}{m\omega}\right)=-i\omega\hat{a}$$

Q63. Two different sets of orthogonal basis vectors $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ and $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix},\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}$ are given for a two dimensional real vector space.

The matrix representation of a linear operator $\hat{A}$ in these basis are related by a unitary transformation. The unitary matrix may be chosen to be

(a) $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ (b) $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ (c) $\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ (d) $\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\1&1\end{pmatrix}$

Ans. : (c)

Solution: $u_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\Rightarrow u=u_1\otimes u_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$

Q64. The Dirac Hamiltonian $H = c\vec{\alpha}.\vec{p} + \beta mc^2$ for a free electron corresponds to the classical relation $E^2 = p^2c^2 + m^2c^4$. The classical energy-momentum relation of a piratical of charge $q$ in a electromagnetic potential $\left(\phi, \vec{A}\right)$ is $\left(E - q\phi\right)^2 = c^2\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + m^2c^4$.

Therefore, the Dirac Hamiltonian for an electron in an electromagnetic field is

(a) $c\vec{\alpha}.p + \frac{e}{c}\vec{A}.\vec{A} + \beta mc^2 - e\phi$ (b) $c\vec{\alpha}.\left(\vec{p} + \frac{e}{c}\vec{A}\right) + \beta mc^2 + e\phi$

(c) $c\left(\vec{\alpha}.\vec{p} + e\phi + \frac{e}{c}\left|\vec{A}\right|\right) + \beta mc^2$ (d) $c\vec{\alpha}.\left(\vec{p} + \frac{e}{c}\vec{A}\right) + \beta mc^2 - e\phi$

Ans. : (d)

Solution: Electromagnetic interaction of Dirac particle

$$H = \left[\left(\vec{P} - \frac{q\vec{A}}{c}\right)^2 c^2 + m^2c^4\right]^{\frac{1}{2}} + q\phi$$

Quantum mechanical Hamiltonian

$$i\hbar\frac{\partial\psi}{\partial t} = \left[c\vec{\alpha}\left(P - \frac{q\vec{A}}{c}\right) + \beta mc^2 + q\phi\right]\psi$$

put $q = -e$

$$H = \left[c\vec{\alpha}.\left(\vec{P} + \frac{e}{c}\vec{A}\right) + \beta mc^2 - e\phi\right]$$

Q65. A particle of energy $E$ scatters off a repulsive spherical potential

$$V(r) = \begin{cases} V_0 & \text{for} \quad r < a \\ 0 & \text{for} \quad r \leq a \end{cases}$$

where $V_0$ and $a$ are positive constants. In the low energy limit, the total scattering cross-section is $\sigma = 4\pi a^2\left(\frac{1}{ka}\tanh ka - 1\right)^2$, where $k^2 = \frac{2m}{h^2}\left(V_0 - E\right) > 0$. In the limit $V_0 \to \infty$ the ratio of $\sigma$ to the classical scattering cross-section off a sphere of radius $a$ is

(a) 4 (b) 3 (c) 1 (d) $\frac{1}{2}$

Ans. : (a)

Solution: $\sigma = 4\pi a^2 \left[\frac{1}{ka}\tanh ka - 1\right]^2$

$ka \to \infty$, $\tanh ka \to 1 \Rightarrow \sigma = 4\pi a^2 \left(\frac{1}{ka} - 1\right)^2$

and $ka \to \infty$, $\lim\limits_{ka\to\infty} \sigma_H = 4\pi a^2$

classically $\sigma_c = \pi a^2$ $\quad\quad \therefore \frac{\sigma_H}{\sigma_c} = 4$

## NET/JRF (DEC-2015)

Q66. A Hermitian operator $\hat{O}$ has two normalized eigenstates $|1\rangle$ and $|2\rangle$ with eigenvalues 1 and 2, respectively. The two states $|u\rangle = \cos\theta|1\rangle + \sin\theta|2\rangle$ and $|v\rangle = \cos\phi|1\rangle + \sin\phi|2\rangle$ are such that $\langle v|\hat{O}|v\rangle = 7/4$ and $\langle u|v\rangle = 0$. Which of the following are possible values of $\theta$ and $\phi$?

(a) $\theta = -\frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$ (b) $\theta = \frac{\pi}{6}$ and $\phi = \frac{\pi}{3}$

(c) $\theta = -\frac{\pi}{4}$ and $\phi = \frac{\pi}{4}$ (d) $\theta = \frac{\pi}{3}$ and $\phi = -\frac{\pi}{6}$

Ans. : (a)

Solution: $|u\rangle = \cos\theta|1\rangle + \sin\theta|2\rangle$, $|v\rangle = \cos\phi|1\rangle + \sin\phi|2\rangle$

it is given $\hat{O}|1\rangle = |1\rangle$, $\hat{O}|2\rangle = 2|2\rangle \Rightarrow \langle v|\hat{O}|v\rangle = \frac{7}{4}$

$\cos^2\phi + 2\sin^2\phi = \frac{7}{4} \Rightarrow \cos^2\phi + \sin^2\phi = 1 \Rightarrow \sin^2\phi = \frac{7}{4} - 1$

$\sin\phi = \frac{\sqrt{3}}{2} \Rightarrow \phi = \frac{\pi}{3}$

$\langle u|v\rangle = 0 \Rightarrow \cos\theta\cos\phi + \sin\theta\sin\phi = 0 \Rightarrow \cos(\theta - \phi) = 0$

$\Rightarrow \theta - \phi = \frac{\pi}{2}$ or $\phi - \theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{2} + \frac{\pi}{3}$ or $\theta = \frac{\pi}{3} - \frac{\pi}{2} \Rightarrow \theta = \frac{5\pi}{6}$ or $\theta = -\frac{\pi}{6}$

Q67. The ground state energy of a particle of mass $m$ in the potential $V(x) = V_0 \cosh\left(\frac{x}{L}\right)$, where $L$ and $V_0$ are constants (with $V_0 >> \frac{\hbar^2}{2mL^2}$) is approximately

(a) $V_0 + \frac{\hbar}{L}\sqrt{\frac{2V_0}{m}}$ (b) $V_0 + \frac{\hbar}{L}\sqrt{\frac{V_0}{m}}$ (c) $V_0 + \frac{\hbar}{4L}\sqrt{\frac{V_0}{m}}$ (d) $V_0 + \frac{\hbar}{2L}\sqrt{\frac{V_0}{m}}$

Ans. : (d)

Solution: $V_0 = \cosh\left(\frac{x}{L}\right) = \frac{V_0}{2}\left(e^{x/L} + e^{-x/L}\right)$

$$= \frac{V_0}{2}\left[1 + \frac{x}{L} + \frac{1}{2!}\left(\frac{x}{L}\right)^2 ....\right] + \frac{V_0}{2}\left[1 - \frac{x}{L} + \frac{1}{2!}\left(\frac{x}{L}\right)^2 + ....\right]$$

$$= \frac{V_0}{2} + \frac{V_0}{2} + \frac{V_0}{2}\left(\frac{x}{L}\right)^2 = V_0 + \frac{1}{2}\left(\frac{V_0}{L^2}\right)x^2$$

$$K = \frac{V_0}{L^2}, \qquad \omega = \sqrt{\frac{V_0}{mL^2}}$$

So, ground state energy is

$$V_0 + \frac{\hbar\omega}{2} = V_0 + \frac{\hbar}{2}\sqrt{\frac{V_0}{mL^2}} = V_0 + \frac{\hbar}{2L}\sqrt{\frac{V_0}{m}}$$

Q68. Let $\psi_{nlm}$ denote the eigenstates of a hydrogen atom in the usual notation. The state

$$\frac{1}{5}\left[2\psi_{200} - 3\psi_{211} + \sqrt{7}\psi_{210} - \sqrt{5}\psi_{21-1}\right]$$

is an eigenstate of

(a) $L^2$, but not of the Hamiltonian or $L_z$ (b) the Hamiltonian, but not of $L^2$ or $L_z$

(c) the Hamiltonian, $L^2$ and $L_z$ (d) $L^2$ and $L_z$, but not of the Hamiltonian

Ans. : (b)

Solution: $|\psi\rangle = \frac{1}{5}\left[2\psi_{200} - 3\psi_{211} + \sqrt{7}\psi_{210} - \sqrt{5}\psi_{21-1}\right]$

$$H|\psi\rangle = -\frac{13.6}{4}|\psi\rangle$$

So $|\psi\rangle$ is eigen state of $H$

But $L^2|\psi\rangle \neq \alpha|\psi\rangle$ and $L_z|\psi\rangle \neq \beta|\psi\rangle$

So $|\psi\rangle$ is not eigen state of $L^2$ and $L_z$

Q69. The Hamiltonian for a spin-$\frac{1}{2}$ particle at rest is given by $H = E_0(\sigma_z + \alpha\sigma_x)$, where $\sigma_x$ and $\sigma_z$ are Pauli spin matrices and $E_0$ and $\alpha$ are constants. The eigenvalues of this Hamiltonian are

(a) $\pm E_0\sqrt{1+\alpha^2}$ (b) $\pm E_0\sqrt{1-\alpha^2}$

(c) $E_0$ (doubly degenerate) (d) $E_0\left(1\pm\frac{1}{2}\alpha^2\right)$

Ans. : (a)

Solution: $H = E_0(\dot{\sigma}_z + \alpha\sigma_x) = E_0\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \alpha\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] \Rightarrow H = E_0\begin{pmatrix}1 & \alpha\\ \alpha & -1\end{pmatrix}$

if $\lambda$ is eigen value, then

$$H - \lambda I = 0 \Rightarrow E_0\begin{pmatrix}(1-\lambda) & \alpha\\ \alpha & -(1+\lambda)\end{pmatrix} = 0, \quad \lambda = \pm E_0\sqrt{1+\alpha^2}$$

Q70. A hydrogen atom is subjected to the perturbation

$$V_{pert}(r) = \in \cos\frac{2r}{a_0}$$

where $a_0$ is the Bohr radius. The change in the ground state energy to first order in $\in$

(a) $\frac{\in}{4}$ (b) $\frac{\in}{2}$ (c) $\frac{-\in}{2}$ (d) $\frac{-\in}{4}$

Ans. : (d)

Solution: For First order perturbation

$$E_1^1 = \langle\phi_{100}|V_p|\phi_{100}\rangle, \phi_{100} = \frac{1}{\sqrt{\pi a_0^3}}e^{\frac{-r}{a}}, \; V_p = \in \cos\left(\frac{2r}{a_0}\right)$$

$$E_1^1 = \int_0^\infty \frac{1}{\pi a_0^3}e^{\frac{-2r}{a_0}} \in \cos\left(\frac{2r}{a_0}\right)4\pi r^2 dr = \frac{4\in}{a_0^3}\int_0^\infty e^{\frac{-2r}{a_0}} \cos\left(\frac{2r}{a_0}\right)r^2 dr$$

$$=\frac{4\in}{a_0^3}\int_0^\infty e^{\frac{-2r}{a_0}}\left[\frac{e^{\frac{i2r}{a_0}}+e^{\frac{-i2r}{a_0}}}{2}\right]r^2dr=\frac{2\in}{a_0^3}\left[\int_0^\infty e^{\frac{-2r}{a_0}(1-i)}r^2dr+\int_0^\infty e^{\frac{-2r}{a_0}(1+i)}r^2dr\right]$$

$$\Rightarrow\quad \frac{2\in}{a_0^3}\left[\frac{2!}{\left[\frac{2}{a_0}(1-i)\right]^3}+\frac{2!}{\left[\frac{2}{a_0}(1+i)\right]^3}\right]\Rightarrow\frac{\in}{2}\left[\frac{1}{(1-i)^3}+\frac{1}{(1+i)^3}\right]$$

$$\Rightarrow\quad \frac{\in}{2}\left[\frac{1}{\left(\sqrt{2}\right)^3\left(\frac{1-i}{\sqrt{2}}\right)^3}+\frac{1}{\left(\sqrt{2}\right)^3\left(\frac{1+i}{\sqrt{2}}\right)^3}\right]\Rightarrow\frac{\in}{4\sqrt{2}}\left[\frac{1}{e^{\frac{-i3\pi}{4}}}+\frac{1}{e^{\frac{i3\pi}{4}}}\right]$$

$$\Rightarrow\quad \frac{\in}{4\sqrt{2}}\left[e^{\frac{i3\pi}{4}}+e^{\frac{-i3\pi}{4}}\right]\Rightarrow\frac{\in}{4\sqrt{2}}\left[2\cos\left(\frac{3\pi}{4}\right)\right]$$

$$\Rightarrow\quad \frac{\in}{4\sqrt{2}}\left[2\left(-\frac{1}{\sqrt{2}}\right)\right]\Rightarrow\frac{-\in}{4}\Rightarrow E_1^1=\frac{-\in}{4}$$

Q71. The product of the uncertainties $(\Delta L_x)(\Delta L_y)$ for a particle in the state $a|1,1\rangle+b|1,-1\rangle$ where $|l,m\rangle$ denotes an eigenstate of $L^2$ and $L_z$ will be a minimum for

(a) $a=\pm ib$ (b) $a=0$ and $b=1$

(c) $a=\frac{\sqrt{3}}{2}$ and $b=\frac{1}{2}$ (d) $a=\pm b$

Ans. : (d)

Solution: $|\psi\rangle=a|1,1\rangle+b|1,-1\rangle$, $L_+|\psi\rangle=\sqrt{2}\hbar b|1,0\rangle$, $L_+^2|\psi\rangle=2\hbar^2 b|1,1\rangle$

$L_-|\psi\rangle=\sqrt{2}\hbar a|1,0\rangle$, $L_-^2|\psi\rangle=2\hbar^2 a|1,-1\rangle$

$$\langle\psi|L^2|\psi\rangle=|a|^2 2\hbar^2+|b|^2 2\hbar^2=\left(|a|^2+|b^2|\right)2\hbar^2$$

$$\langle\psi|L_z^2|\psi\rangle=\left(|a|^2+|b^2|\right)\hbar^2$$

$$\langle L_x\rangle=0,\langle L_y\rangle=0$$

$$\langle L_x^2\rangle=\frac{1}{4}\left\langle\left[L_+^2+L_-^2+2\left(L^2-L_z^2\right)\right]\right\rangle=\frac{1}{4}\left[\left(a^*b+b^*a\right)2\hbar^2+2\left(2\hbar^2-\hbar^2\right)\left(|a|^2+|b|^2\right)\right]$$

$$\langle L_x^2\rangle = \frac{\hbar^2}{2}\left[\left(a^*b+b^*a\right)+|a|^2+|b|^2\right]$$

$$L_y^2 = \frac{2\left(L^2-L_2^2\right)-L_+^2-L_-^2}{4}$$

$$\langle L_y^2\rangle = \frac{\hbar^2}{2}\left[|a|^2+|b|^2-\left(a^*b+b^*a\right)\right]$$

$$\Delta L_x \Delta L_y = \frac{\hbar^2}{2}\sqrt{\left(\left(|a|^2+|b|^2\right)^2-\left(a^*b+b^*a\right)^2\right)} \qquad \because |a|^2+|b|^2=1$$

$$\Delta L_x \Delta L_y = \frac{\hbar^2}{2}\sqrt{1-\left(a^*b+b^*a\right)^2} \qquad \text{(i)}$$

Now check option (a) $a=\pm ib \Rightarrow a=\frac{1}{\sqrt{2}}, b=\frac{-i}{\sqrt{2}} \Rightarrow \Delta L_x\Delta L_y = \frac{\hbar^2}{2}$

Option (b) $a=0,\ b=1 \Rightarrow \Delta L_x\Delta L_y = \frac{\hbar^2}{2}$

Option (c) $a=\frac{\sqrt{3}}{2}, b=\frac{1}{2} \Rightarrow \Delta L_x\Delta L_y = \frac{\hbar^2}{4}$

Option (d) $a=\pm b \Rightarrow a=\frac{1}{\sqrt{2}}, b=\frac{1}{\sqrt{2}} \Rightarrow \Delta L_x\Delta L_y = 0$ option (d) is correct

Q72. The ground state energy of a particle in potential $V(x)=g|x|$, estimated using the trail wavefunction

$$\psi(x)=\begin{cases}\sqrt{\frac{c}{a^5}}\left(a^2-x^2\right), & x<|a| \\ 0, & x\geq|a|\end{cases}$$

(where $g$ and $c$ are constants) is

(a) $\frac{15}{16}\left(\frac{\hbar^2 g^2}{m}\right)^{1/3}$ (b) $\frac{5}{6}\left(\frac{\hbar^2 g^2}{m}\right)^{1/3}$ (c) $\frac{3}{4}\left(\frac{\hbar^2 g^2}{m}\right)^{1/3}$ (d) $\frac{7}{8}\left(\frac{\hbar^2 g^2}{m}\right)^{1/3}$

Ans. : (a)

Solution: $\int_{-a}^{a}\psi^*\psi dx = 1 \Rightarrow c=\frac{15}{16}$

$$\langle T\rangle = \frac{-\hbar^2}{2m}\left(\frac{15}{16a^2}\right)\int_{-a}^{a}\left(a^2-x^2\right)\frac{\partial^2}{\partial x^2}\left(a^2-x^2\right)dx \Rightarrow \langle T\rangle = \frac{10\hbar^2}{4ma^2}$$

$$\langle V\rangle = \frac{15\times 2g}{16a^5}\int_0^a x\left(a^2-x^2\right)dx \Rightarrow \langle V\rangle = \frac{5}{16}ga$$

$$E=\langle T\rangle+\langle V\rangle \qquad \text{(i)}$$

$$E = \frac{10\hbar^2}{4ma^2} + \frac{5ga}{16}$$

$$\frac{dE}{da} = 0 \Rightarrow a^3 = \frac{8\hbar}{mg} \Rightarrow a = 2\left(\frac{\hbar^2}{mg}\right)^{\frac{1}{3}}$$

put the value of $a$ in equation (i)

$$E = \frac{15}{16}\left(\frac{\hbar^2 g^2}{m}\right)^{\frac{1}{3}}$$

**NET/JRF (JUNE-2016)**

Q73. The state of a particle of mass $m$ in a one dimensional rigid box in the interval $0$ to $L$ is given by the normalized wavefunction $\psi(x) = \sqrt{\frac{2}{L}}\left(\frac{3}{5}\sin\left(\frac{2\pi x}{L}\right) + \frac{4}{5}\sin\left(\frac{4\pi x}{L}\right)\right)$. If its energy is measured the possible outcomes and the average value of energy are, respectively

(a) $\frac{h^2}{2mL^2}, \frac{2h^2}{mL^2}$ and $\frac{73}{50}\frac{h^2}{mL^2}$

(b) $\frac{h^2}{8mL^2}, \frac{h^2}{2mL^2}$ and $\frac{19}{40}\frac{h^2}{mL^2}$

(c) $\frac{h^2}{2mL^2}, \frac{2h^2}{mL^2}$ and $\frac{19}{10}\frac{h^2}{mL^2}$

(d) $\frac{h^2}{8mL^2}, \frac{2h^2}{mL^2}$ and $\frac{73}{200}\frac{h^2}{mL^2}$

Ans. : (a)

Solution: $\psi(x) = \sqrt{\frac{2}{L}}\left(\frac{3}{5}\sin\left(\frac{2\pi x}{L}\right) + \frac{4}{5}\sin\left(\frac{4\pi x}{L}\right)\right)$

Measurement $E = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$\because n = 2 \Rightarrow E_2 = \frac{h^2}{2mL^2}$ and $n = 4 \Rightarrow E_4 = \frac{2h^2}{mL^2}$

Probability $p(E_2) = \frac{9}{25}$ and $p(E_4) = \frac{16}{25}$

Now, average value of energy is

$$\langle E\rangle = \sum a_n p(a_n) = \frac{9}{25}\times\frac{h^2}{2mL^2} + \frac{16}{25}\times\frac{2h^2}{mL^2} = \frac{73h^2}{50mL^2}$$

Q74. If $\hat{L}_x, \hat{L}_y, \hat{L}_z$ are the components of the angular momentum operator in three dimensions the commutator $\left[\hat{L}_x, \hat{L}_x\hat{L}_y\hat{L}_z\right]$ may be simplified to

(a) $i\hbar L_x\left(\hat{L}_z^2 - \hat{L}_y^2\right)$

(b) $i\hbar\hat{L}_z\hat{L}_y\hat{L}_x$

(c) $i\hbar L_x\left(2\hat{L}_z^2 - \hat{L}_y^2\right)$

(d) 0

Ans. : (a)

Solution: $\left[L_x, L_xL_yL_z\right] = L_x\left[L_x, L_yL_z\right] + \left[L_x, L_x\right]L_yL_z$

$$= L_x\left[L_x, L_y\right]L_z + L_xL_y\left[L_x, L_z\right] + 0 = L_x\left[i\hbar L_z\right]L_z + L_xL_y\left(-i\hbar L_y\right)$$

$$= i\hbar L_xL_z^2 - i\hbar L_xL_y^2 = i\hbar L_x\left(L_z^2 - L_y^2\right)$$

Q75. Suppose that the Coulomb potential of the hydrogen atom is changed by adding an inverse-square term such that the total potential is $V(\vec{r}) = -\frac{Ze^2}{r} + \frac{g}{r^2}$, where $g$ is a constant. The energy eigenvalues $E_{nlm}$ in the modified potential

(a) depend on $n$ and $l$, but not on $m$

(b) depend on $n$ but not on $l$ and $m$

(c) depend on $n$ and $m$, but not on $l$

(d) depend explicitly on all three quantum numbers $n$, $l$ and $m$

Ans. : (b)

Solution: $V(r) = -\frac{ze^2}{r} + \frac{g}{r^2}$ is central potential

So angular momentum is conserve then eigen value $E_{n,l,m}$ will depend only on $n$, which is principal quantum number.

Q76. The eigenstates corresponding to eigenvalues $E_1$ and $E_2$ of a time independent Hamiltonian are $|1\rangle$ and $|2\rangle$ respectively. If at $t = 0$, the system is in a state $|\psi(t=0)\rangle = \sin\theta|1\rangle + \cos\theta|2\rangle$, then the value of $\langle\psi(t)|\psi(t)\rangle$ at time $t$ will be

(a) 1

(b) $\frac{\left(E_1\sin^2\theta + E_2\cos^2\theta\right)}{\sqrt{E_1^2 + E_2^2}}$

(c) $e^{iE_1t/\hbar}\sin\theta + e^{iE_2t/\hbar}\cos\theta$

(d) $e^{-iE_1t/\hbar}\sin^2\theta + e^{-iE_2t/\hbar}\cos^2\theta$

Ans. : (a)

Solution: $\left|\psi(t=0)\right\rangle = \sin\theta\left|1\right\rangle + \cos\theta\left|2\right\rangle$

$$\left|\psi(t)\right\rangle = \sin\theta\left|1\right\rangle e^{\frac{-iE_1t}{\hbar}} + \cos\theta\left|2\right\rangle e^{\frac{-iE_2t}{\hbar}}$$

$$\left\langle\psi(t)\middle|\psi(t)\right\rangle = \sin^2\theta\left\langle 1|1\right\rangle + \cos^2\theta\left\langle 2|2\right\rangle + 2\,\mathrm{Re}\, e^{\frac{-i(E_1-E_2)t}{\hbar}} \sin\theta\cdot\cos\theta\left\langle 1|2\right\rangle$$

$$= \sin^2\theta + \cos^2\theta + 0 = 1 \qquad \left(\because \left\langle 1|2\right\rangle = 0\right)$$

Q77. Consider a particle of mass $m$ in a potential $V(x) = \frac{1}{2}m\omega^2x^2 + g\cos kx$. The change in the ground state energy, compared to the simple harmonic potential $\frac{1}{2}m\omega^2x^2$, to first order in $g$ is

(a) $g\exp\left(-\frac{k^2\hbar}{2m\omega}\right)$ (b) $g\exp\left(\frac{k^2\hbar}{2m\omega}\right)$ (c) $g\exp\left(-\frac{2k^2\hbar}{m\omega}\right)$ (d) $g\exp\left(-\frac{k^2\hbar}{4m\omega}\right)$

Ans. : (d)

Solution: Ground state wavefunction

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

The perturbation term is $H_p = g\cos kx$

First order correction $E_0^1 = \int_{-\infty}^{\infty}\psi_0^*(x)H_P\psi_0(x)\,dx$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} g\int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{\hbar}}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx = \frac{g}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\left[\int_{-\infty}^{\infty} e^{\frac{-m\omega x^2}{\hbar}}.e^{ikx}dx + \int_{-\infty}^{\infty} e^{\frac{-m\omega x^2}{\hbar}}.\,e^{-ikx}dx\right]$$

$$= \frac{g}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{\hbar}+ikx}dx + \frac{g}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{\frac{-m\omega x^2}{\hbar}-ikx}dx$$

From 1st term, we have

$$= \frac{g}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}\left[x^2+\frac{2ikx\hbar}{2m\omega}+\left(\frac{ik\hbar}{2m\omega}\right)^2-\left(\frac{ik\hbar}{2m\omega}\right)^2\right]}dx = \frac{g}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}\left(x+\frac{ik\hbar}{2m\omega}\right)^2} e^{-\frac{k^2\hbar}{4m\omega}}dx$$

$$= \frac{g}{2} e^{-\frac{k^2\hbar}{4m\omega}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{\frac{m\omega}{\hbar}\left(x+\frac{ik\hbar}{2m\omega}\right)^2} dx = e^{-\frac{k^2\hbar}{4m\omega}}$$

Similarly, from term (ii), $\frac{g}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{m\omega x^2}{\hbar}-ikx} dx$

$$= \frac{g}{2} e^{-\frac{k^2\hbar}{4m\omega}} \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}\left(x-\frac{ik\hbar}{2m\omega}\right)^2} dx = e^{-\frac{k^2\hbar}{4m\omega}}$$

Hence, $E_0^1 = \frac{g}{2}\left[e^{-\frac{k^2\hbar}{4m\omega}} + e^{-\frac{k^2\hbar}{4m\omega}}\right] = ge^{-\frac{k^2\hbar}{4m\omega}}$

Q78. The energy levels for a particle of mass $m$ in the potential $V(x) = \alpha|x|$, determined in the $WKB$ approximation

$$\sqrt{2m}\int_a^b \sqrt{E - V(x)}dx = \left(n + \frac{1}{2}\right)\hbar\pi$$

(where $a, b$ are the turning points and $n = 0, 1, 2...$), are

(a) $E_n = \left[\frac{h\pi\alpha}{4\sqrt{m}}\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}$ (b) $E_n = \left[\frac{3h\pi\alpha}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}$

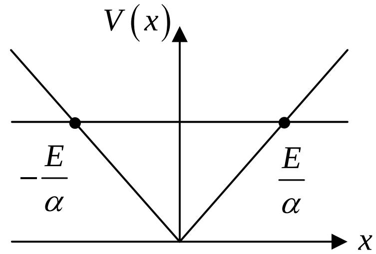
(c) $E_n = \left[\frac{3h\pi\alpha}{4\sqrt{m}}\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}$ (d) $E_n = \left[\frac{h\pi\alpha}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}$

Ans. : (b)

Solution: $V(x) = \alpha|x|$

$$\Rightarrow V(x) = \begin{cases} -\alpha x, & x < 0 \\ = \alpha x, & x > 0 \end{cases}$$

$$\sqrt{2m}\int_a^b \sqrt{E - V(x)}dx = \left(n + \frac{1}{2}\right)\pi\hbar$$




From figure, $a = \left(-\frac{E}{\alpha}\right), b = \left(\frac{E}{\alpha}\right) \Rightarrow \sqrt{2m}\int_{-\frac{E}{\alpha}}^{\frac{E}{\alpha}} \sqrt{E - V(x)}dx = \left(n + \frac{1}{2}\right)\pi\hbar$

$$\Rightarrow \sqrt{2m}\int_{-\frac{E}{\alpha}}^{0}\sqrt{E+\alpha x}dx+\int_{0}^{\frac{E}{\alpha}}\sqrt{E-\alpha x}dx=\left(n+\frac{1}{2}\right)\pi\hbar \Rightarrow 2\sqrt{2m}\int_{0}^{\frac{E}{\alpha}}\sqrt{E-\alpha x}\left(dx\right)=\left(n+\frac{1}{2}\right)\pi\hbar$$

put $E-\alpha x=t$, $\quad dx=-\frac{dt}{\alpha}$

limit $x\to 0 \quad \Rightarrow t\to E$, $\quad x\to\frac{E}{\alpha}\Rightarrow t\to 0$

$$2\sqrt{2m}\int_{E}^{0}\sqrt{t}\left(\frac{-dt}{\alpha}\right)=\left(n+\frac{1}{2}\right)\pi\hbar$$

$$\Rightarrow -\frac{2\sqrt{2m}}{\alpha}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_{E}^{0}=\left(n+\frac{1}{2}\right)\pi\hbar \Rightarrow \frac{2\sqrt{2m}}{\alpha}\frac{2}{3}.E^{\frac{3}{2}}=\left(n+\frac{1}{2}\right)\pi h$$

$$\Rightarrow \quad E^{\frac{3}{2}}=\left(n+\frac{1}{2}\right)\frac{3\pi\hbar\alpha}{4\sqrt{2m}}\Rightarrow E_n=\left[\frac{3\hbar\pi\alpha}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{\frac{2}{3}}$$

Q79. A particle of mass $m$ moves in one dimension under the influence of the potential $V\left(x\right)=-\alpha\delta\left(x\right)$, where $\alpha$ is a positive constant. The uncertainty in the product $\left(\Delta x\right)\left(\Delta p\right)$ in its ground state is

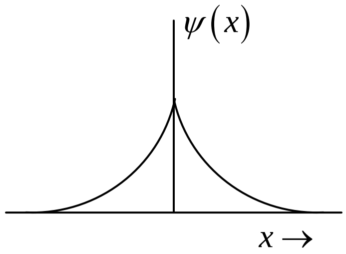
(a) $2\hbar$ (b) $\frac{\hbar}{2}$ (c) $\frac{\hbar}{\sqrt{2}}$ (d) $\sqrt{2}\hbar$

Ans. : (c)

Solution: $V\left(x\right)=-\alpha\delta\left(x\right)$

For this potential wavefunction

$$\psi\left(x\right)=\begin{cases}\sqrt{\alpha}e^{\alpha x}, & x<0\\ \sqrt{\alpha}e^{-\alpha x}, & x>0\end{cases}$$




which evenfunction about $x=0$

so $\langle x\rangle=0,\langle p\rangle=0$

now $\langle x^2\rangle=2\alpha\int_0^{\infty}x^2e^{-2\alpha x}dx=\frac{1}{2\alpha^2}\Rightarrow \Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\frac{1}{\sqrt{2}\alpha}$

$$\langle p^2\rangle = -\hbar^2\int_{-\infty}^{\infty}\psi^*\frac{d^2}{dx^2}\psi\,dx = -\hbar^2\int_{-\infty}^{0}\sqrt{\alpha}e^{\alpha x}\frac{d^2}{dx^2}\sqrt{\alpha}e^{\alpha x}dx - \hbar^2\int_{0}^{\infty}\sqrt{\alpha}e^{-\alpha x}\frac{d^2}{dx^2}\sqrt{\alpha}e^{-\alpha x}dx$$

$$= -\hbar^2\alpha^3\int_{-\infty}^{0}e^{2\alpha x}dx - \hbar^2\alpha^3\int_{0}^{\infty}e^{-2\alpha x}dx = -\frac{\hbar^2\alpha^3}{2\alpha} - \frac{\hbar^2\alpha^3}{2\alpha} = -\hbar^2\alpha^2$$, which is not possible

so, we will use the formula $\langle p\rangle^2 = \hbar^2\int_{-\infty}^{\infty}\left|\frac{d\psi}{dx}\right|^2 dx = \hbar^2\alpha^2$, $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \hbar\alpha$

now, $\Delta x.\Delta p = \frac{1}{\sqrt{2}\alpha}.\hbar\alpha = \frac{\hbar}{\sqrt{2}}$

Q80. The ground state energy of a particle of mass $m$ in the potential $V(x) = \frac{\hbar^2\beta}{6m}x^4$, estimated using the normalized trial wavefunction $\psi(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{\frac{-\alpha x^2}{2}}$, is

[use $\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} dx x^2 e^{-\alpha x^2} = \frac{1}{2\alpha}$ and $\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty} dx\ x^4 e^{-\alpha x^2} = \frac{3}{4\alpha^2}$ ]

(a) $\frac{3}{2m}\hbar^2\beta^{\frac{1}{3}}$ (b) $\frac{8}{3m}\hbar^2\beta^{\frac{1}{3}}$ (c) $\frac{2}{3m}\hbar^2\beta^{\frac{1}{3}}$ (d) $\frac{3}{8m}\hbar^2\beta^{\frac{1}{3}}$

Ans. : (d)

Solution: $\langle E\rangle = \langle T\rangle + \langle V\rangle$, for $\psi(x) = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\alpha x^2}{2}}$, $\langle T\rangle = \frac{\hbar^2\alpha}{4m}$

$$\langle V\rangle = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty}\frac{\hbar^2\beta}{6m}x^4 e^{-\alpha x^2}dx = \left(\frac{\alpha}{\pi}\right)^{\frac{1}{2}}\frac{\hbar^2\beta}{6m}\int_{-\infty}^{\infty}x^4 e^{-\alpha x^2}dx = \frac{\hbar^2\beta}{6m}.\frac{3}{4\alpha^2} = \frac{\hbar^2\beta}{8m\alpha^2}$$

$$\langle E\rangle = \frac{\hbar^2\alpha}{4m} + \frac{\hbar^2\beta}{8m\alpha^2} \qquad \text{(i)}$$

$$\frac{dE}{d\alpha} = \frac{\hbar^2}{4m} - \frac{2\hbar^2\beta}{8m\alpha^3} = 0 \Rightarrow \frac{\hbar^2}{4m}\left(1 - \frac{\beta}{\alpha^3}\right) = 0 \Rightarrow \alpha = (\beta)^{\frac{1}{3}}$$

Putting the value of $\alpha$ in equation (i),

$$\langle E\rangle = \frac{\hbar^2}{4m}(\beta)^{\frac{1}{3}} + \frac{\hbar^2\beta}{8m(\beta)^{\frac{2}{3}}} = \frac{\hbar^2}{4m}\left[(\beta)^{\frac{1}{3}} + \frac{(\beta)^{\frac{1}{3}}}{2}\right] = \frac{3}{8m}\hbar^2\beta^{\frac{1}{3}}$$

**NET/JRF (DEC-2016)**

Q81. Consider the two lowest normalized energy eigenfunctions $\psi_0(x)$ and $\psi_1(x)$ of a one dimensional system. They satisfy $\psi_0(x)=\psi_0^*(x)$ and $\psi_1(x)=\alpha\frac{d\psi_0}{dx}$, where $\alpha$ is a real constant. The expectation value of the momentum operator in the state $\psi_1$ is

(a) $-\frac{\hbar}{\alpha^2}$ (b) 0 (c) $\frac{\hbar}{\alpha^2}$ (d) $\frac{2\hbar}{\alpha^2}$

Ans. : (b)

Solution: $\psi_1(x)=\alpha\frac{d\psi_0}{dx}$

$$\langle p_x\rangle=\int_{-\infty}^{\infty}\psi_1^* p_x\psi\,dx=\int_{-\infty}^{\infty}\psi_1^*\left(-i\hbar\frac{\partial\psi_1}{\partial x}\right)dx=\int_{-\infty}^{\infty}\alpha^*\frac{d\psi_0}{dx}(-i\hbar\alpha)\frac{d^2\psi_0}{dx^2}dx$$

$$=-i\hbar|\alpha|^2\int_{-\infty}^{\infty}\frac{d\psi_0}{dx}\frac{d^2\psi_0}{dx^2}dx$$

Integrate by parts

$$I=-i\hbar|\alpha|^2\left(\frac{d\psi_0}{dx}\frac{d\psi_0}{dx}\bigg|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}\frac{d^2\psi_0}{dx^2}\frac{d\psi_0}{dx}dx\right)=0-(-i\hbar)|\alpha|^2\int_{-\infty}^{\infty}\frac{d\psi_0}{dx}\frac{d^2\psi_0}{dx^2}dx$$

$$I=0-(-i\hbar)|\alpha|^2\int_{-\infty}^{\infty}\frac{d\psi_0}{dx}\frac{d^2\psi_0}{dx^2}dx$$

$$\frac{d\psi_0}{dx}\to\frac{\psi_0}{\alpha},\quad \psi_0=0,\,x\to\infty$$

$$I=0-I\Rightarrow 2I=0\Rightarrow I=0\quad\Rightarrow\langle p_x\rangle=0$$

Q82. Consider the operator, $a=x+\frac{d}{dx}$ acting on smooth function of $x$. Then commutator $[\alpha,\cos x]$ is

(a) $-\sin x$ (b) $\cos x$ (c) $-\cos x$ (d) 0

Ans. : (a)

Solution: $a=x+\frac{d}{dx}$

$$[a,\cos x]=\left[x+\frac{d}{dx},\cos x\right]=[x,\cos x]+\left[\frac{d}{dx},\cos x\right]=0+\left[\frac{d}{dx},\cos x\right]$$

$$\left[\frac{d}{dx},\cos x\right]\psi(x)=\frac{d}{dx}\cos x\psi(x)-\cos x\frac{d\psi}{dx}$$

$$=\cos x\frac{d\psi}{dx}+(-\sin x)\psi-\frac{\cos x d\psi}{dx}=-\sin x\psi$$

$$[a,\cos x]\psi(x)=-\sin x\psi$$

$$[a,\cos x]=-\sin x$$

Q83. Consider the operator $\vec{\pi}=\vec{p}-q\vec{A}$ , where $\vec{p}$ is the momentum operator, $\vec{A}=(A_x,A_y,A_z)$ is the vector potential and $q$ denotes the electric charge. If $\vec{B}=(B_x,B_y,B_z)$ denotes the magnetic field, the $z$ -component of the vector operator $\vec{\pi}\times\vec{\pi}$ is

(a) $iq\hbar B_z+q(A_xp_y-A_yp_x)$ (b) $-iq\hbar B_z-q(A_xp_y-A_yp_x)$

(c) $-iq\hbar B_z$ (d) $iq\hbar B_z$

Ans. : (d)

Solution: $\vec{\pi}=\vec{p}-q\vec{A}$

$$(\vec{\pi}\times\vec{\pi})\psi=(\vec{p}-q\vec{A})\times(\vec{p}-q\vec{A})\psi=\vec{p}\times\vec{p}\psi-q\vec{p}\times\vec{A}\psi-q\vec{A}\times\vec{p}\psi+q^2\vec{A}\times\vec{A}\psi$$

$$\vec{p}\times\vec{p}\psi=0$$

$$-q\vec{p}\times\vec{A}\psi=-q(-i\hbar\vec{\nabla}\times\vec{A})\psi=qi\hbar\vec{B}\psi$$

$$q\vec{A}\times\vec{p}\psi=q(\vec{A}(-i\hbar\vec{\nabla}))\psi=0$$

$$q^2\vec{A}\times\vec{A}\psi=0$$

$$\vec{\pi}\times\vec{\pi}=qi\hbar\vec{B}$$

So, $z$ component is given by $qi\hbar B_z$

Q84. The dynamics of a free relativistic particle of mass $m$ is governed by the Dirac Hamiltonian $H = c\vec{\alpha}.\vec{p} + \beta mc^2$, where $\vec{p}$ is the momentum operator and $\vec{\alpha} = (\alpha_x, \alpha_y, \alpha_z)$ and $\beta$ are four $4\times4$ Dirac matrices. The acceleration operator can be expressed as

(a) $\frac{2ic}{\hbar}(c\vec{p} - \vec{\alpha}H)$ (b) $2ic^2\vec{\alpha}\beta$

(c) $\frac{ic}{\hbar}H\vec{\alpha}$ (d) $-\frac{2ic}{\hbar}(c\vec{p} + \vec{\alpha}H)$

Ans. : (a)

Solution: $H = c\vec{\alpha}.\vec{p} + \beta mc^2$

If $v_x$ velocity of $x$ direction

From the Ehrenfest theorem

$$v_x = \frac{dx}{dt} = \frac{1}{i\hbar}[x, H] + \frac{\partial x}{\partial t} = \frac{1}{i\hbar}\left[x, c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z + \beta mc^2\right] + 0$$

$$= \frac{c}{i\hbar}[x, \alpha_x p_x] = c\alpha_x$$

Similarly, acceleration is given by

$$a_x = \frac{dv_x}{dt} = \frac{1}{i\hbar}[c\alpha_x, H] = \frac{c}{i\hbar}\left[\alpha_x, c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z + \beta mc^2\right]$$

Using relation $\alpha_i\alpha_j + \alpha_j\alpha_i = 0$ , $\alpha_i\beta + \beta\alpha_i = 0$ and $\left[\alpha_i, p_j\right] = 0$

$$[\alpha_x, c\alpha_x p_x] = 0$$

$$\left[\alpha_x, c\alpha_y p_y\right] = c\left[\alpha_x\alpha_y - \alpha_y\alpha_x\right]p_y + \alpha_y c\left[\alpha_x, p_y\right] = \left[c\alpha_x\alpha_y - (-c\alpha_x\alpha_y)\right]p_y + 0 = 2c\alpha_x\alpha_y p_y$$

$$[\alpha_x, c\alpha_z p_z] = [c\alpha_x\alpha_z - c\alpha_z\alpha_x]p_z + \alpha_z[c\alpha_x, p_z] = \left[c\alpha_x\alpha_z - (-c\alpha_x\alpha_z)\right]p_z + 0 = 2c\alpha_x\alpha_z p_z$$

$$\left[\alpha_x, \beta mc^2\right] = [\alpha_x\beta - \beta\alpha_x]mc^2 = 2mc^2\alpha_x\beta$$

$$a_x = \frac{c}{i\hbar}\left[2c\alpha_x\alpha_y p_y + 2c\alpha_x\alpha_z p_z + 2\alpha_x\beta mc^2\right]$$

$$a_x = \frac{2\alpha_x c}{i\hbar}\left[c\alpha_y p_y + c\alpha_z p_z + \beta mc^2 + c\alpha_x p_x - c\alpha_x p_x\right]$$

$$a_x = \frac{2\alpha_x c}{i\hbar}\left[c\alpha_x p_x + c\alpha_y p_y + c\alpha_z p_z + \beta mc^2 - c\alpha_x p_x\right]$$

$$a_x = \frac{2c}{i\hbar}\left[\alpha_x.H - c\alpha_x\alpha_x p_x\right] = \frac{2ic}{\hbar}\left[c\alpha_x\alpha_x p_x - \alpha_x.H\right],\ \left(\alpha_x^2 = x\right)$$

$$\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = \left[\frac{2ic}{\hbar}\left(c\vec{p} - \vec{\alpha}.\vec{H}\right)\right]$$

Q85. A particle of charge $q$ in one dimension is in a simple harmonic potential with angular frequency $\omega$. It is subjected to a time- dependent electric field $E(t) = Ae^{-\left(\frac{t}{\tau}\right)^2}$, where $A$ and $\tau$ are positive constants and $\omega\tau \gg 1$. If in the distant past $t \to -\infty$ the particle was in its ground state, the probability that it will be in the first excited state as $t \to +\infty$ is proportional to

(a) $e^{-\frac{1}{2}(\omega\tau)^2}$ (b) $e^{\frac{1}{2}(\omega\tau)^2}$ (c) 0 (d) $\frac{1}{(\omega\tau)^2}$

Ans. : (a)

Solution: Transition probability is proportional to $P_{if} \propto \left|\int_{-\infty}^{\infty} e^{-\frac{t^2}{\tau^2}} e^{i\omega_{fi}t}\right|^2$ where

$$\omega_{fi} = \frac{\frac{3}{2}\hbar\omega - \frac{1}{2}\hbar\omega}{\hbar} = \omega$$

$$P_{if} = \left|\int_{-\infty}^{\infty} \exp -\frac{t^2}{\tau^2} + i\omega t\, dt\right|^2$$

Now calculate $\int_{-\infty}^{\infty} \exp\left(-\frac{t^2}{\tau^2} + i\omega t\right)dt = \int_{-\infty}^{\infty} \exp -\frac{1}{\tau^2}\left(t^2 - i\omega t\tau^2 + \left(\frac{i\omega\tau^2}{2}\right)^2 - \left(\frac{i\omega\tau^2}{2}\right)^2\right)$

$$= \exp\left(-\frac{\omega^2\tau^2}{4}\right)\int_{-\infty}^{\infty} \exp\frac{1}{\tau^2}\left(t - \frac{i\omega t}{2}\right)^2 dt$$

$$P_{if} = \left|\int_{-\infty}^{\infty} \exp -\frac{t^2}{\tau^2} + i\omega t\, dt\right|^2$$

$$P_{if} = \left|\exp\left(-\frac{\omega^2\tau^2}{4}\right)\int_{-\infty}^{\infty} \exp\frac{1}{\tau^2}\left(t - \frac{i\omega t}{2}\right)^2 dt\right|^2$$

$$P_{if} \propto \exp -\frac{\omega^2\tau^2}{2}$$

Q86. A particle is scattered by a central potential $V(r)=V_0 re^{-\mu r}$, where $V_0$ and $\mu$ are positive constants. If the momentum transfer $\vec{q}$ is such that $q=|\vec{q}|\gg\mu$, the scattering cross-section in the Born approximation, as $q\to\infty$, depends on $q$ as

[You may use $\int x^n e^{ax}dx=\frac{d^n}{da^n}\int e^{ax}dx$]

(a) $q^{-8}$ (b) $q^{-2}$ (c) $q^2$ (d) $q^6$

Ans. : (a)

Solution: The form factor is given for high energy as $q\to\infty$

$$f(\theta,\phi)=\frac{-2m}{\hbar^2 q}\int_0^\infty rV(r)\sin qr\,dr=\frac{-2m}{\hbar^2 q}\int_0^\infty r^2V_0e^{-\mu v}\sin qr\,dr$$

$$=\frac{-2m}{\hbar^2 q}V_0\int_0^\infty r^2e^{-\mu r}\frac{e^{iqr}-e^{-iqr}}{2i}dr=\frac{mV_0}{\hbar^2 q}i\left[\int_0^\infty r^2e^{-r(\mu-iq)}dr-\int_0^\infty r^2e^{-r(\mu+iq)}dr\right]$$

$$=\frac{mV_0 i}{\hbar^2 q}\left[\frac{\lfloor 2}{(\mu-iq)^3}-\frac{\lfloor 2}{(\mu+iq)^3}\right]=\frac{2mV_0 i}{\hbar^2 q}\left[\frac{\left((\mu+iq)^3-(\mu-iq)^3\right)}{(\mu+iq)^3(\mu-iq)^3}\right]$$

$$=\frac{2mV_0}{\hbar^2 q}\frac{i\left[\left(\mu^3-iq^3+3\mu^2iq-3\mu q^2\right)-\left(\mu^3+iq^3-3\mu^2iq-3\mu q^2\right)\right]}{\left(\mu^2+q^2\right)^3}$$

$$=\frac{2mV_0 i}{\hbar^2 q}\left[\frac{6\mu^2iq-2iq^3}{\left(\mu^2+q^2\right)^3}\right]=\frac{2mV_0}{\hbar^2 q}\left[\frac{2q^3-6\mu^2q}{\left(\mu^2+q^2\right)^3}\right]$$

$$\propto\frac{q^3}{q}\left(2-\frac{6\mu^2}{q^2}\right)\times\frac{1}{q^6\left(\frac{\mu^2}{q^2}+1\right)^3}\propto q^2\times\frac{1}{q^6}\propto\frac{1}{q^4}\qquad\left(\because\frac{\mu^2}{q^2}<<1\right)$$

$$\sigma(\theta)\propto|f(\theta)|^2\propto\left(q^{-4}\right)^2=q^{-8}$$

Q87. A particle in one dimension is in a potential $V(x)=A\delta(x-a)$. Its wavefunction $\psi(x)$ is continuous everywhere. The discontinuity in $\frac{d\psi}{dx}$ at $x=a$ is

(a) $\frac{2m}{\hbar^2}A\psi(a)$ (b) $A(\psi(a)-\psi(-a))$

(c) $\frac{\hbar^2}{2m}A$ (d) 0

Ans. : (a)

Solution: $-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+A\delta(x-a)\psi(x)=E\psi(x)$

Integrates both side within limit

$a-\in$ to $a+\in$

$$-\frac{\hbar^2}{2m}\int_{a-\in}^{a+\in}\frac{d^2\psi}{dx^2}dx+\int_{a-\in}^{a+\in}A\delta(x-a)\psi dx=E\int_{a-\in}^{a+\in}\psi(x)dx$$

$$-\frac{\hbar^2}{2m}\left(\frac{d\psi_{II}}{dx}-\frac{d\psi_I}{dx}\right)+A\psi(a)=0$$

$$\frac{d\psi_{II}}{dx}-\frac{d\psi_I}{dx}=\frac{2mA}{\hbar^2}\psi(a)$$

so discontinues in $\frac{d\psi}{dx}$ at $x=a$ is $\frac{2mA}{\hbar^2}\psi(a)$.

**NET/JRF (JUNE-2017)**

Q88. If the root-mean-squared momentum of a particle in the ground state of a one-dimensional simple harmonic potential is $p_0$, then its root-mean-squared momentum in the first excited state is

(a) $p_0\sqrt{2}$ (b) $p_0\sqrt{3}$ (c) $p_0\sqrt{2/3}$ (d) $p_0\sqrt{3/2}$

Ans. : (b)

Solution: $P=\sqrt{m\omega\hbar}\hat{P}=\sqrt{m\omega\hbar}\frac{(a-a^\dagger)}{\sqrt{2}i}$

$$P^2=-\frac{m\omega\hbar}{2}\left(a^2+a^{\dagger 2}-(2N+1)\right)$$

$$\langle P^2\rangle=-\frac{m\omega\hbar}{2}\left(\langle a^2\rangle+\langle a^{\dagger 2}\rangle-\langle 2N+1\rangle\right)$$

For any state $|n\rangle$,

$\langle a^2\rangle=0,\langle a^{\dagger 2}\rangle=0$ and $\langle 2N+1\rangle=2n+1$

$\langle P^2\rangle=(2n+1)\frac{m\omega\hbar}{2}$ and $\langle P\rangle=0$

$P_{rms}=\sqrt{\langle P^2\rangle-\langle P\rangle^2}\Rightarrow P_{rms}=\sqrt{\frac{m\omega\hbar}{2}}\sqrt{2n+1}$

For ground stat $n=0,\ P_{rms}=\sqrt{\frac{m\omega\hbar}{2}}=P_0$

So, for $n=1,\ P_{rms}=\sqrt{\frac{m\omega\hbar}{2}}\sqrt{3}$

$P_{rms}=\sqrt{3}P_0$

Q89. Consider a potential barrier $A$ of height $V_0$ and width $b$, and another potential barrier $B$ of height $2V_0$ and the same width $b$. The ratio $T_A/T_B$ of tunnelling probabilities $T_A$ and $T_B$, through barriers $A$ and $B$ respectively, for a particle of energy $V_0/100$ is best approximated by

(a) $\exp\left[\left(\sqrt{1.99}-\sqrt{0.99}\right)\sqrt{8mV_0b^2/\hbar^2}\right]$ (b) $\exp\left[\left(\sqrt{1.98}-\sqrt{0.98}\right)\sqrt{8mV_0b^2/\hbar^2}\right]$

(c) $\exp\left[\left(\sqrt{2.99}-\sqrt{0.99}\right)\sqrt{8mV_0b^2/\hbar^2}\right]$ (d) $\exp\left[\left(\sqrt{2.98}-\sqrt{0.98}\right)\sqrt{8mV_0b^2/\hbar^2}\right]$

Ans. : (a)

Solution: $T\alpha e^{-\sqrt{2m(V-E)}}$, where $E=\frac{V_0}{100}$

For potential $A$, $V=V_0$

$T_A\alpha e^{-\sqrt{\frac{2m}{\hbar^2}\left(V_0-\frac{V_0}{100}\right)}}\Rightarrow T_A\alpha e^{-\sqrt{\frac{2m}{\hbar^2}\left(\frac{99}{100}V_0\right)}}\alpha e^{-\sqrt{2m(0.99V_0)}}$

For Potential $B$, $V=2V_0$ and $E=\frac{V_0}{100}$

$T_B\alpha\ e^{-\sqrt{\frac{2m}{\hbar^2}\left(2V_0-\frac{V_0}{100}\right)}}\Rightarrow T_B\alpha\ e^{-\sqrt{\frac{2m}{\hbar^2}\left(\frac{199V_0}{100}\right)}}\alpha\ e^{-\sqrt{2m(1.99V_0)}}$

$\frac{T_A}{T_B}=\frac{e^{-\sqrt{0.99V_0}}}{e^{-\sqrt{1.99V_0}}}$

$\frac{T_A}{T_B}=\left(e^{\sqrt{1.99V_0}}-e^{-\sqrt{0.99V_0}}\right)$

Q90. A constant perturbation $H'$ is applied to a system for time $\Delta t$ (where $H'\Delta t << \hbar$) leading to a transition from a state with energy $E_i$ to another with energy $E_f$. If the time of application is doubled, the probability of transition will be

(a) unchanged (b) doubled (c) quadrupled (d) halved

Ans. : (c)

Solution: For constant potential transition probability

$$p_{if} = 4\frac{\left|\langle\psi_f|v|\psi_i\rangle\right|^2}{h^2\omega_{fi}^2}\left(\sin^2\frac{\omega_{fi}t_i}{2}\right)$$

at $t_i = 2t_i$,

$$p_{if} = \frac{4\left|\langle\psi_f|v|\psi_i\rangle\right|^2}{h^2\omega_{fi}^2}\sin^2\frac{\omega_{fi}t_i}{2}$$

at $t_i = 2t_i$,

$$p_{ff} = \frac{4\left|\langle\psi_f|v|\psi_i\rangle\right|^2}{h^2\omega_{fi}^2}\sin^2\left(\frac{\omega_{fi}2t_i}{2}\right) = \frac{4\left|\langle\psi_f|v|\psi_i\rangle\right|^2}{h^2\omega_{fi}^2}\sin\left(\omega_{fi}t_i\right)$$

$$\frac{p_{if}}{p_{ff}} = \frac{\sin^2\left(\omega_{fi}t_i\right)}{\sin^2\left(\frac{\omega_{fi}t_i}{2}\right)} \Rightarrow \frac{\frac{\sin^2\left(\omega_{fi}t_i\right)}{\omega_{fi}^2t_i^2}\omega_{fi}^2t_i^2}{\frac{\sin^2\left(\frac{\omega_{fi}t_i}{2}\right)\left(\frac{\omega_{fi}t_i}{2}\right)^2}{\frac{\omega_{fi}^2t_i^2}{2}}} \qquad t_1 \to 0$$

$$= \frac{4\omega_{fi}^2t_i^2}{\omega_{fi}^2t_i^2} = 4$$

$$\frac{p_{if(2)}}{p_{if(1)}} = 4 \Rightarrow p_{if(2)} = 4p_{if(1)}$$

Q91. The two vectors $\begin{pmatrix} a \\ 0 \end{pmatrix}$ and $\begin{pmatrix} b \\ c \end{pmatrix}$ are orthonormal if

(a) $a = \pm 1,\ b = \pm 1/\sqrt{2},\ c = \pm 1/\sqrt{2}$ (b) $a = \pm 1,\ b = \pm 1,\ c = 0$

(c) $a = \pm 1,\ b = 0,\ c = \pm 1$ (d) $a = \pm 1,\ b = \pm 1/2,\ c = 1/2$

Ans. : (c)

Solution: $|\phi_1\rangle = \begin{pmatrix} a \\ 0 \end{pmatrix}$, $|\phi_2\rangle = \begin{pmatrix} b \\ c \end{pmatrix}$

$$\langle \phi_1 | \phi_1 \rangle = 1 \quad \Rightarrow a = \pm 1$$

$$\langle \phi_2 | \phi_2 \rangle = 1 \quad \Rightarrow |b|^2 + |c|^2 = 1$$

$$\langle \phi_1 | \phi_2 \rangle = 0 \quad \Rightarrow (a \;\; 0)\binom{b}{c} = 0$$

$$a.b + 0 \cdot c = 0 \Rightarrow a \cdot b = 0$$

$$\text{so } b = 0$$

$$|c|^2 = 1, \quad c = \pm 1$$

$$a = \pm 1, \quad b = 0, \quad c = \pm 1$$

Q92. Consider the potential

$$V(\vec{r}) = \sum_i V_0 a^3 \delta^{(3)}(\vec{r} - \vec{r}_i)$$

where $\vec{r}_i$ are the position vectors of the vertices of a cube of length $a$ centered at the origin and $V_0$ is a constant. If $V_0 a^2 << \frac{\hbar^2}{m}$, the total scattering cross-section, in the low-energy limit, is

(a) $16a^2\left(\frac{mV_0a^2}{\hbar^2}\right)$ (b) $\frac{16a^2}{\pi^2}\left(\frac{mV_0a^2}{\hbar^2}\right)^2$

(c) $\frac{64a^2}{\pi}\left(\frac{mV_0a^2}{\hbar^2}\right)^2$ (d) $\frac{64a^2}{\pi^2}\left(\frac{mV_0a^2}{\hbar^2}\right)$

Ans. : (c)

Solution: $V(r) = \sum_i V_0 a^3 \delta^3(\vec{r} - \vec{r}_i)$

$$= \sum_i V_0 a^3 \delta(x - x_i)\delta(y - y_i)\delta(z - z_i)$$

where $x_i, y_i, z_i$ are co-ordinate at 8 corner cube whose center is at origin.

$$f(\theta) = -\frac{m}{2\pi\hbar^2}\int V(r) d^3r$$

$$= \frac{-m}{2\pi\hbar^2} V_0 a^3 \int \int_{-\infty}^{\infty} \int \sum_{i=1}^{8} \delta(x - x_i)\delta(y - y_i)\delta(z - z_i) dxdydz$$

$$= \frac{-m}{2\pi\hbar^2} V_0 a^3 [1+1+1+1+1+1+1+1]$$

$$= \frac{-8mV_0a^3}{2\pi\hbar^2} = \frac{-4mV_0a^3}{\pi\hbar^2}$$

total scattering cross section $\sigma = \int |f(\theta)|^2 \sin\theta d\theta d\phi.$

Differential scattering cross section $D(\theta) = |f(\theta)|^2 = \frac{16m^2V_0^2a^6}{\pi^2\hbar^4}$

$$= \frac{16m^2V_0^2a^6}{\pi^2\hbar^4} 4\pi = \frac{64a^2}{\pi}\left(\frac{m^2V_0^2a^4}{h^4}\right)$$

$$\sigma = \frac{64a^2}{\pi}\left(\frac{mV_0a^2}{\hbar^2}\right)^2$$

Q93. The Coulomb potential $V(r) = -e^2/r$ of a hydrogen atom is perturbed by adding $H' = bx^2$ (where $b$ is a constant) to the Hamiltonian. The first order correction to the ground state energy is

(The ground state wavefunction is $\psi_0 = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$)

(a) $2ba_0^2$ (b) $ba_0^2$ (c) $ba_0^2/2$ (d) $\sqrt{2}ba_0^2$

Ans. : (b)

Solution: $H' = bx^2$ put $x = r\sin\theta\cos\phi$

$H' = br^2 \sin^2\theta \cos^2\phi.$

$$E_1^1 = \langle\psi_1|H'|\psi_1\rangle, |\psi_1\rangle = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

$$= \int \psi_1^* H' \psi_1 r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$= \frac{b}{\pi a_0^3} \int_0^{\infty} r^2 e^{-\frac{2r}{a_0}} r^2 dr \int_0^{\pi} \sin^3\theta d\theta \int_0^{2\pi} \cos^2\phi d\phi = ba_0^2$$

Q94. Using the trial function

$$\psi(x)=\begin{cases}A\left(a^2-x^2\right), & -a<x<a\\ 0 & \text{otherwise}\end{cases}$$

the ground state energy of a one-dimensional harmonic oscillator is

(a) $\hbar\omega$ (b) $\sqrt{\frac{5}{14}}\hbar\omega$ (c) $\frac{1}{2}\hbar\omega$ (d) $\sqrt{\frac{5}{7}}\hbar\omega$

Ans. : (b)

Solution: $\psi(x)=\begin{cases}A\left(a^2-x^2\right), & -a<x<a\\ 0 \quad , & \text{otherwise}\end{cases}$

For normalization

$$\int\psi^*\psi dx=1$$

$$A^2=\frac{15}{16a^5}\Rightarrow A=\sqrt{\frac{15}{16a^5}}$$

$$\langle T\rangle=\frac{-\hbar^2}{2m}\int_{-a}^{a}\psi^*\frac{\partial^2}{\partial x^2}\psi dx=\frac{-\hbar^2}{2m}\frac{15}{16a^5}\cdot(-2)(2)\int_0^a\left(a^2-x^2\right)dx$$

$$\langle T\rangle=\frac{5\hbar^2}{4ma^2}$$

$$\langle V\rangle=\int_{-a}^{a}\psi^*V\psi dx\text{, where } V(x)=\frac{1}{2}m\omega^2x^2=\frac{1}{2}m\omega^2\frac{15}{16a^5}2\int_0^a x^2\left(a^2-x^2\right)^2dx.$$

$$\langle V\rangle=\frac{m\omega^2a^2}{14}$$

$$E=T+V=\frac{5\hbar^2}{4ma^2}+\frac{m\omega^2a^2}{14}$$

$$\frac{dE}{da}=0\Rightarrow\frac{5\times(-2)\hbar^2}{4ma^3}+\frac{m\omega^2a}{7}=0\Rightarrow a^4=\frac{35}{2}\left(\frac{\hbar^2}{m^2\omega^2}\right).$$

$$a^2=\left(\frac{35}{2}\right)^{1/2}\left(\frac{\hbar}{m\omega}\right).$$

$$E=\frac{5}{4}\times\frac{\hbar^2}{m}.\frac{m\omega}{\hbar}\sqrt{\frac{2}{35}}+\frac{m\omega^2}{14}\sqrt{\frac{35}{2}}\frac{\hbar}{m\omega}.$$

$$=\frac{\hbar\omega}{2}\left(\frac{5}{2}\sqrt{\frac{2}{35}}+\frac{1}{7}\sqrt{\frac{35}{2}}\right)=\frac{\hbar\omega}{2}\left(\sqrt{\frac{5}{14}}+\sqrt{\frac{5}{14}}\right)=\hbar\omega\sqrt{\frac{5}{14}}$$

**NET/JRF (DEC - 2017)**

Q95. Let $x$ denote the position operator and $p$ the canonically conjugate momentum operator of a particle. The commutator

$$\left[\frac{1}{2m}p^2+\beta x^2, \frac{1}{m}p^2+\gamma x^2\right]$$

where $\beta$ and $\gamma$ are constants, is zero if

(a) $\gamma=\beta$ (b) $\gamma=2\beta$ (c) $\gamma=\sqrt{2}\beta$ (d) $2\gamma=\beta$

Ans. : (b)

Solution: $\left[\frac{1}{2m}p^2+\beta x^2, \frac{1}{m}p^2+\gamma x^2\right]=0\Rightarrow\frac{1}{2m}\gamma\left[p^2,x^2\right]+\frac{\beta}{m}\left[x^2,p^2\right]=0$

$$-\frac{\gamma}{2m}\left[x^2,p^2\right]+\frac{\beta}{m}\left[x^2,p^2\right]=0\Rightarrow\frac{1}{m}\left[x^2,p^2\right]\left[\frac{-\gamma}{2}+\beta\right]=0\Rightarrow\gamma=2\beta$$

Q96. The normalized wavefunction of a particle in three dimensions is given by $\psi(r,\theta,\varphi)=\frac{1}{\sqrt{8\pi a^3}}e^{-r/2a}$ where $a>0$ is a constant. The ratio of the most probable distance from the origin to the mean distance from the origin, is

[You may use $\int_0^\infty dx\, x^n e^{-x}=n!$]

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{3}{2}$ (d) $\frac{2}{3}$

Ans. : (d)

Solution: $\psi(r,\theta,\varphi)=\frac{1}{\sqrt{8\pi a^3}}e^{\frac{-r}{2a}}$

$$\langle r\rangle=\iiint r\psi^*\psi r^2 dr\sin\theta d\theta d\phi=\frac{3}{2}(2a)=3a$$

one can compare the wave function at hydrogen atom with Bohr radius $a_0=2a$

most probable distance,

$$\frac{d}{dr}r^2 e^{-r/a} = 0$$
$$r_P = 2a$$
$$\frac{r_p}{\langle r\rangle} = \frac{2a}{3a} = \frac{2}{3}$$

Q97. The state vector of a one-dimensional simple harmonic oscillator of angular frequency $\omega$, at time $t = 0$, is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |2\rangle\right]$, where $|0\rangle$ and $|2\rangle$ are the normalized ground state and the second excited state, respectively. The minimum time $t$ after which the state vector $|\psi(t)\rangle$ is orthogonal to $|\psi(0)\rangle$, is

(a) $\frac{\pi}{2\omega}$ (b) $\frac{2\pi}{\omega}$ (c) $\frac{\pi}{\omega}$ (d) $\frac{4\pi}{\omega}$

Ans. : (a)

Solution: $\langle\psi(0)| = \frac{1}{\sqrt{2}}\left[|0\rangle + |2\rangle\right]$

$$E_2 = \frac{5}{2}\hbar\omega \qquad |\psi(t)\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle e^{\frac{-\hbar\omega t}{2\hbar}} + |2\rangle e^{\frac{-5\hbar}{2}\frac{\omega t}{\hbar}}\right]$$

$$E_0 = \frac{\hbar\omega}{2} \cdot \langle\psi(0)\psi|\psi(t)\rangle = 0 \Rightarrow t = \frac{\hbar}{E_2 - E_0}\cos^{-1}(-1)$$

$$t = \frac{\hbar}{\left(\frac{5\hbar\omega}{2} - \frac{1}{2}\hbar\omega\right)}\cos^{-1}(-1) = \frac{\hbar}{2\hbar\omega/2}\pi = \frac{\pi}{2\omega}.$$

Q98. The normalized wavefunction in the momentum space of a particle in one dimension is $\phi(p) = \frac{\alpha}{p^2 + \beta^2}$, where $\alpha$ and $\beta$ are real constants. The uncertainty $\Delta x$ in measuring its position is

(a) $\sqrt{\pi}\frac{\hbar\alpha}{\beta^2}$ (b) $\sqrt{\pi}\frac{\hbar\alpha}{\beta^3}$ (c) $\frac{\hbar}{\sqrt{2}\beta}$ (d) $\sqrt{\frac{\pi}{\beta}}\frac{\hbar\alpha}{\beta}$

Ans. : (c)

Solution: $\phi(p) = \frac{\alpha}{p^2 + \beta^2}$

From inverse Fourier transformation

Normalize, $\psi(x)=\sqrt{\frac{\beta}{\hbar}}e^{-\frac{\beta|x|}{\hbar}}$

$\langle x\rangle = 0$,

$$\langle x^2\rangle = \frac{\beta}{h}\int_{-\infty}^{\infty} x^2 e^{-2\frac{\beta|x|}{\hbar}}dx = \frac{\hbar^2}{2\beta^2}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{\hbar}{\sqrt{2}\beta}$$

Q99. A phase shift of $30^0$ is observed when a beam of particles of energy $0.1MeV$ is scattered by a target. When the beam energy is changed, the observed phase shift is $60^0$. Assuming that only $s$-wave scattering is relevant and that the cross-section does not change with energy, the beam energy is

(a) 0.4 $MeV$ (b) 0.3 $MeV$ (c) 0.2 $MeV$ (d) 0.15 $MeV$

Ans. : (b)

Solution: $\sigma = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$

only $s$-wave scattering is relevant $l=0$ $k=\sqrt{\frac{2mE}{\hbar^2}}$

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi\hbar^2}{2mE}\sin^2\delta_0$$

According to problem $\frac{\sin^2 30}{0.1MeV} = \frac{\sin^2 60}{E} \Rightarrow E = \frac{\sin^2 60}{\sin^2 30}\times 0.1MeV = 0.3MeV$

Q100. The Hamiltonian of a two-level quantum system is $H=\frac{1}{2}\hbar\omega\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ possible initial state in which the probability of the system being in that quantum state does not change with time, is

(a) $\begin{pmatrix}\cos\frac{\pi}{4}\\ \sin\frac{\pi}{4}\end{pmatrix}$ (b) $\begin{pmatrix}\cos\frac{\pi}{8}\\ \sin\frac{\pi}{8}\end{pmatrix}$ (c) $\begin{pmatrix}\cos\frac{\pi}{2}\\ \sin\frac{\pi}{2}\end{pmatrix}$ (d) $\begin{pmatrix}\cos\frac{\pi}{6}\\ \sin\frac{\pi}{6}\end{pmatrix}$

Ans. : (b)

Q101. Consider a one-dimensional infinite square well

$$V(x)=\begin{cases}0 & \text{for } 0<x<a,\\ \infty & \text{otherwise}\end{cases}$$

If a perturbation

$$\Delta V(x)=\begin{cases}V_0 & \text{for } 0<x<a/3,\\ 0 & \text{otherwise}\end{cases}$$

is applied, then the correction to the energy of the first excited state, to first order in $\Delta V$, is nearest to

(a) $V_0$ (b) $0.16\, V_0$ (c) $0.2\, V_0$ (d) $0.33\, V_0$

Ans. : (d)

Solution: $\Delta V = \int_0^{a/3} \Delta V_x \phi_2^* \phi_2 dx$

$$\Delta V = \int_0^{a/3} V_0 \frac{2}{a}\sin^2\left(\frac{2\pi x}{a}\right)dx = \frac{2}{a}V_0\int_0^{a/3}\frac{1}{2}\left[1-\cos\frac{4\pi x}{a}\right]dx$$

$$=\frac{2}{a}V_0\left[\frac{a}{6}-\frac{\sin\frac{4\pi}{3}}{\frac{4\pi}{a}}\right]=V_0\left[\frac{1}{3}+\frac{\sqrt{3}}{4\pi}\right]\simeq 0.33\, V_0$$

Q102. The energy eigenvalues $E_n$ of a quantum system in the potential $V = cx^6$ (where $c>0$ is a constant), for large values of the quantum number $n$, varies as

(a) $n^{4/3}$ (b) $n^{3/2}$ (c) $n^{5/4}$ (d) $n^{6/5}$

Ans. : (b)

Solution: We can use Bohr Somerfield theory

$V(x) = cx^6$

$$\oint Pdx = nh = 4\int_0^{\left(\frac{E}{C}\right)^{1/6}} \sqrt{2m\left(E-cx^6\right)}dx = nh = \sqrt{2mE}\left(\frac{E}{c}\right)^{1/6}\int_0^t \sqrt{1-t^6}\,dt = nh$$

$$E^{1/2+1/6} \propto n = E^{\frac{3+1}{6}} \propto n\, , E \propto n^{3/2}$$

Therefore, correct option is (b)

Q103. The Hamiltonian of a two-level quantum system is $H = \frac{1}{2}\hbar\omega\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ possible initial state in which the probability of the system being in that quantum state does not change with time, is

(a) $\begin{pmatrix}\cos\frac{\pi}{4}\\ \sin\frac{\pi}{4}\end{pmatrix}$ (b) $\begin{pmatrix}\cos\frac{\pi}{8}\\ \sin\frac{\pi}{8}\end{pmatrix}$ (c) $\begin{pmatrix}\cos\frac{\pi}{2}\\ \sin\frac{\pi}{2}\end{pmatrix}$ (d) $\begin{pmatrix}\cos\frac{\pi}{6}\\ \sin\frac{\pi}{6}\end{pmatrix}$

Ans. : (b)

**<u>NET/JRF (JUNE-2018)</u>**

Q104. The Hamiltonian of a spin $\frac{1}{2}$ particle in a magnetic field $\vec{B}$ is given by $H = -\mu.\vec{B}.\vec{\sigma}$, where $\mu$ is a real constant and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli spin matrices. If $\vec{B} = (B_0, B_0, 0)$ and the spin state at time $t = 0$ is an eigenstate of $\sigma_x$, then of the expectation values $\langle\sigma_x\rangle, \langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$

(a) only $\langle\sigma_x\rangle$ changes with time (b) only $\langle\sigma_y\rangle$ changes with time

(c) only $\langle\sigma_z\rangle$ changes with time (d) all three change with time

Ans. : (d)

Solution: $\langle\sigma_x\rangle, \langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$ will changes with time because Eigen state of $\sigma_x$ ie $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}$ and $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\end{pmatrix}$ and can be written in basis of eigen state of $H = -\mu.\vec{B}.\vec{\sigma} = -B_0\begin{pmatrix}0 & 1-i\\ 1+i & 0\end{pmatrix}$

Q105. A particle of mass $m$ is constrained to move in a circular ring of radius $R$. When a perturbation $V' = \frac{a}{R^2}\cos^2\phi$ (where $a$ is a real constant) is added , the shift in energy of the ground state, to first order in $a$, is

(a) $\frac{a}{R^2}$ (b) $\frac{2a}{R^2}$ (c) $\frac{a}{2R^2}$ (d) $\frac{a}{(\pi R^2)}$

Ans. : (c)

Solution: $V' = \frac{a}{R^2}\cos^2\phi$ where $|\phi_0\rangle = \frac{1}{\sqrt{2\pi}}$

$$\langle\phi_0|V'|\phi_0\rangle = \frac{a}{R^2}\int_0^{2\pi}\frac{1}{2\pi}\cos^2\phi$$

$$= \frac{a}{2\pi R^2}\int_0^{2\pi}\frac{1}{2}(1+\cos 2\phi)\,d\phi = \frac{a}{2\pi R^2}\frac{2\pi}{2} = \frac{a}{2R^2}$$

Q106. A particle of mass $m$ is confined in a three-dimensional box by the potential

$$V(x,y,z) = \begin{cases} 0, & 0 \le x,y,z \le a \\ \infty & \text{otherwise} \end{cases}$$

The number of eigenstates of Hamiltonian with energy $\frac{9\hbar^2\pi^2}{2ma^2}$ is

(a) 1 (b) 6 (c) 3 (d) 4

Ans. : (c)

Solution: $E_{n_x,n_y,n_z} = \frac{9\pi^2\hbar^2}{2ma^2}$

| $n_x$ | $n_y$ | $n_z$ |
|---|---|---|
| 1 | 2 | 2 |
| 2 | 2 | 1 |
| 2 | 1 | 2 |

where $E_{xx,xy,xz} = (n_x^2 + n_y^2 + n_z^2)\frac{\pi^2\hbar^2}{2ma^2}$

Q107. The $n^{th}$ energy eigenvalues $E_n$ of a one-dimensional Hamiltonian $H = \frac{p^2}{2m} + \lambda x^4$ (where $\lambda > 0$ is a constant) in the WEB approximation, is proportional to

(a) $\left(n+\frac{1}{2}\right)^{4/3}\lambda^{1/3}$ (b) $\left(n+\frac{1}{2}\right)^{4/3}\lambda^{2/3}$ (c) $\left(n+\frac{1}{2}\right)^{5/3}\lambda^{1/3}$ (d) $\left(n+\frac{1}{2}\right)^{5/3}\lambda^{2/3}$

Ans. : (a)

Solution: From W.K.B approximation

$$4.\int_0^x P\,dx \propto \left(n+\frac{1}{2}\right)h$$

$$4\int_0^{\left(\frac{E}{\lambda}\right)^{1/4}} \sqrt{2m\left(E-\lambda x^4 dx\right)} \propto \left(n+\frac{1}{2}\right)h$$

making the integration dimensional

$$4\times\left(2mE\right)^{1/2}\left(\frac{E}{\lambda}\right)^{1/4}\int_0^1\sqrt{1-t^4}\,dt \propto \left(n+\frac{1}{2}\right) \Rightarrow E^{3/4}\propto\left(n+\frac{1}{2}\right)\lambda^{1/4} \Rightarrow E\propto\left(n+\frac{1}{2}\right)^{4/3}\lambda^{1/3}$$

Q108. The differential scattering cross-section $\frac{d\sigma}{d\Omega}$ for the central potential $V\left(r\right)=\frac{\beta}{r}e^{-\mu r}$, where $\beta$ and $\mu$ are positive constants, is calculated in thee first Born approximation. Its dependence on the scattering angle $\theta$ is proportional to ($A$ is a constant below)

(a) $\left(A^2+\sin^2\frac{\theta}{2}\right)$

(b) $\left(A^2+\sin^2\frac{\theta}{2}\right)^{-1}$

(c) $\left(A^2+\sin^2\frac{\theta}{2}\right)^{-2}$

(d) $\left(A^2+\sin^2\frac{\theta}{2}\right)^{2}$

Ans. : (c)

Solution: $f\left(\theta\right)\propto\int_0^\infty V\left(r\right)\sin krdr \quad \Rightarrow D\left(\theta\right)\propto\left|f\left(\theta\right)\right|^2$

$$f\left(\theta\right)\propto\frac{1}{k}\int_0^\infty \beta\frac{e^{-\mu r}}{r}r\sin krdr$$

$$f\left(\theta\right)\propto\frac{1}{k}\int_0^\infty\frac{e^{-\mu r}}{r}r\left(\frac{e^{ikr}-e^{-ikr}}{2i}\right)dr \Rightarrow \frac{1}{2ik}\int_0^\infty e^{-\mu r}e^{ikr}dr-\int_0^\infty e^{-\mu r}e^{-ikr}dr$$

$$\Rightarrow\frac{1}{2ik}\left(\int_0^\infty e^{-r(\mu-ik)}dr-\int e^{-r(\mu+ikr)}dr\right)\Rightarrow\frac{1}{2ik}\left[\frac{\mu+ik-\mu+ik}{\mu^2+k^2}\right]=\frac{2ik}{2ik}\left(\mu^2+k^2\right)^{-1}$$

$$f\left(\theta\right)\propto\frac{1}{\left(\mu^2+k^2\right)}, \qquad D\left(\theta\right)=\left(\frac{1}{\mu^2+k^2}\right)^2$$

$$D\left(\theta\right)=\left(\mu^2+k^2\right)^{-2}, \text{ where } k\propto\sin\frac{\theta}{2}$$

$$D\left(\theta\right)\propto\left(\mu^2+\sin^2\frac{\theta}{2}\right)^{-2} \text{ or } D\left(\theta\right)=\left(A^2+\sin^2\frac{\theta}{2}\right)^{-2}$$

Q109. At $t=0$, the wavefunction of an otherwise free particle confined between two infinite walls at $x=0$ and $x=L$ is $\psi(x,t=0)=\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)$. Its wave function at a later time $t=\frac{mL^2}{4\pi\hbar}$ is

(a) $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)e^{i\pi/6}$ (b) $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L}+\sin\frac{3\pi x}{L}\right)e^{-i\pi/6}$

(c) $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L}-\sin\frac{3\pi x}{L}\right)e^{-i\pi/8}$ (d) $\sqrt{\frac{2}{L}}\left(\sin\frac{\pi x}{L}+\sin\frac{3\pi x}{L}\right)e^{-i\pi/6}$

Ans. : (d)

Solution: $\psi(x,t=0)=\left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}-\sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}\right)$

$\psi(x,t=0)=|\varphi_1\rangle-|\varphi_3\rangle$

$\psi(x,t)=|\varphi_1\rangle e^{\frac{-iE_1t}{\hbar}}-|\varphi_3\rangle e^{\frac{-iE_3t}{\hbar}}$

$E_1=\frac{\pi^2\hbar^2}{2mL^2}E_3=\frac{9\pi^2\hbar^2}{2mL^2}t=\frac{mL^2}{4\pi\hbar}$

$\psi(x,t)=|\varphi_1\rangle e^{\frac{-i\pi}{8}}-|\varphi_3\rangle e^{\frac{=9i\pi}{8}}=e^{\frac{-i\pi}{8}}\left(|\varphi_1\rangle-|\varphi_3\rangle e^{-i\pi}\right)$

$=e^{\frac{-i\pi}{8}}\left(|\varphi_1\rangle+|\varphi_3\rangle\right)=e^{\frac{-i\pi}{8}}\left(\sqrt{\frac{2}{L}}\sin\frac{\pi x}{L}+\sqrt{\frac{2}{L}}\sin\frac{3\pi x}{L}\right)$

**NET/JRF (DEC - 2018)**

Q110. The ground state energy of an anisotropic harmonic oscillator described by the potential $V(x,y,z)=\frac{1}{2}m\omega^2x^2+2m\omega^2y^2+8m\omega^2z^2$ (in units of $\hbar\omega$) is

(a) $\frac{5}{2}$ (b) $\frac{7}{2}$ (c) $\frac{3}{2}$ (d) $\frac{1}{2}$

Ans. : (b)

Solution: $V(x,y,z)=\frac{1}{2}m\omega^2x^2+\frac{1}{2}m(2\omega)^2y^2+\frac{1}{2}m(4\omega)^2z^2$

$\omega_x=\omega \qquad \omega_y=2\omega \qquad \omega_z=4\omega$

$E_{n_x,n_y,n_z}=\left(n_x+\frac{1}{2}\right)\hbar\omega_x+\left(n_y+\frac{1}{2}\right)\hbar\omega_y+\left(n_z+\frac{1}{2}\right)\hbar\omega_z$

For ground state

$n_x=0, n_y=0, n_z=0$

$=\frac{1}{2}\hbar\omega+\frac{1}{2}\hbar 2\omega+\frac{1}{2}\hbar 4\omega=\frac{1}{2}\hbar\omega(1+2+4)=\frac{7}{2}\hbar\omega$

Q111. The product $\Delta x\,\Delta p$ of uncertainties in the position and momentum of a simple harmonic oscillator of mass $m$ and angular frequency $\omega$ in the ground state $|0\rangle$, is $\frac{\hbar}{2}$. The value of the product $\Delta x\,\Delta p$ in the state, $e^{-i\hat{p}\ell/\hbar}|0\rangle$ (where $\ell$ is a constant and $\hat{p}$ is the momentum operator) is

(a) $\frac{\hbar}{2}\sqrt{\frac{m\omega\ell^2}{\hbar}}$ (b) $\hbar$ (c) $\frac{\hbar}{2}$ (d) $\frac{\hbar^2}{m\omega\ell^2}$

Ans. : (c)

Q112. Let the wavefunction of the electron in a hydrogen atom be

$$\psi(\vec{r})=\frac{1}{\sqrt{6}}\phi_{200}(\vec{r})+\sqrt{\frac{2}{3}}\phi_{21-1}(\vec{r})-\frac{1}{\sqrt{6}}\phi_{100}(\vec{r})$$

where $\phi_{nlm}(\vec{r})$ are the eigenstates of the Hamiltonian in the standard notation. The expectation value of the energy in this state is

(a) $-10.8\ eV$ (b) $-6.2\ eV$ (c) $-9.5\ eV$ (d) $-5.1\ eV$

Ans. : (d)

Solution: $\psi = \frac{1}{\sqrt{6}}\phi_{2,0,0} + \sqrt{\frac{2}{3}}\phi_{2,1,-1} - \frac{1}{\sqrt{6}}\phi_{(1,0,0)}$

$P\left(\frac{-13.6}{4}\right) = \frac{1}{6} + \frac{2}{3} = \frac{1+4}{6} = \frac{5}{6}$

$P(-3.4) = \frac{5}{6}$ and $P(-13.6) = \frac{1}{6}$

$\langle E \rangle = (-3.4)\times\frac{5}{6} + (-13.6)\times\frac{1}{6} = \frac{1}{6}(-17.00 - 13.6)eV = -\frac{30.60}{6} = -5.1eV$

Q113. Three identical spin $\frac{1}{2}$ particles of mass $m$ are confined to a one-dimensional box of length $L$, but are otherwise free. Assuming that they are non-interacting, the energies of the lowest two energy eigen states, in units of $\frac{\pi^2\hbar^2}{2mL^2}$, are

(a) 3 and 6 (b) 6 and 9 (c) 6 and 11 (d) 3 and 9

Ans. : (b)

Solution: Put $\frac{\pi^2\hbar^2}{2mL^2} = E_0$

For ground state configuration 2 particle has engine $E_0$ and 1 particle has engine $4E_0$

Total energy is $2\times E_0 + 1\times 4E_0 = 6E_0$

For first excited state configuration, 1 particles has engine $E_0$ and 2 particle has engine $4E_0$

Total energy $1\times E_0 + 2\times 4E_0 = 9E_0$

Lowest two energy levels are $6E_0, 9E_0$ respectively, where $E_0 = \frac{\pi^2\hbar^2}{2mL^2}$

Q114. Consider the operator $A_x = L_y p_z - L_z p_y$, where $L_i$ and $p_i$ denote, respectively, the components of the angular momentum and momentum operators. The commutator $[A_x, x]$, where $x$ is the $x$-component of the position operator, is

(a) $-i\hbar(zp_z + yp_y)$ (b) $-i\hbar(zp_z - yp_y)$ (c) $i\hbar(zp_z + yp_y)$ (d) $i\hbar(zp_z - yp_y)$

Ans. : (a)

Solution: $A_x = L_y p_z - L_z p_y$, $L_y = zp_x - xp_z$, $L_z = xp_y - yp_x$

$$[A_x, x] = [L_y p_z, x] - [L_z p_y, x] = [L_y, x] p_z - [L_z, x] p_y$$

$$= [zp_x, x] p_z + [yp_x, x] p_y = z[p_x, x] p_z + y[p_x, x] p_y$$

$$= (-i\hbar zp_z) + (-i\hbar yp_y) = -i\hbar (zp_z + yp_y)$$

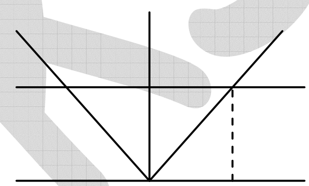
Q115. A one-dimensional system is described by the Hamiltonian $H = \frac{p^2}{zm} + \lambda |x|$ (where $\lambda > 0$).

The ground state energy varies as a function of $\lambda$ as

(a) $\lambda^{5/3}$ (b) $\lambda^{2/3}$ (c) $\lambda^{4/3}$ (d) $\lambda^{1/3}$

Ans. : (a)

Solution: Using Bohr-Sommerfield theory,



$$\oint pdx = nh = 4 \int_0^{x_0 = \frac{E}{\lambda}} \sqrt{2m(E - \lambda x)}\, dx = nh$$

where $x_0$ is turning point $x_0 = \frac{E}{\lambda}$ $x_0 = \frac{E}{\lambda}$

$$\Rightarrow 4 \times \sqrt{2mE} \times \frac{E}{\lambda} \int_0^1 \sqrt{1 - t}\, dt = nh$$

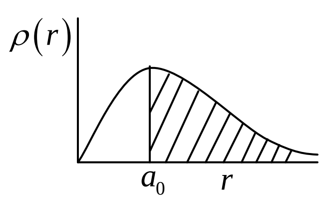
$$\frac{E^{3/2}}{\lambda} \propto n \Rightarrow E \propto \lambda^{2/3}$$

Q116. If the position of the electron in the ground state of a Hydrogen atom is measured, the probability that it will be found at a distance $r \geq a_0$ ($a_0$ being Bohr radius) is nearest to

(a) 0.91 (b) 0.66 (c) 0.32 (d) 0.13

Ans. : (b)

Solution: $P(a_0 \leq r < \infty) = \int_{a_0}^{\infty} r^2 |R_{10}|^2 dr$




$$R_{10} = \frac{2}{a_0^{3/2}}$$

$$P(a_0 \leq r < \infty) = \frac{4}{a_0^3} \int_{a_0}^{\infty} r^2 e^{-\frac{2r}{a_0}} dr = 0.66$$

Q117. A system of spin $\frac{1}{2}$ particles is prepared to be in the eigenstate of $\sigma_z$ with eigenvalue $+1$.

The system is rotated by at angle of $60^0$ about the $x$-axis. After the rotation, the fraction of the particles that will be measured to be in the eigenstate of $\sigma_z$ with eigenvalue $+1$ is

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$

Ans. : (d)

Solution: Rotation with angle $\theta$ about $x$ axis

$$U\left[R(\theta)\right] = \exp\left(-i\theta.\frac{\sigma}{2}\right)$$

$$U\left[R(\theta)\right] = \cos\left(\frac{\theta}{2}\right)I - i\sin\left(\frac{\theta}{2}\right)\hat{\theta}\cdot\sigma$$

$$U\left[R_x(\theta)\right] = \cos\frac{\theta}{2}I - i\sin\left(\frac{\theta}{2}\right)\hat{\theta}\cdot\sigma_x$$

$$R_x(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \quad \text{Put } \theta = \frac{\pi}{3}$$

$$\left|\psi(\theta)\right\rangle = R_x(\theta)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{i}{2} \\ \frac{-i}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\left|\psi\right\rangle = \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{i}{2} \end{pmatrix} = \frac{\sqrt{3}}{2}\begin{pmatrix}1\\0\end{pmatrix} - \frac{i}{2}\begin{pmatrix}0\\1\end{pmatrix}$$

If $\sigma_z$ is measure on $\left|\psi\right\rangle$, the measurement is $+1$ with probability $\frac{3}{4}$ and $-1$ with probability $\frac{1}{4}$