GATE 2019 (Solution)

SECTION: GENERAL APTITUDE

Q1 – Q5 carry one mark each.

Q1 The fishermen, __________ the flood victims owed their lives, were rewarded by the government.

(a) whom (b) to which (c) to whom (d) that

Ans. : (c)

Q2 Some students were not involved in the strike.

If the above statement is true, which of the following conclusions is/are logically necessary?

1. Some who were involved in the strike were students
2. No student was involved in the strike
3. At least one student was involved in the strike
4. Some who were not involved in the strike were students

(a) 1 and 2 (b) 3 (c) 4 (d) 2 and 4

Ans. : (c)

Q3. The radius as well as the height of a circular cone increases by 10 %. The percentage increase in its volume is __________.

(a) 17.1 (b) 21.0 (c) 33.1 (d) 72.8

Ans. : (c)

Q4 Five numbers 10, 7, 5, 4 and 2 are to be arranged in a sequence from left to right following the directions given below:

1. No two odd or even numbers are next to each other
2. The second number from the left is exactly half of the left-most number
3. The middle number is exactly twice the right-most number

Which of the second number from the right?

(a) 2 (b) 4 (c) 7 (d) 10

Ans. : (c)

Q5 Until Iran came along, India had never been _________ in kabaddi.

(a) defeated (b) defeating (c) defeat (d) defeatist

Ans. : (a)

Q6 – Q10 carry two marks each.

Q6 Since the last one year, after a 125 basis point reduction in repo rate by the Reserve Bank of India, banking institutions have been making a demand to reduce interest rates on small saving schemes. Finally, the government announced yesterday a reduction in interest rates on small saving schemes to bring them on par with fixed deposit interest rates.

Which one of the following statements can be inferred from the given passage?

(a) Whenever the Reserve Bank of India reduces the repo rate, the interest rates on small saving schemes are also reduced

(b) Interest rates on small saving schemes are always maintained on par with fixed deposit interest rates

(c) The government sometimes takes into consideration the demands of banking institutions before reducing the interest rates on small saving schemes

(d) A reduction in interest rates on small saving schemes follow only after a reduction in repo rate by the Reserve Bank of India.

Ans. : (c)

Q7. In a country of 1400 million population 70% own mobile phones. Among the mobile phone owners, only 294 million access the Internet. Among these Internet users, only half buy goods from e-commerce portals. What is the percentage of these buyers in the country?

(a) 10.50 (b) 14.70 (c) 15.00 (d) 50.00

Ans. : (a)

Q8. The nomenclature of Hindustani music has changed over the centuries. Since the medieval period dhrupad styles were identified as baanis. Terms like gayaki and baaj were used to refer to vocal and instrumental styles, respectively. With the institutionalization of music education the term gharana became acceptable. Gharana originally referred to hereditary musicians from a particular lineage, including disciples and grand disciples.

Which one of the following pairings is NOT correct?

(a) dhupad, baani (b) gayaki, vocal (c) baaj, institution (d) gharana, lineage

Ans. : (c)

Q9. Two trains started at 7 AM from the same point. The first train travelled north at a speed of $80\,km/h$ and the second train travelled south at a speed of $100\,km/h$. The time at which they were $540\,km$ apart is __________A.M

(a) 9 (b) 10 (c) 11 (d) 11.30

Ans. : (b)

Q10. "I read somewhere that in ancient times the prestige of a kingdom depended upon the number of taxes that it was able to levy on its people. It was very much like the prestige of a head-hunter in his won community."

Based on the paragraph above, the prestige of a head-hunter depended upon _____________

(a) the prestige of the kingdom (b) the prestige of the heads

(c) the number of taxes he could levy (d) the number of heads he could gather

Ans. : (d)

SECTION: PHYSICS

Q1 – Q25 carry one mark each.

Q1. The relative magnetic permeability of a type-I super conductor is

(a) 0 (b) -1 (c) 2π (d) $\frac{1}{4\pi}$

Ans.: (a)

Solution: $B=\mu_0\left(H+M\right)=\mu_0\left(H+\chi H\right)=\mu_0\left(1+\chi\right)H=\mu H$

$$\therefore \quad \mu=\mu_0\left(1+\chi\right)\Rightarrow \mu_r=\frac{\mu}{\mu_0}=1+\chi$$

For type-I superconductor: $\chi=-1$

$$\therefore \mu_r=1-1=0$$

Q2. Considering baryon number and lepton number conservation laws, which of the following process is/are allowed?

(i) $p\rightarrow \pi^0+e^++\nu_e$

(ii) $e^++\nu_e\rightarrow \mu^++\nu_\mu$

(a) both (i) and (ii) (b) only (i) (c) only (ii) (d) neither (i) nor (ii)

Ans. : (c)

Solution: (i) $P \to \pi^0 + e^+ + \nu_e$

$B: +1 \quad 0 \quad 0 \quad 0$: Not conserved

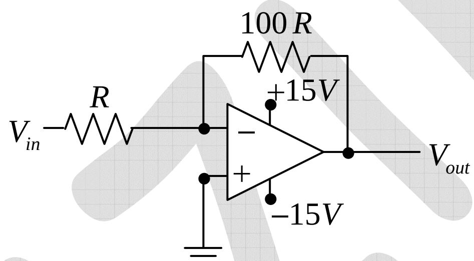
Therefore, this is not an allowed process

(ii) $e^+ + \nu_e \to \mu^+ + \nu_\mu$

	e^+	ν_e	μ^+	ν_μ	
q:	$+1$	0	$+1$	0	: conserved
$spin$:	$1/2$	$1/2$	$1/2$	$1/2$	: conserved
L_e:	-1	$+1$	0	0	: conserved
L_μ:	0	0	-1	$+1$	: conserved

Since neutrino is involve, therefore parity is violated. This is allowed through weak interaction

Q3. For the following circuit, what is the magnitude of V_{out} if $V_{\text{in}} = 1.5V$?



(a) $0.015V$ (b) $0.15V$ (c) $15V$ (d) $150V$

Ans. : (c)

Solution: $V_{out} = -\frac{100R}{R} \times 1.5 = -150V \Rightarrow |V_0| = 15V$

Q4. For the differential equation $\frac{d^2y}{dx^2} - n(n+1)\frac{y}{x^2} = 0$, where n is a constant, the product of its two independent solutions is

(a) $\frac{1}{x}$ (b) x (c) x^n (d) $\frac{1}{x^{n+1}}$

Ans. : (b)

Solution: This is a Euler-Cauchy by differential equation whose characteristic equation is

$$m^2 - m - n(n+1) = 0$$

Therefore, $m = \dfrac{1 \pm \sqrt{1+4n(n+1)}}{2}$ or $m = \dfrac{1 \pm \sqrt{(2n+1)^2}}{2} = \dfrac{1 \pm (2n+1)}{2}$

or $m = 1+n$, or $m = -n$

Therefore two independent solution are $y_1 = x^{1+n}$ and $y_2 = x^{-n}$

Therefore, $y_1 y_2 = x^{1+n-n} = x$

Q5. Consider a one-dimensional gas of N non-interacting particles of mass m with the Hamiltonian for a single particle given by

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 \left(x^2 + 2x\right)$$

The high temperature specific heat in units of $R = Nk_B$ (k_B is the Boltzmann constant) is

(a) 1 (b) 1.5 (c) 2 (d) 2.5

Ans. : (c)

Solution: $\langle H \rangle = \left\langle \dfrac{p^2}{2m} \right\rangle + \dfrac{1}{2} m\omega^2 \langle x^2 \rangle + \dfrac{1}{2} m\omega^2 \langle 2x \rangle = \dfrac{NkT}{2} + \dfrac{NkT}{2} + U_0$

$$\langle H \rangle = NkT$$

$$C_V = \frac{\partial H}{\partial T} = NkT$$

Q6. An electric field $\vec{E} = E_0 \hat{z}$ is applied to a Hydrogen atom in $n = 2$ excited state. Ignoring spin the $n = 2$ state is fourfold degenerate, which in the $|l, m\rangle$ basis are given by $|0,0\rangle, |1,1\rangle, |1,0\rangle$ and $|1,-1\rangle$. If H' is the interaction Hamiltonian corresponding to the applied electric field, which of the following matrix elements is nonzero?

(a) $\langle 0,0 | H' | 0,0 \rangle$ (b) $\langle 0,0 | H' | 1,1 \rangle$

(c) $\langle 0,0 | H' | 1,0 \rangle$ (d) $\langle 0,0 | H' | 1,-1 \rangle$

Ans. : (c)

Q7. A large number N of ideal bosons, each of mass m, are trapped in a three-dimensional potential $V(r)=\frac{m\omega^2 r^2}{2}$. The bosonic system is kept at temperature T which is much lower than the Bose-Einstein condensation temperature T_C. The chemical potential (μ) satisfies

(a) $\mu \le \frac{3}{2}\hbar\omega$ (b) $2\hbar\omega > \mu > \frac{3}{2}\hbar\omega$

(c) $3\hbar\omega > \mu > 2\hbar\omega$ (d) $\mu = 3\hbar\omega$

Ans. : (a)

Q8. During a rotation, vectors along the axis of rotation remain unchanged. For the rotation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$, the unit vector along the axis of rotation is

(a) $\frac{1}{3}\left(2\hat{i}-\hat{j}+2\hat{k}\right)$ (b) $\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}-\hat{k}\right)$

(c) $\frac{1}{\sqrt{3}}\left(\hat{i}-\hat{j}-\hat{k}\right)$ (d) $\frac{1}{3}\left(2\hat{i}+2\hat{j}-\hat{k}\right)$

Ans. : (b)

Solution: Since the vector along the axis of rotation remain unchanged during rotation then

$$A\mathrm{x} = \mathrm{x} \quad \text{.... (i)}$$

When $A=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$

Equation (i) is a standard eigenvalue-eigenvector relation for $\lambda=1$. Equation (i) can be written

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

or $x_2 = x_1$ .... (ii)

or $x_2 = -x_3$ .... (iii)

or $x_1 = -x_3$ .... (iv)

Using these relations we see that the general eigenvector is

General eigenvector $= \begin{pmatrix} k \\ k \\ -k \end{pmatrix}$

Therefore a unit eigenvector along the axis of rotation is

Unit eigenvector $= \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} - \hat{k}\right)$

Q9. For a spin $\frac{1}{2}$ particle, let $|\uparrow\rangle$ and $|\downarrow\rangle$ denote its spin up and spin down states, respectively. If $|a\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$ and $|b\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$ are composite states of two such particles, which of the following statements is true for their total spin S?

(a) $S = 1$ for $|a\rangle$ and $|b\rangle$ is not an eigenstate of the operator $\hat{S}^2$

(b) $|a\rangle$ is not an eigenstate of the operator $\hat{S}^2$ and $S = 0$ for $|b\rangle$

(c) $S = 0$ for $|a\rangle$, and $S = 1$ for $|b\rangle$

(d) $S = 1$ for $|a\rangle$, and $S = 0$ for $|b\rangle$

Ans. : (d)

Solution: $S = 1$ is triplet $|a\rangle$, and $S = 0$ for singlet for $|b\rangle$

Q10. Consider a transformation from one set of generalized coordinate and momentum (q, p) to another set (Q, P) denoted by,

$$Q = pq^s; \qquad P = q^r$$

where s and r are constants. The transformation is canonical if

(a) $s = 0$ and $r = 1$

(b) $s = 2$ and $r = -1$

(c) $s = 0$ and $r = -1$

(d) $s = 2$ and $r = 1$

Ans. : (b)

Solution: $\frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} = 1 \Rightarrow 0 - q^s r q^{r-1} = 1$

$-rq^{r+s-1} = 1 \Rightarrow \; s = 2$ and $r = -1$

Q11. In order to estimate the specific heat of phonons, the appropriate method to apply would be

(a) Einstein model for acoustic phonons and Debye model for optical phonons

(b) Einstein model for optical phonons and Debye model for acoustic phonons

(c) Einstein model for both optical and acoustic phonons

(d) Debye model for both optical and acoustic phonons

Ans.: (b)

Solution: At low temperature, the optical branch phonons have energies higher than $k_B T$ and therefore, optical branch waves are not excited. And Debye model is not suitable for optical branch instead it is suitable for acoustical branch. Whereas Einstein model is useful for high temperature and therefore can be applied to optical branch.

Q12. The pole of the function $f(z) = \cot z$ at $z = 0$ is

(a) a removablesingularity (b) an essential singularity

(c) a simple pole (d) a second order pole

Ans. : (c)

Solution: $f(z) = \cot z$ at $z = 0$

$f(z) = \frac{1}{\tan z}$ $z = 0$ is a simple pole $f(z) = \frac{1}{z}\left[1 - \frac{1}{3}z^2 + ...\right]$

Q13. A massive particle X in free space decays spontaneously into two photons. Which of the following statements is true for X ?

(a) X is charged

(b) Spin of X must be greater than or equal to 2

(c) X is a boson

(d) X must be a baryon

Ans.: (c)

Solution: $X \rightarrow r + r$

$q:$ 0 0 0

$spin:$ $0,1,2$ 1 1

Thus spin of X can be either $0,1$ or 2. (integer)

Therefore, option (b) is wrong while option (c) is correct.

Q14. The electric field of an electromagnetic wave is given by $\vec{E} = 3\sin(kz - \omega t)\hat{x} + 4\cos(kz - \omega t)\hat{y}$. The wave is

(a) linearly polarized at an angle $\tan^{-1}\left(\frac{4}{3}\right)$ from the x - axis

(b) linearly polarized at an angle $\tan^{-1}\left(\frac{3}{4}\right)$ from the x - axis

(c) elliptically polarized in clockwise direction when seen travelling towards the observer

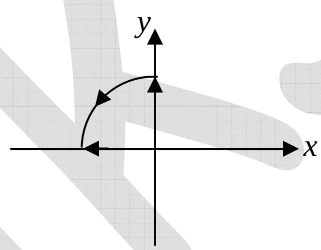
(d) elliptically polarized in counter-clockwise direction when seen travelling towards the observer

Ans. : (d)

Solution: At $z = 0,\ E_x = -3\sin\omega t,\ E_y = 4\cos\omega t$

At $\omega t = 0,\ E_x = 0,\ E_y = 4$

At $\omega t = \frac{\pi}{2},\ E_x = -3,\ E_y = 0$



Q15. The nuclear spin and parity of ${}^{40}_{20}Ca$ in its ground state is

(a) 0^+ (b) 0^- (c) 1^+ (d) 1^-

Ans.: (a)

Solution: ${}^{40}_{20}Ca$ is an even-even nuclei, therefore $I = 0, P = +ve$

$\therefore$ Spin-parity $= 0^+$

Q16. An infinitely long thin cylindrical shell has its axis coinciding with the z-axis. It carries a surface charge density $\sigma_0 \cos\phi$, where ϕ is the polar angle and σ_0 is a constant. The magnitude of the electric field inside the cylinder is

(a) 0 (b) $\frac{\sigma_0}{2\in_0}$ (c) $\frac{\sigma_0}{3\in_0}$ (d) $\frac{\sigma_0}{4\in_0}$

Ans. : (b)

Solution: $dE = \frac{d\lambda}{2\pi\in_0 R} = \frac{(\sigma_0\cos\phi)(Rd\phi)}{2\pi\in_0 R} = \frac{\sigma_0\cos\phi}{2\pi\in_0}d\phi$

Along axis of cylinder $dE_x = dE\cos\phi \Rightarrow E_x = \frac{\sigma_0}{2\pi\in_0}\int_0^{2\pi}\cos^2\phi\, d\phi = \frac{\sigma_0}{2\in_0}$

Q17. Consider a three-dimensional crystal of N inert gas atoms. The total energy is given by $U(R)=2N\in\left[p\left(\frac{\sigma}{R}\right)^{12}-q\left(\frac{\sigma}{R}\right)^{6}\right]$, where $p=12.13,\ q=14.45$ and R is the nearest neighbour distance between two atoms. The two constants, $\in$ and R, have the dimensions of energy and length, respectively. The equilibrium separation between two nearest neighbour atoms in units of σ (rounded off to two decimal places) is____________

Ans.: 1.09

Solution: $U(R)=2N\in\left[p\left(\frac{\sigma}{R}\right)^{12}-q\left(\frac{\sigma}{R}\right)^{6}\right]$

$$\frac{dU}{dR}=0\Rightarrow 2N\in\left[12p\left(\frac{\sigma}{R}\right)^{11}\cdot\left(\frac{-\sigma}{R^2}\right)-6q\left(\frac{\sigma}{R}\right)^{5}\left(-\frac{\sigma}{R^2}\right)\right]=0$$

$$\Rightarrow 12p\frac{\sigma^{12}}{R^{13}}-6q\frac{\sigma^{6}}{R^{7}}=0\Rightarrow 12p\frac{\sigma^{12}}{R^{13}}=6q\frac{\sigma^{6}}{R^{7}}\Rightarrow R^6=\frac{12p}{6q}\sigma^6$$

$$\Rightarrow R=\left(\frac{2p}{q}\right)^{1/6}\sigma \qquad \text{given } p=12.13,\ q=14.45$$

$$\therefore R=\left(\frac{2\times 12.13}{14.45}\right)^{1/6}\sigma=(1.679)^{1/6}\sigma=1.09\sigma$$

Thus $\frac{R}{\sigma}=1.09$

Q18. The energy-wavevector $(E-k)$ dispersion relation for a particle in two dimensions is $E=Ck$, where C is a constant. If its density of states $D(E)$ is proportional to E^p then the value of p is____________

Ans.: 1

Solution: For $E(k)\propto k^s$. The density of states in d - dimension is $D(E)\propto E^{\left(\frac{d}{s}-1\right)}$

Given, $E=Ck$ $\qquad\therefore\quad s=1, d=2$

Thus $D(E)\propto E^{\left(\frac{2}{1}-1\right)}$

$\propto E^1$

Q19. A circular loop made of a thin wire has radius $2\,cm$ and resistance 2Ω. It is placed perpendicular to a uniform magnetic field of magnitude $|\vec{B}_0| = 0.01$ Tesla. At time $t=0$ the field starts decaying as $\vec{B} = \vec{B}_0 e^{-t/t_0}$, where $t_0 = 1s$. The total charge that passes through a cross section of the wire during the decay is Q. The value of Q in μC (rounded off to two decimal places) is____________

Ans. : 6.28

Solution: $\varepsilon = -\frac{d\phi}{dt} = -\frac{AdB}{dt}$, $I = \frac{\varepsilon}{R} = -\frac{d\phi}{dt}\frac{1}{R}$

$$\Rightarrow -\frac{d\phi}{dt} = -\pi r^2 \frac{d}{dt}\left(B_0 e^{-t/t_0}\right) = \pi r^2 B_0 e^{-t}\left(t_0 = 1\right)$$

$$Q = \int_0^\infty I(t)\,dt = \int_0^\infty \frac{\pi r^2}{R} B_0 e^{-t} dt = \frac{\pi r^2 B_0}{R}\left|\frac{e^{-t}}{-1}\right|_0^\infty = 3.14\times\left(2\times10^{-2}\right)^2\times 0.01 = 6.28\mu C$$

Q20. The electric field of an electromagnetic wave in vacuum is given by

$$\vec{E} = E_0 \cos\left(3y + 4z - 1.5\times10^9 t\right)\hat{x}$$

The wave is reflected from the $z=0$ surface. If the pressure exerted on the surface is $\alpha \in E_0^2$, the value of α (rounded off to one decimal place) is___________

Ans. : 0.8

Solution: $\vec{K} = 3\hat{y} + 4\hat{z} \Rightarrow \tan\theta_R = \frac{K_y}{K_z} = \frac{3}{4}$

$$P = 2\frac{I}{c}\cos\theta_R = \frac{2}{c}\times\frac{1}{2}\in_0 cE_0^2\times\frac{4}{5} \qquad \Rightarrow P = 0.8\in_0 E_0^2$$

Q21. The Hamiltonian for a quantum harmonic oscillator of mass m in three dimensions is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

where ω is the angular frequency. The expectation value of r^2 in the first excited state of the oscillator in units of $\frac{\hbar}{m\omega}$ (rounded off to one decimal place) is__________

Ans. : 2.5

Solution: $\langle r^2\rangle=\langle x^2\rangle+\langle y^2\rangle+\langle z^2\rangle=\frac{\hbar}{2m\omega}\left[(2n_x+1)+(2n_y+1)+(2n_z+1)\right]$

For first excited state $n_x=1, n_y=0, n_z=0$

Hence it is triply degenerate one can take

$n_x=0,\ n_y=1,\ n_z=0$ or $n_x=0, n_y=0,\ n_z=1$

putting any one combination, expectation value of $r^2=\frac{5}{2}\frac{\hbar}{m\omega}=2.5\frac{\hbar}{m\omega}$

Q22. The Hamiltonian for a particle of mass m is $H=\frac{p^2}{2m}+kqt$ where q and p are the generalized coordinate and momentum, respectively, t is time and k is a constant. For the initial condition, $q=0$ and $p=0$ at $t=0, q(t)\propto t^{\alpha}$. The value of α is ________

Ans. : 3

Solution: $$\frac{\partial H}{\partial p}=\dot{q}=\frac{p}{m} \qquad ....(1)$$

$$\frac{\partial H}{\partial q}=-\dot{p}=kt\Rightarrow p=-\frac{kt^2}{2} \qquad ....(2)$$

$$\frac{dq}{dt}=-\frac{kt^2}{2}\Rightarrow q=-\frac{kt^3}{6}\Rightarrow q\propto t^3 \quad \text{so } \alpha=3$$

Q23. At temperature T Kelvin (K), the value of the Fermi function at an energy $0.5eV$ above the Fermi energy is 0.01. Then T, to the nearest integer, is __________

($k_B=8.62\times10^{-5}eV/K$)

Ans. : 1262

Solution: $F(E)=\frac{1}{e^{(E-E_F)/k_BT}+1}\Rightarrow e^{(E-E_F)/k_BT}+1=\frac{1}{F(E)}$

$$\Rightarrow e^{-(E-E_F)/k_BT}=\frac{1-F}{F}\Rightarrow\frac{E-E_F}{k_BT}=\ln\left(\frac{1-F}{F}\right)\Rightarrow T=\frac{E-E_F}{k_B\ln\left(\frac{1-F}{F}\right)}$$

$$\therefore\quad T=\frac{0.5}{8.62\times10^{-5}\ln\left(\frac{0.99}{0.01}\right)}=\frac{0.5}{8.62\times\ln(99)}=\frac{0.5\times10^5}{8.62\times4.595}=1262.3\ K$$

Q24. Let $|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|\psi_2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ represent two possible states of a two-level quantum system. The state obtained by the incoherent superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by a density matrix that is defined as $\rho \equiv c_1|\psi_1\rangle\langle\psi_1| + c_2|\psi_2\rangle\langle\psi_2|$. If $c_1 = 0.4$ and $c_2 = 0.6$, the matrix element ρ_{22} (rounded off to one decimal place) is __________

Ans. : 0.6

Solution: $\rho_{2,2} = \langle\psi_2|\rho|\psi_2\rangle = \rho \equiv c_1\langle\psi_2|\psi_1\rangle\langle\psi_1|\psi_2\rangle + c_2\langle\psi_2|\psi_2\rangle\langle\psi_2|\psi_2\rangle$

$\Rightarrow c_2 = 0.6$

Q25. A conventional type-I superconductor has a critical temperature of $4.7\,K$ at zero magnetic field and a critical magnetic field of 0.3 Tesla at $0\,K$. The critical field in Tesla at $2K$ (rounded off to three decimal places) is__________

Ans.: 0.246

Solution: $H_c(T) = H_0\left[1 - \left(\frac{T}{T_c}\right)^2\right] = 0.3\left[1 - \left(\frac{2}{4.7}\right)^2\right] = 0.3\left[1 - (0.426)^2\right]$

$= 0.3[1 - 0.181] = 0.3 \times 0.819 = 0.246\,Atm$

Q26 – Q55 carry two marks each.

Q26. Consider the following Boolean expression:

$$\left(\bar{A} + \bar{B}\right)\left[\overline{A\left(B + C\right)}\right] + A\left(\bar{B} + \bar{C}\right)$$

It can be represented by a single three-input logic gate. Identify the gate

(a) AND (b) OR (c) XOR (d) NAND

Ans. : (d)

Solution: $Y = \left(\bar{A} + \bar{B}\right)\left[\overline{A\left(B + C\right)}\right] + A\left(\bar{B} + \bar{C}\right)$

$= \left(\bar{A} + \bar{B}\right)\left[\bar{A} + \overline{\left(B + C\right)}\right] + A\bar{B} + A\bar{C}$

$= \left(\bar{A} + \bar{B}\right)\left[\bar{A} + \bar{B}\bar{C}\right] + A\bar{B} + A\bar{C}$

$= \bar{A} + \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B} + \bar{B}\bar{C} + A\bar{B} + A\bar{C}$

$$= \bar{A} + \bar{A}\bar{B}\bar{C} + \bar{B}\bar{C} + \bar{A}\bar{B} + A\bar{B} + A\bar{C}$$

$$= \bar{A} + \bar{B}\bar{C} + \bar{B} + A\bar{C} = \bar{A} + \bar{B} + A\bar{C}$$

$$Y = \left(\bar{A} + A\bar{C}\right) + \bar{B} = \left(\bar{A} + \bar{\bar{A}}\bar{C}\right) + \bar{B} = \bar{A} + \bar{C} + \bar{B}$$

$$\Rightarrow Y = \overline{ABC}$$

Q27. The value of the integral $\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + a^2} dx$, where $k > 0$ and $a > 0$, is

(a) $\frac{\pi}{a} e^{-ka}$ (b) $\frac{2\pi}{a} e^{-ka}$ (c) $\frac{\pi}{2a} e^{-ka}$ (d) $\frac{3\pi}{2a} e^{-ka}$

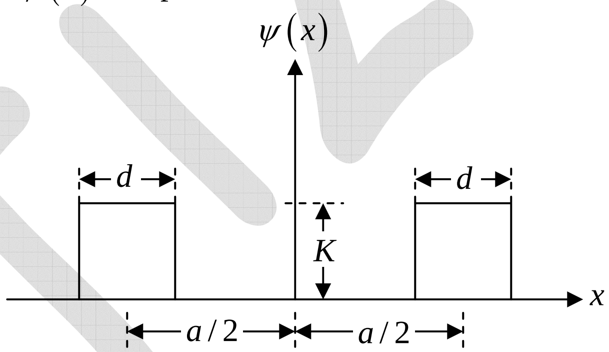
Ans. : (a)

Solution: $\int_{-\infty}^{\infty} \frac{\cos kx}{x^2 + a^2} dx$

$$f(z) = \frac{e^{ikx}}{z^2 + a^2} = \frac{e^{ikz}}{(z + ia)(z - ia)}$$

$$I = \text{Re}.2\pi i \times \frac{e^{ik(ia)}}{2ia} = \frac{\pi e^{-ka}}{a}$$

Q28. The wave function $\psi(x)$ of a particle is as shown below



Here K is a constant, and $a > d$. The position uncertainty (Δx) of the particle is

(a) $\sqrt{\frac{a^2 + 3d^2}{12}}$ (b) $\sqrt{\frac{3a^2 + d^2}{12}}$ (c) $\sqrt{\frac{d^2}{6}}$ (d) $\sqrt{\frac{d^2}{24}}$

Ans. : (b)

Solution: $\psi(x)=\begin{cases} k, & -\frac{a}{2}-\frac{d}{2}<x<-\frac{a}{2}+\frac{d}{2} \\ 0, & -\frac{a}{2}+\frac{d}{2}<x<\frac{a}{2}-\frac{d}{2} \\ k, & \frac{a}{2}-\frac{d}{2}<x<\frac{a}{2}+\frac{d}{2} \\ 0, & \frac{a}{2}+\frac{d}{2}>0 \end{cases}$

$$\langle\psi|\psi\rangle=1$$

$$k^2\int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}dx+k^2\int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}dx=1$$

$$k^2\left[\left(-\frac{a}{2}+\frac{d}{2}\right)-\left(-\frac{a}{2}-\frac{d}{2}\right)\right]+k^2\left[\left(\frac{a}{2}+\frac{d}{2}\right)-\left(\frac{a}{2}-\frac{d}{2}\right)\right]=1$$

$$k^2\left[\frac{d}{2}+\frac{d}{2}+\frac{d}{2}+\frac{d}{2}\right]=1\Rightarrow k=\frac{1}{\sqrt{2d}}$$

Hence wavefunction is symmetric about $x=0$, so $\langle x\rangle=0$

$$\langle x^2\rangle=k^2\int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}x^2dx+k^2\int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}x^2dx$$

$$=\frac{k^2}{3}\left[\left[x^3\right]_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}+\left[x^3\right]_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}\right]$$

$$=\frac{k^2}{3\times 8}\left[(-a+d)^3-(-a-d)^3+(a+d)^3-(a-d)^3\right]$$

$$=\frac{k^2}{24}\left\{\left(-a^3+d^3-3a^2d+3ad^2\right)+\left(a^3+d^3+3a^2d+3ad^2\right)+\left(a^3+d^3+3a^2d+3ad^2\right)\right.$$
$$\left.-\left\{a^3-d^3-3ad(a-d)\right\}\right]$$

$$\langle x^2\rangle=\frac{k^2}{24}\left[4a^3+12a^2d\right]=\frac{4d\left(d^2+3a^2\right)}{24\times 2d}=\frac{3a^2+d^2}{12}$$

$$\Delta x=\sqrt{\langle x^2\rangle-\langle x\rangle^2}=\sqrt{\frac{3a^2+d^2}{12}}$$

Q29. A solid cylinder of radius R has total charge Q distributed uniformly over its volume. It is rotating about its axis with angular speed ω. The magnitude of the total magnetic moment of the cylinder is

(a) $QR^2\omega$ (b) $\frac{1}{2}QR^2\omega$ (c) $\frac{1}{4}QR^2\omega$ (d) $\frac{1}{8}QR^2\omega$

Ans. : (c)

Solution: Magnetic moment due to disc $\mu = \frac{\pi\sigma\omega R^4}{4}$

Due to cylinder $d\mu = \frac{\pi\omega R^4}{4}(\rho dz)$ $(\sigma \to \rho dz)$

$$\mu = \frac{\pi\omega R^4}{4}\int_0^L \frac{Q}{\pi R^2 L}dz = \frac{Q\omega R^4}{4}$$

Q30. Consider the motion of a particle along the x- axis in a potential $V(x) = F|x|$. Its ground state energy E_0 is estimated using the uncertainty principle. Then E_0 is proportional to

(a) $F^{1/3}$ (b) $F^{1/2}$ (c) $F^{2/5}$ (d) $F^{2/3}$

Ans. : (d)

Solution: $E = \frac{p^2}{2m} + F|x|$ $E = \frac{p^2}{2m} + Fx$ for $x > 0$ $E = \frac{p^2}{2m} - Fx$ < 0 from uncertainty theory

$$\Delta x.\Delta p = \hbar \Rightarrow \Delta p = \frac{\hbar}{\Delta x}$$

$$E = \frac{(\Delta p)^2}{2m} + F(\Delta x) \Rightarrow E = \frac{\hbar^2}{2m(\Delta x)^2} + F\Delta x$$

For minimum energy,

$$\frac{dE}{d\Delta x} = -\frac{\hbar^2}{m(\Delta x)^3} + F = 0 \Rightarrow \Delta x = \left(\frac{\hbar^2}{mF}\right)^{1/3} \frac{\hbar^2}{2m}\left(\frac{mF}{\hbar^2}\right)^{2/3} + F\left(\frac{\hbar^2}{mF}\right)^{1/3} \Rightarrow E \propto F^{2/3}$$

Q31. A 3- bit analog-to-digital converter is designed to digitize analog signals ranging from $0V$ to $10V$. For this converter, the binary output corresponding to an input of $6\ V$ is

(a) 011 (b) 101 (c) 100 (d) 010

Ans. : (c)

Solution:

$$0 \rightarrow (000) \rightarrow 0V$$
$$1 \rightarrow (001) \rightarrow \frac{10}{7} = 1.42V$$
$$2 \rightarrow (010) \rightarrow \frac{20}{7} = 2.8V$$
$$3 \rightarrow (011) \rightarrow \frac{30}{7} = 4.28V$$
$$4 \rightarrow (100) \rightarrow \frac{40}{7} = 5.71V$$
$$5 \rightarrow (101) \rightarrow \frac{50}{7} = 7.14V$$
$$6 \rightarrow (110) \rightarrow \frac{60}{7} = 8.57V$$
$$7 \rightarrow (111) \rightarrow \frac{70}{7} = 10V$$

Q32. The Hamiltonian operator for a two-level quantum system is $H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$. If the state of the system at $t = 0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $|\langle\psi(0)|\psi(t)\rangle|^2$ at a later time t is

(a) $\frac{1}{2}\left(1 + e^{-(E_1 - E_2)t/\hbar}\right)$ (b) $\frac{1}{2}\left(1 - e^{-(E_1 - E_2)t/\hbar}\right)$

(c) $\frac{1}{2}\left(1 + \cos[E_1 - E_2]t/\hbar\right)$ (d) $\frac{1}{2}\left(1 - \cos[E_1 - E_2]t/\hbar\right)$

Ans. : (c)

Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad |\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp - \frac{iE_1 t}{\hbar} \\ \exp - \frac{iE_2 t}{\hbar} \end{pmatrix}$

$$|\langle\psi(0)|\psi(t)\rangle|^2 = \frac{1}{4}\left|\exp - \frac{iE_1 t}{\hbar} + \exp - \frac{iE_2 t}{\hbar}\right|^2 = \frac{1}{2}\left(1 + \cos[E_1 - E_2]t/\hbar\right)$$

Q33. A particle of mass m moves in a lattice along the x - axis in a periodic potential $V(x)=V(x+d)$ with periodicity d. The corresponding Brillouin zone extends from $-k_0$ to k_0 with these two k - points being equivalent. If a weak force F in the x - direction is applied to the particle, it starts a periodic motion with the time period T. Using the equation of motion $F=\frac{dp_{\text{crystal}}}{dt}$ for a particle moving in a band, where p_{crystal} is the crystal momentum of the particle, the period T is found to be (h is Planck constant)

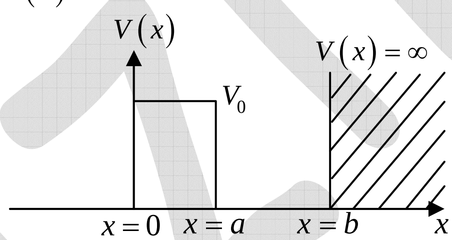
(a) $\sqrt{\frac{2md}{F}}$ (b) $2\sqrt{\frac{2md}{F}}$ (c) $\frac{2h}{Fd}$ (d) $\frac{h}{Fd}$

Ans. : (d)

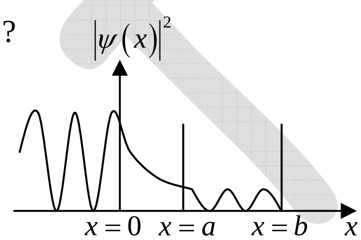
Solution: $\Delta E = E\int_0^d F dx = F[x]_0^d = Fd$

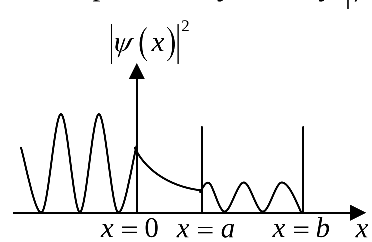
Using Heisenberg uncertainty $\Delta E\cdot\Delta t = h$, $T=\Delta t=\frac{h}{\Delta E}=\frac{h}{Fd}$. Thus correct option is (d)

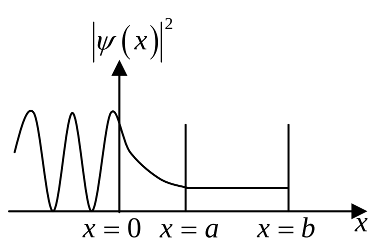
Q34. Consider a potential barrier $V(x)$ of the form:

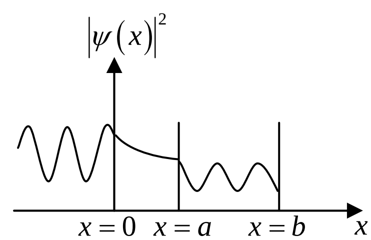


where V_0 is a constant. For particles of energy $E<V_0$ incident on this barrier from the left which of the following schematic diagrams best represents the probability density $|\psi(x)|^2$ as a function of x?

(a)


(b)


(c)


(d)


Ans. : (a)

Q35. The spin-orbit interaction term of an electron moving in a central field is written as $f(r)\vec{l}\cdot\vec{s}$, where r is the radial distance of the electron from the origin. If an electron moves inside a uniformly charged sphere, then

(a) $f(r)=$ constant (b) $f(r)\propto r^{-1}$ (c) $f(r)\propto r^{-2}$ (d) $f(r)\propto r^{-3}$

Ans. : (a)

Solution: The electric potential of a uniformly charged sphere at $r<R$ is

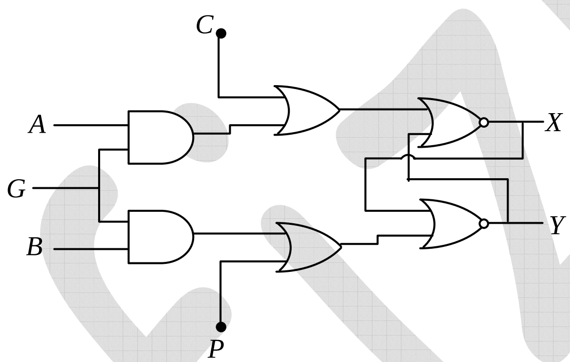
$$V=\frac{kQ}{2R}\left(3-\frac{r^2}{R^2}\right)$$

where Q is the electric charge on the sphere of radius R and k is a constant.

The interaction energy is $W=f(r)\vec{l}\cdot\vec{s}$, where for central potential V, $f(r)=\frac{1}{r}\left(\frac{\partial V}{\partial r}\right)$

$\therefore\ f(r)=\frac{1}{r}\left[\frac{-kQr}{R^3}\right]=\frac{-kQ}{R^3}=$ constant. Thus option (a) is correct.

Q36. For the following circuit, the correct logic values for the entries X_2 and Y_2 in the truth table are



G	A	B	P	C	X	Y
1	0	1	0	0	0	1
0	0	0	1	0	X_2	Y_2
1	0	0	0	1	0	1

(a) 1 and 0 (b) 0 and 0 (c) 0 and 1 (d) 1 and 1

Ans. : (a)

Q37. In a set of N successive polarizers, the m^{th} polarizer makes an angle $\left(\frac{m\pi}{2N}\right)$ with the vertical. A vertically polarized light beam of intensity I_0 is incident on two such sets with $N=N_1$ and $N=N_2$, where $N_2>N_1$. Let the intensity of light beams coming out be $I(N_1)$ and $I(N_2)$, respectively. Which of the following statements is correct about the two outgoing beams?

(a) $I(N_2)>I(N_1)$; the polarization in each case is vertical

(b) $I(N_2)<I(N_1)$; the polarization in each case is vertical

(c) $I(N_2)>I(N_1)$; the polarization in each case is horizontal

(d) $I(N_2)<I(N_1)$; the polarization in each case is horizontal

Ans. : (c)

Solution: $I(N_1) = I_0\left[\cos\left(\frac{n/2}{N_1}\right)\right]^{2N_1}$, $I(N_2) = I_0\left[\cos\left(\frac{n/2}{N_2}\right)\right]^{2N_2}$

$I(N_2) > I(N_1)$

For last polarization, pass axis will be horizontal.

Ex: $N_1 = 5$

$I(5) = I_0\left[\cos\left(18^*\right)\right]^{10} = 0.605\,I_0$

$N_2 = 10$

$I(10) = I_0\left[\cos\left(9^*\right)\right]^{20} = 0.780\,I_0$

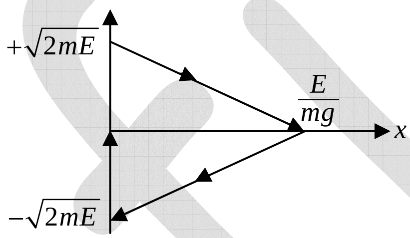
$I(10) > I(5)$

Q38. A ball bouncing of a rigid floor is described by the potential energy function

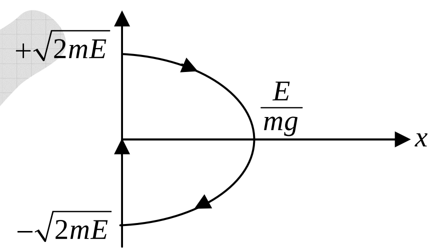
$$V(x) = \begin{cases} mgx & \text{for} \quad x > 0 \\ \infty & \text{for} \quad x \le 0 \end{cases}$$

Which of the following schematic diagrams best represents the phase space plot of the ball?

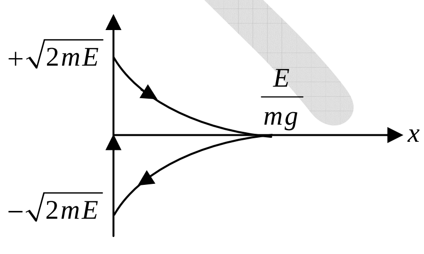
(a)



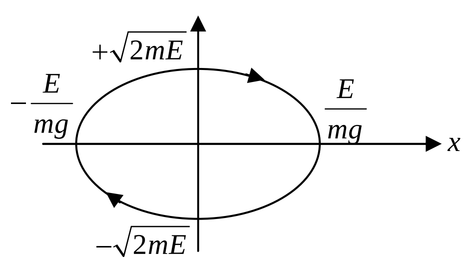
(b)



(c)



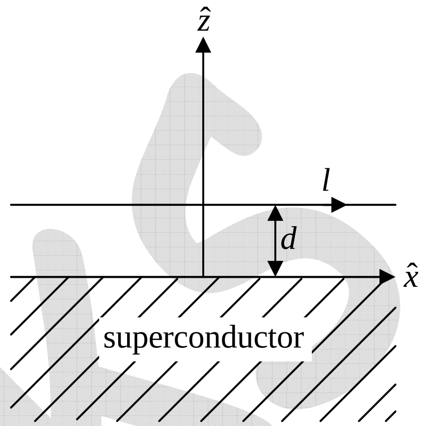
(d)



Ans. : (b)

Solution: $E = \frac{p^2}{2m} + mgx \Rightarrow p^2 = 2m(E - mgx)$ which is equation of parabola

Q39. An infinitely long wire parallel to the x-axis is kept at $z = d$ and carries a current I in the positive x direction above a superconductor filling the region $z \le 0$ (see figure). The magnetic field $\vec{B}$ inside the superconductor is zero so that the field just outside the superconductor is parallel to its surface. The magnetic field due to this configuration at a point $(x, y, z > 0)$ is



(a) $\left(\frac{\mu_0 I}{2\pi}\right)\frac{-(z-d)\hat{j}+y\hat{k}}{\left[y^2+(z-d)^2\right]}$

(b) $\left(\frac{\mu_0 I}{2\pi}\right)\left[\frac{-(z-d)\hat{j}+y\hat{k}}{y^2+(z-d)^2}+\frac{(z+d)\hat{j}-y\hat{k}}{y^2+(z+d)^2}\right]$

(c) $\left(\frac{\mu_0 I}{2\pi}\right)\left[\frac{-(z-d)\hat{j}+y\hat{k}}{y^2+(z-d)^2}-\frac{(z+d)\hat{j}-y\hat{k}}{y^2+(z+d)^2}\right]$

(d) $\left(\frac{\mu_0 I}{2\pi}\right)\left[\frac{y\hat{j}+(z-d)\hat{k}}{y^2+(z-d)^2}+\frac{y\hat{j}-(z+d)\hat{k}}{y^2+(z+d)^2}\right]$

Ans. : (b)

Solution: Verify that $\vec{B} = 0$, when $d = 0$

Q40. The vector potential inside a long solenoid with n turns per unit length and carrying current I, written in cylindrical coordinates is $\vec{A}(s,\phi,z) = \frac{\mu_0 nI}{2} s\hat{\phi}$. If the term $\frac{\mu_0 nI}{2} s\left(\alpha\cos\phi\,\hat{\phi} + \beta\sin\phi\,\hat{s}\right)$, where $\alpha \ne 0,\ \beta \ne 0$ is added to $\vec{A}(s,\phi,z)$, the magnetic field remains the same if

(a) $\alpha = \beta$ (b) $\alpha = -\beta$ (c) $\alpha = 2\beta$ (d) $\alpha = \frac{\beta}{2}$

$$\left[\begin{array}{c}\text{Useful formulae:}\quad \vec{\Delta}t = \frac{\partial t}{\partial S}\hat{S} + \frac{1}{S}\frac{\partial t}{\partial \phi}\hat{\phi} + \frac{\partial t}{\partial z}\hat{z}; \\ \vec{\nabla}\times\vec{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial (s v_\phi)}{\partial s} - \frac{\partial v_s}{\partial \phi}\right)\hat{z}\end{array}\right]$$

Ans. : (d)

Solution: $\vec{B}=\vec{\nabla}\times\vec{A}=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & 0\end{vmatrix}=\mu_0 nI\hat{z}$

$$\vec{B}'=\vec{\nabla}\times\vec{A}'=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\phi} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z}\\ A_r & rA_\phi & 0\end{vmatrix}=\mu_0 nI\left[\left(\alpha\cos\phi+1\right)-\frac{\beta\cos\phi}{2}\right]\hat{z}$$

Equate $\vec{B}'=\vec{B}\Rightarrow\left[\left(\alpha\cos\phi+1\right)-\frac{\beta\cos\phi}{2}\right]\mu_0 nI=\mu_0 nI$

$$\Rightarrow\alpha\cos\phi=\frac{\beta}{2}\cos\phi\Rightarrow\alpha=\frac{\beta}{2}$$

Q41. Low energy collision (s- wave scattering) of pion (π^+) with deuteron (d) results in the production of two proton ($\pi^+ + d\rightarrow p+p$). The relative orbital angular momentum (in units of $\hbar$) of the resulting two-proton system for this reaction is

(a) 0 (b) 1 (c) 2 (d) 3

Ans.: (b)

Solution: $\pi^+ + d \quad\rightarrow\quad p+p$

Parity: $(-1)\times(+1) \qquad (-1)^l\ \pi_p\pi_p$

$\therefore\ (-1)^l\ \pi_p\pi_p=-1$

Since $\pi_p=+1 \qquad \therefore\ (-1)^l=-1$

Thus, $l=1$.

Q42. Consider the Hamiltonian $H(q,p)=\frac{\alpha p^2q^4}{2}+\frac{\beta}{q^2}$, where α and β are parameters with appropriate dimensions, and q and p are the generalized coordinate and momentum, respectively. The corresponding Lagrangian $L(q,\dot{q})$ is

(a) $\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4}-\frac{\beta}{q^2}$ (b) $\frac{2}{\alpha}\frac{\dot{q}^2}{q^4}+\frac{\beta}{q^2}$ (c) $\frac{1}{\alpha}\frac{\dot{q}^2}{q^4}+\frac{\beta}{q^2}$ (d) $-\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4}+\frac{\beta}{q^2}$

Ans. : (a)

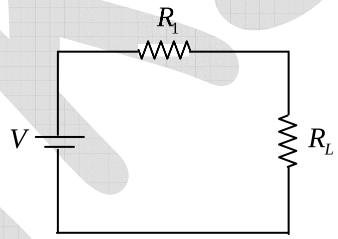
Solution: $L = p\dot{q} - H \Rightarrow p\dot{q} - \frac{ap^2q^4}{2} - \frac{\beta}{q^2}$ from Hamiltonian equation of motion $\frac{\partial H}{\partial p} = \dot{q} \Rightarrow p = \frac{\dot{q}}{aq^4}$

$$L = \frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} - \frac{\beta}{q^2}$$

Q43. For a given load resistance $R_L = 4.7$ ohm, the power transfer efficiencies $\left(\eta = \frac{P_{load}}{P_{total}}\right)$ of a dc voltage source and a dc current source with internal resistances R_1 and R_2, respectively, are equal. The product R_1R_2 in units of ohm^2 (rounded off to one decimal place) is________

Ans. : 22.09

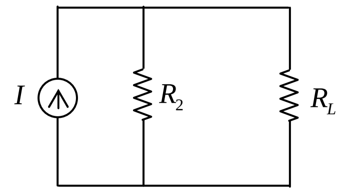
Solution: For dc voltage source



$$P_{total} = \frac{V^2}{R_1 + R_L} \text{ and } P_{R_L} = \left(\frac{V}{R_1 + R_L}\right)^2 R_L$$

$$\eta_{dc\ vol} = \frac{P_{R_L}}{P_{total}} = \frac{R_L}{R_1 + R_L}$$

For dc current source



$$P_{total} = I^2\left(\frac{R_2R_L}{R_2 + R_L}\right) \text{ and } P_{R_L} = I_L^2R_L = \left(\frac{R_2I}{R_2 + R_L}\right)^2 R_L$$

$$\eta_{dc\ curr} = \frac{P_{R_L}}{P_{total}} = \frac{R_2}{R_2 + R_L}$$

Since $\eta_{dc\ vol} = \eta_{dc\ curr}$

$$\Rightarrow \frac{R_L}{R_1 + R_L} = \frac{R_2}{R_2 + R_L} \Rightarrow R_L\left(R_2 + R_L\right) = R_2\left(R_1 + R_L\right) \Rightarrow R_1R_2 = R_L^2$$

$$\Rightarrow R_1R_2 = \left(4.7\right)^2 = 22.09\ \Omega^2$$

Q44. The ground state electronic configuration of the rare-earth ion (Nd^{3+}) is $[Pd]4f^{3}5s^{2}5p^{6}$. Assuming LS coupling, the Lande g - factor of this ion is $\frac{8}{11}$. The effective magnetic moment in units of Bohr magneton μ_B (rounded off to two decimal places) is ____________.

Ans.: 3.62

Solution: For $4f^{3}$ $M_L =$

−3	−2	−1	0	+1	+2	+3
↑	↑	↑				

$L = 6, \quad S = 3/2, \quad J = 9/2$

$$\therefore \mu = g_J \mu_B \sqrt{J(J+1)} = \frac{8}{11} \times \mu_B \times \sqrt{\frac{9}{2}\left(\frac{9}{2}+1\right)}$$

$$= \frac{8}{11}\sqrt{\frac{9}{2} \times \frac{11}{2}} \mu_B = 3.62\, \mu_B$$

Q45. A projectile of mass $1kg$ is launched at an angle of 30^0 from the horizontal direction at $t = 0$ and takes time T before hitting the ground. If its initial speed is 10 ms^{-1}, the value of the action integral for the entire flight in the units of kgm^2s^{-1} (round off to one decimal place) is____________. [Take $g = 10\, ms^{-2}$]

Ans. : 33.3

Solution: $T = \dfrac{2v\sin\theta}{g} = 1\sec$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$\dot{x} = v\cos\theta = 5\sqrt{3}ms^{-1} \quad \dot{y} = v\sin\theta - gt = 5 - 10t$$

$$y = ut - \frac{1}{2}gt^2 = v\sin\theta t - \frac{1}{2}gt^2 = 10.\frac{1}{2}t - \frac{1}{2}10t^2 = 5t - 5t^2$$

$$L = \frac{1}{2}1\left(\left(5\sqrt{3}\right)^2 + (5-10t)^2\right) - 1\times 10 \times \left(5t - 5t^2\right)$$

$$L = 100t^2 - 100t + 50$$

$$A = \int_0^T L dt = \int_0^1 \left(100t^2 - 100t + 50\right) dt = 33.3$$

Q46. Let θ be a variable in the range $-\pi \le \theta < \pi$. Now consider a function

$$\psi(\theta) = \begin{cases} 1 & \text{for } \frac{-\pi}{2} \le \theta < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

if its Fourier-series is written as $\psi(\theta) = \sum_{m=-\infty}^{\infty} C_m e^{-im\theta}$, then the value of $|C_3|^2$ (rounded off to three decimal places) is___________.

Ans. : 0.011

Solution: The Fourier coefficient C_n is $C_m = \frac{1}{2l}\int_{-\pi}^{\pi} f(x) e^{im\theta} d\theta$

Here $2l = 2\pi$,

Therefore, $C_m = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} (1) e^{im\theta} d\theta = \frac{1}{2\pi}\cdot\frac{1}{im}\left\{e^{im\theta}\right\}_{-\pi/2}^{\pi/2} = \frac{1}{2\pi\, im}\left[e^{\pi im/2} - e^{-\pi im/2}\right]$

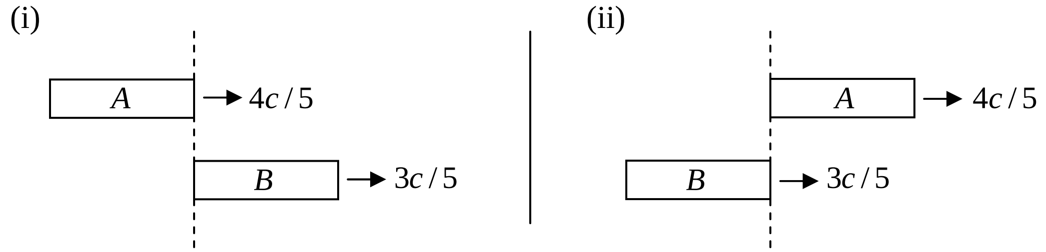
For $m = 3$

$$C_3 = \frac{1}{6\pi i}\left[e^{3\pi i/2} - e^{-3\pi i/2}\right]$$

or, $C_3 = \frac{1}{6\pi i}\left(2i\sin\frac{3\pi}{2}\right) = \frac{1}{3\pi}\sin\frac{3\pi}{2} = \frac{1}{3\pi}(-1) = -\frac{1}{3\pi}$

Therefore, $|C_3|^2 = \left(\frac{1}{3\pi}\right)^2 = \frac{1}{9\pi^2} = 0.011$

Q47. Two spaceships A and B, each of the same rest length L, are moving in the same direction with speeds $\frac{4c}{5}$ and $\frac{3c}{5}$, respectively, where c is the speed of light. As measured by B, the time taken by A to completely overtake B [see figure below] in units of L/c (to the nearest integer) is ____________



Ans. : 5

Solution: $$u_{A,B}=\frac{\frac{4}{5}c-\frac{3}{5}c}{1-\frac{4}{5}c.\frac{3}{5}c.\frac{1}{c^2}}=\frac{\frac{c}{5}}{\frac{13}{25}}=\frac{5}{13}c$$

Kinematic equation is given by

$$\frac{5}{13}c\times t=L\sqrt{1-\frac{25}{169}}+L\Rightarrow t=\frac{5L}{c}\Rightarrow\alpha=5$$

Q48. A radioactive element X has a half-life of 30 hours. It decays via alpha, beta and gamma emissions with the branching ratio for beta decay being 0.75. The partial half-life for beta decay in unit of hours is ____________

Ans.: 40

Solution: Branching ratio is the fraction of particles (here β) which decays by an individual decay mode with respect to the total number of particles which decays

$$BR=\frac{\left(\frac{dN}{dt}\right)_x}{\left(\frac{dt}{dt}\right)_\beta}=\frac{(T_{1/2})_x}{(T_{1/2})_\beta}\Rightarrow(T_{1/2})_\beta=\frac{(T_{1/2})_x}{BR}=\frac{30}{0.75}=40\text{ hours}$$

Q49. In a thermally insulated container, $0.01\,kg$ of ice at $273\,K$ is mixed with $0.1\,kg$ of water at $300\,K$. Neglecting the specific heat of the container, the change in the entropy of the system in J/K on attaining thermal equilibrium (rounded off to two decimal places) is____________

Ans. : 1.03

Solution: $T_{eq}=290.29\,K$ (Heat gain $=$ Heat lost)

$$m_{ice}L+m_{ice}C(T-273)=m_\omega C(300-T)$$

$$T=290.29\,K$$

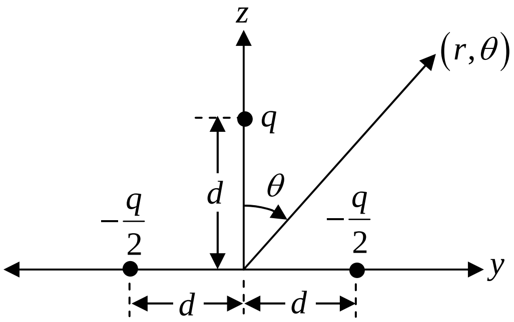
$$\Delta s=\Delta s_{ice}+\Delta s_{water}$$

$$(\Delta s)_{ice}=\frac{m_{ice}L}{T_{ice}}+m_{ice}C\ln\frac{T_i}{T_{ice}}=14.85\ J/K$$

$$(\Delta S)_{water}=m_\omega C\ln\frac{290.29}{300}=-13.82\ J/K$$

$$\Delta S=1.03\ J/K$$

Q50. Consider a system of three charges as shown in the figure below:



For $r = 10\ m; \theta = 60$ degrees; $q = 10^{-6}$ Coulomb, and $d = 10^{-3}\ m$, the electric dipole potential in volts (rounded off to three decimal places) at a point (r, θ) is ________

[Use: $\frac{1}{4\pi \in_0} = 9 \times 10^9 \frac{Nm^2}{C^2}$]

Ans. : 0.045

Solution: Monopole moment $= -\frac{q}{2} - \frac{q}{2} + q = 0$

$$\vec{p} = -\frac{q}{2} \times (-d\hat{y}) - \frac{q}{2}(d\hat{y}) + q(d\hat{z}) \Rightarrow \vec{p} = qd\hat{z}$$

$$V(r,\theta) = \frac{1}{4\pi \in_0} \frac{\vec{p} \cdot r}{r^2} = \frac{1}{4\pi \in_0} \frac{qd\cos\theta}{r^2}$$

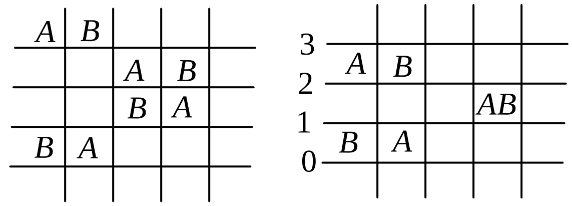
$$V(r,\theta) = 9 \times 10^9 \times \frac{10^{-6} \times 10^{-3} \times \cos 60^0}{(10)^2} = 9 \times 10^9 \times \frac{10^{-9}}{2 \times 100} = 0.045$$

Q51. Consider two system A and B each having two distinguishable particles. In both the systems, each particle can exist in states with energies $0, 1, 2$ and 3 units with equal probability. The total energy of the combined system is 5 units. Assuming that the system A has energy 3 units and the system B has energy 2 units, the entropy of the system is $k_B \ln \lambda$. The value of λ is__________

Ans. : 12

Solution:

$E_A = 3$ $\qquad$ $E_B = 2$



$\Omega_A = 4$ $\qquad$ $\Omega_B = 3$

$\Omega = 4 \times 3 = 12$

$S = \ln \Omega = k_B \ln 12$

$\lambda = 12$.

Q52. Electrons with spin in the z - direction $(\hat{z})$ are passed through a Stern-Gerlach (SG) set up with the magnetic field at $\theta = 60^0$ from $\hat{z}$. The fraction of electrons that will emerge with their spin parallel to the magnetic field in the SG set up (rounded off to two decimal places) is____________

$$\left[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

Ans. : 0.25

Solution: $|\psi\rangle = \begin{pmatrix} \cos 60^0 \\ \sin 60^0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$ state related to up state is $\left|\frac{1}{2},\frac{1}{2}\right\rangle = |\chi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The fraction of electrons that will emerge with their spin parallel to the magnetic field $|\langle \chi|\psi\rangle|^2 = \frac{1}{4} = 0.25$

Q53. The Hamiltonian of a system is $H = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & -1 \end{pmatrix}$ with $\varepsilon \ll 1$. The fourth order contribution to the ground state energy of H is $\gamma\varepsilon^4$. The value of γ (rounded off to three decimal places) is__________.

Ans. : 0.125

Solution: $H = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & -1 \end{pmatrix}$ the eigen value of the Hamiltonian is $E_g = -\sqrt{1-\varepsilon^2}, E_f = +\sqrt{1-\varepsilon^2}$

The ground state is $E_g = -\sqrt{1-\varepsilon^2}$

Taylor expansion of $-\sqrt{1-\varepsilon^2} = -\left(1 - \frac{\varepsilon^2}{2} - \frac{\varepsilon^4}{8}.....\right) = -1 + \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{8}.....$

$\gamma = \frac{1}{8} = 0.125$

Q54. Two events, one on the earth and the other one on the Sun, occur simultaneously in the earth's frame. The time difference between the two events as seen by an observer in a spaceship moving with velocity $0.5c$ in the earth's frame along the line joining the earth to the Sun is Δt, where c is the speed of light. Given that light travels from the Sun to the earth in 8.3 minutes in the earth's frame, the value of $|\Delta t|$ in minutes (rounded off to two decimal places) is____________

(Take the earth's frame to be inertial and neglect the relative motion between the earth and the sun)

Ans. : 4.77

Solution: $t_2' - t_1' = 0 \quad x_2' - x_1' = 8.3 \times 3 \times 10^8 \times 60 \quad v = 0.5c$

$$\Delta t = t_2 - t_1 = \left(\frac{t_2' + \frac{v x_2'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) - \left(\frac{t_1' + \frac{v x_1'}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \left(\frac{t_2' - t_1'}{\sqrt{1 - \frac{v^2}{c^2}}} \right) + \frac{v}{c^2} \frac{(x_2' - x_1')}{\sqrt{1 - \frac{v^2}{c^2}}} = 4.77 \text{ min}$$

Q55. In a certain two-dimensional lattice, the energy dispersion of the electrons is

$$\varepsilon(\vec{k}) = -2t \left[\cos k_x a + 2 \cos \frac{1}{2} k_x a \cos \frac{\sqrt{3}}{2} k_y a \right]$$

where $\vec{k} = (k_x, k_y)$ denotes the wave vector, a is the lattice constant and t is a constant in units of eV. In this lattice the effective mass tensor m_{ij} of electrons calculated at the center of the Brillouin zone has the form $m_{ij} = \frac{\hbar^2}{ta^2} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$. The value of α (rounded off to two decimal places) is ____________

Ans.: 0.33

Solution: Effective mass tensor matrix 4

$$m_{ij} = \begin{bmatrix} \frac{1}{m_{xx}} & \frac{1}{m_{xy}} \\ \frac{1}{m_{yx}} & \frac{1}{m_{yy}} \end{bmatrix} = \begin{bmatrix} \frac{1}{m_{xx}} & 0 \\ 0 & \frac{1}{m_{yy}} \end{bmatrix}$$

When $m_{xx} = \frac{\hbar^2}{\partial^2 E / \partial k_x^2}$ and $m_{yy} = \frac{\hbar^2}{\partial^2 E / \partial k_y^2}$

Now $\frac{\partial E}{\partial k_x} = 2t\left[a \sin k_x a + a \sin\left(\frac{1}{2}k_x a\right)\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$

$$\frac{\partial^2 E}{dk_x^2} = 2t\left[a^2 \cos(k_x a) + \frac{a^2}{2}\cos\left(\frac{1}{2}k_x a\right)\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$$

At the Brillouin zone centre i.e. at $k_x = k_y = 0$

$$\therefore \qquad \frac{\partial^2 E}{\partial k_x^2} = 2ta^2\left(1 + \frac{1}{2}\right) = 3ta^2$$

Similarly, $\frac{\partial E}{\partial k_y} = 2t\left[\sqrt{3}a \cos\left(\frac{1}{2}k_x a\right)\sin\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$

$$\frac{\partial^2 E}{\partial k_y^2} = 2t\left[\frac{3a^2}{2}\cos\left(\frac{1}{2}k_x a\right)\cos\left(\frac{\sqrt{3}}{2}k_y a\right)\right]$$

At the Brillouin zone centre i.e. at $k_x = k_y = 0$

$$\frac{\partial^2 E}{\partial_y^2} = 3ta^2$$

Thus $m_{xx} = \frac{\hbar^2}{\partial^2 E / \partial k_x^2} = \frac{\hbar^2}{3ta^2}$ and $m_{yy} = \frac{\hbar^2}{\partial^2 E / \partial k_y^2} = \frac{\hbar^2}{3ta^2}$

$$m_{ij} = \begin{bmatrix} \frac{\hbar^2}{3ta^2} & 0 \\ 0 & \frac{\hbar^2}{3ta^2} \end{bmatrix} = \frac{\hbar^2}{ta^2}\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

Thus $\alpha = \frac{1}{3} = 0.333$.