JEST 2019 (Booklet Series-A)

Part-A: 1-Mark Questions

Q1. Let $\vec{r}$ be the position vector of a point on a closed contour C. What is the value of the line integral $\oint \vec{r} \cdot d\vec{r}$?

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) π

Ans. : (a)

Solution: $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \Rightarrow \vec{\nabla} \times \vec{r} = 0 \Rightarrow \oint \vec{r} \cdot d\vec{r} = 0$

Q2. A dc voltage of 80 Volt is switched on across a circuit containing a resistance of 5Ω in series with an inductance of $20\,H$. What is the rate of change of current at the instant when the current is $12\,A$?

(a) $0\,A/s$ (b) $1\,A/s$ (c) $5\,A/s$ (d) $80\,A/s$

Ans. : (b)

Solution: $i(t) = \frac{V}{R}\left(1 - e^{-\frac{R}{L}t}\right) \Rightarrow \frac{di}{dt} = \frac{V}{L}e^{-\frac{R}{L}t} \Rightarrow \frac{di}{dt} = \frac{V}{L}\left(1 - \frac{R}{V}i\right)$

$$\Rightarrow \frac{di}{dt} = \frac{80}{20}\left(1 - \frac{5}{80} \times 12\right) = \frac{80}{20}\left(\frac{20}{80}\right) = 1\,A/s$$

Q3. Consider the function $f(x, y) = |x| - i|y|$. In which domain of the complex plane is this function analytic?

(a) First and second quadrants (b) Second and third quadrants

(c) Second and fourth quadrants (d) Nowhere

Ans. : (c)

Solution: $f(x, y) = |x| - i|y|$

$f(x, y) = x - iy = \bar{z}$

$f(x, y) = -x - iy = -z$

$f(x, y) = -x + iy = -\bar{z}$

$f(x, y) = x + iy = z$

We know $\bar{z}$ is not analytic and z and $-z$ are analytic. So answer is (c).

Q4. Consider the following transformation of the phase space coordinates $(q, p) \rightarrow (Q, P)$

$$Q = q^a \cos bp \quad P = q^a \sin bp$$

For what values of a and b will the transformation be canonical?

(a) $1,1$ (b) $\frac{1}{2}, \frac{1}{2}$ (c) $2, \frac{1}{2}$ (d) $\frac{1}{2}, 2$

Ans. : (d)

Solution: For canonical transformation $\frac{\partial Q}{\partial q} \cdot \frac{\partial P}{\partial p} - \frac{\partial P}{\partial q} \cdot \frac{\partial Q}{\partial p} = 1 \Rightarrow abq^{2a-1}\left(\cos^2 bp + \sin^2 bp\right) = 1$

$$a = \frac{1}{2}, b = 2$$

Q5. What is the binding energy of an electron in the ground state of a He^+ ion?

(a) $6.8\,eV$ (b) $13.6\,eV$ (c) $27.2\,eV$ (d) $54.4\,eV$

Ans. : (d)

Solution: $E = -\frac{13.6}{n^2} z^2 (eV)$

$He^+ : z = 2$

$$\therefore E = \frac{-13.6 \times 4}{n^2}(eV)$$

The binding energy of an electron in ground state is

$$E = \frac{-13.6 \times 4}{(1)^2}(eV) = 54.4\,eV$$

Q6. A collimated white light source illuminates the slits of a double slit interference setup and forms the interference pattern on a screen. If one slit is covered with a blue filter, which one of the following statements is correct?

(a) No interference pattern is observed after the slit is covered with the blue filter

(b) Interference pattern remains unchanged with and without the blue filter

(c) A blue interference pattern is observed

(d) The central maximum is blue with coloured higher order maxima

Ans. : (c)

Solution: Because to form stationary interference pattern light from two coherent source should be of same frequency and wavelength.

Q7. Consider a system of N distinguishable particles with two energy levels for each particle, a ground state with energy zero and an excited state with energy $\varepsilon > 0$. What is the average energy per particle as the system temperature $T \to \infty$?

(a) 0 (b) $\frac{\varepsilon}{2}$ (c) ε (d) ∞

Ans. : (b)

Solution: $\langle E \rangle = \sum_i P_i E_i \Rightarrow P_i = \frac{e^{\beta E_i}}{z}$

$$\langle E \rangle = 0 \times \frac{01}{1+e^{-\beta\varepsilon}} + \varepsilon \times \frac{1}{1+e^{-\beta\varepsilon}}$$

$$= \frac{\varepsilon}{1+e^{-\varepsilon/k_B T}} = \frac{\varepsilon}{2} \text{ at } T \to \infty$$

Q8. Consider a diatomic molecule with an infinite number of equally spaced non-degenerate energy levels. The spacing between any two adjacent levels is ε and the ground state energy is zero. What is the single particle partition function Z?

(a) $Z = \frac{1}{1-\frac{\varepsilon}{k_B T}}$ (b) $Z = \frac{1}{1-e^{\frac{\varepsilon}{k_B T}}}$

(c) $Z = \frac{1}{1-e^{\frac{2\varepsilon}{k_B T}}}$ (d) $Z = \frac{1-\frac{\varepsilon}{k_B T}}{1+\frac{\varepsilon}{k_B T}}$

Ans. : No option is matched

Solution: $Z = \sum_i g_i e^{-\beta\varepsilon_i}$

$$g_i = 1$$

$$Z = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + \ldots\ldots$$

$$Z = \frac{1}{1-e^{-\beta\varepsilon}}$$

Q9. A very long solenoid (axis along z direction) of n turns per unit length carries a current which increases linearly with time, $i = Kt$. What is the magnetic field inside the solenoid at a given time t?

(a) $B = \mu_0 nKt\hat{z}$ (b) $B = \mu_0 nK\hat{z}$

(c) $B = \mu_0 nKt(\hat{x}+\hat{y})$ (d) $B = \mu_0 cnKt\hat{z}$

Ans. : (a)

Q10. Suppose $\psi\vec{A}$ is a conservative vector, $\vec{A}$ is a non-conservative vector and ψ is non-zero scalar everywhere. Which one of the following is true?

(a) $(\nabla\times\vec{A})\cdot\vec{A} = 0$ (b) $\vec{A}\times\nabla\psi = \vec{0}$

(c) $\vec{A}\cdot\nabla\psi = 0$ (d) $(\nabla\times\vec{A})\times\vec{A} = \vec{0}$

Ans. : (a)

Solution: Divergence of a curl is always zero.

Q11. Consider two $n\times n$ matrices, A and B such that $A+B$ is invertible. Define two matrices, $C = A(A+B)^{-1}B$ and $D = B(A+B)^{-1}A$. Which of the following relations always hold true?

(a) $C = D$ (b) $C^{-1} = D$ (c) $BCA = ADB$ (d) $C \neq D$

Ans. : (a)

Solution: $C^{-1} = \left[A(A+B)^{-1}B\right]^{-1} = B^{-1}(A+B)A^{-1}$

$$= B^{-1}AA^{-1} + B^{-1}BA^{-1} = B^{-1} + A^{-1}$$

$$\Rightarrow C^{-1} = B^{-1} + A^{-1}$$

$$D^{-1} = \left[B(A+B)^{-1}A\right]^{-1} = A^{-1}(A+B)B^{-1}$$

$$= A^{-1}AB^{-1} + A^{-1}B^{-1} = B^{-1} + A^{-1}$$

or $D^{-1} = B^{-1} + A^{-1}$

From equation (i) and (ii)

$$C^{-1} = D^{-1}$$

or $CC^{-1}D = CD^{-1}D$ or $D = C$

Therefore, option (a) is correct.

Q12. The refractive index (n) of the entire environment around a double slit interference setup is changed from $n=1$ to $n=2$. Which one of the following statements is correct about the change in the interference pattern?

(a) The fringe pattern disappears

(b) The central bright maximum turns dark, i.e. becomes a minimum

(c) Fringe width of the pattern increases by a factor 2

(d) Fringe width of the pattern decreases by a factor 2

Ans. : (d)

Solution: $\beta = \frac{D}{2d}\left(\frac{\lambda}{n}\right)$

Q13. A cyclotron can accelerate deuteron to $16\,MeV$. If the cyclotron is used to accelerate α - particles, what will be their energy? Take the mass of deuteron to be twice the mass of proton and mass of alpha particles to be four times the mass of protons.

(a) $8\,MeV$ (b) $16\,MeV$ (c) $32\,MeV$ (d) $64\,MeV$

Ans. : (c)

Solution: Energy gain in cyclotron is

$$E = \frac{q^2B^2R^2}{2m}$$

Let E_d, m_d, E_α and m_α are the energy of mass of deuteron and α - particle

$$\therefore \frac{E_d}{E_\alpha} = \frac{m_\alpha}{m_d}$$

$$\Rightarrow E_\alpha = \frac{m_d}{m_\alpha}E_d = \frac{2m_\alpha}{m_\alpha}\times 16MeV$$

$$E_\alpha = 32\,MeV$$

Q14. Consider a hypothetical world in which the electron has spin $\frac{3}{2}$ instead of $\frac{1}{2}$. What will be the electronic configuration for an element with atomic number $Z=5$?

(a) $1s^4, 2s^1$ (b) $1s^4, 2s^2, 2p^1$ (c) $1s^5$ (d) $1s^3, 2s^1, 2p^1$

Ans. : (a)

Solution: The degeneracy of level j is $d = 2j+1$

For s - orbit, $d = 2s+1 = 2\times\frac{3}{2}+1 = 4$

$\therefore$ The electronic configuration for $z = 5$ is

$1s^4, 2s^1$

Thus correct option is (a)

Q15. Two objects of unit mass are thrown up vertically with a velocity of $1ms^{-1}$ at latitudes $45^0 N$ and $45^0 S$, respectively. The angular velocity of the rotation of Earth is given to be $7.29\times10^{-5} s^{-1}$. In which direction will the objects deflect when they reach their highest point (due to Coriolis force)? Assume zero air resistance.

(a) to the east in Northern hemisphere and west in Southern Hemisphere

(b) to the west in Northern hemisphere and east in Southern Hemisphere

(c) to the east in both hemispheres

(d) to the west in both hemispheres

Ans. : (d)

Q16. Which one of the following vectors lie along the line of intersection of the two planes $x+3y-z=5$ and $2x-2y+4z=3$?

(a) $10\hat{i}-2\hat{j}+5\hat{k}$ (b) $10\hat{i}-6\hat{j}-8\hat{k}$

(c) $10\hat{i}+2\hat{j}+5\hat{k}$ (d) $10\hat{i}-2\hat{j}-5\hat{k}$

Ans. : (b)

Solution: Unit vector normal to $x+3y-z=5$ is $\hat{n}_1 = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{\hat{i}+3\hat{j}-\hat{k}}{\sqrt{1+9+1}} = \frac{\hat{i}+3\hat{j}-\hat{k}}{\sqrt{11}}$

Unit vector normal to $2x-2y+4z=3$ is $\hat{n}_2 = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|} = \frac{2\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{4+4+16}} = \frac{2\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{24}}$

Check for option (b) $\hat{n} = 10\hat{i}-6\hat{j}-8\hat{k}$

$\hat{n}_1.\hat{n} = \frac{10-18+8}{\sqrt{11}} = 0$ and $\hat{n}_2.\hat{n} = \frac{20+12-32}{\sqrt{24}} = 0$

Q17. What is the value of the integral $\int_{-\infty}^{+\infty} dx\delta\left(x^2-\pi^2\right)\cos x$?

(a) π (b) $-\frac{1}{2\pi}$ (c) $-\frac{1}{\pi}$ (d) 0

Ans. : (c)

Solution: $\delta\left(x^2-\pi^2\right)=\frac{1}{|\pi-(-\pi)|}\left[\delta(x-\pi)+\delta(x+\pi)\right]$

$$=\frac{1}{2\pi}\left[\delta(x-\pi)+\delta(x+\pi)\right]$$

Therefore, $\int_{-\infty}^{\infty} dx\,\delta\left(x^2-\pi^2\right)\cos x=\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\left[\delta(x-\pi)+\delta(x+\pi)\right]\cos x$

$$=\frac{1}{2\pi}\left[\cos\pi+\cos(-\pi)\right]=\frac{1}{2\pi}(-1-1)=-\frac{1}{\pi}$$

Q18. Two joggers A and B are running at a steady pace around a circular track. A takes T_A minutes whereas B takes $T_B\left(>T_A\right)$ minutes to complete one round. Assuming that they have started together, what will be time taken by A to overtake B for the first time?

(a) $\frac{2\pi}{T_A-T_B}$ (b) $\frac{1}{T_A}-\frac{1}{T_B}$ (c) $\frac{1}{T_A+T_B}$ (d) $\left(\frac{1}{T_A}-\frac{1}{T_B}\right)^{-1}$

Ans. : (d)

Solution: $v_{relative}=v_A-v_B\Rightarrow T\left(v_A-v_B\right)=2\pi R$

$$TR\left(\omega_A-\omega_B\right)=2\pi R\Rightarrow TR\left(\frac{2\pi}{T_A}-\frac{2\pi}{T_B}\right)=2\pi R\Rightarrow T=\left(\frac{1}{T_A}-\frac{1}{T_B}\right)^{-1}$$

Q19. The magnetic field (Gaussian units) in an empty space is described by

$$B=B_0\exp(ax)\sin(ky-\omega t)\hat{z}$$

What is the y - component of the electric field?

(a) $-\frac{ac}{\omega}B_0\sin(ky-\omega t)$ (b) $-\frac{ac}{\omega}B_0\exp(ax)\cos(ky-\omega t)$

(c) $-B_0\sin(ky-\omega t)$ (d) 0

Ans. : (d)

Q20. Let A be a hermitian matrix, and C and D be the unitary matrices. Which one of the following matrices is unitary?

(a) $C^{-1}AC$ (b) $C^{-1}DC$ (c) $C^{-1}AD$ (d) $A^{-1}CD$

Ans. : (b)

Solution: $\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger}=C^{-1}DCC^{\dagger}D^{\dagger}\left(C^{-1}\right)^{\dagger}$

Since C is unitary $CC^{-1}=I$, therefore $\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger}=C^{-1}DD^{\dagger}\left(C^{-1}\right)^{\dagger}$

Since D is unitary $DD^{\dagger}=I$, therefore, $\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger}=C^{-1}\left(C^{-1}\right)^{\dagger}$

Since for any invertible matrix $\left(C^{-1}\right)^{\dagger}=\left(C^{+}\right)^{-1}$ we have

$\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger}=C^{-1}\left(C^{\dagger}\right)^{-1}$

Since C is unitary $C^{\dagger}=C^{-1}$, therefore,

$\left(C^{-1}DC\right)\left(C^{-1}DC\right)^{\dagger}=C^{-1}\left(C^{-1}\right)^{-1}=C^{-1}C=I$

Therefore, $C^{-1}DC$ is a unitary matrix.

Q21. The wave function $\psi(x)=A\exp\left(-\frac{b^2x^2}{2}\right)$ (for real constants A and b) is a normalized eigen-function of the Schrodinger equation for a particle of mass m and energy E in a one dimensional potential $V(x)$ such that $V(x)=0$ at $x=0$. Which of the following is correct?

(a) $V=\frac{\hbar^2b^4x^2}{m}$ (b) $V=\frac{\hbar^2b^4x^2}{2m}$ (c) $E=\frac{\hbar^2b^2}{4m}$ (d) $E=\frac{\hbar^2b^2}{m}$

Ans. : (b)

Solution: Comparing with harmonic oscillator $\psi(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$ the potential is

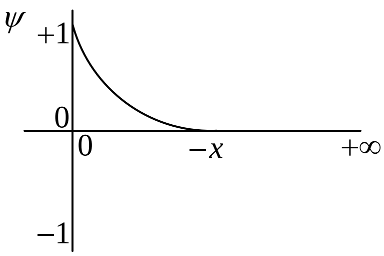
$V(x)=\frac{1}{2}m\omega^2x^2$ and energy is $E=\frac{\hbar\omega}{2}$

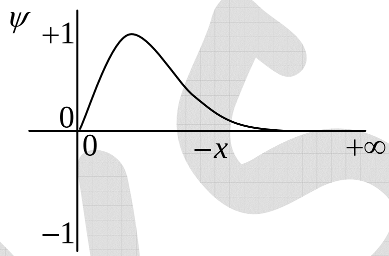
$\psi(x)=A\exp\left(-\frac{b^2x^2}{2}\right)$ $\omega=\frac{b^2\hbar}{m}$ so $V(x)=\frac{b^4\hbar^2x^2}{2m}$ and energy $E=\frac{\hbar\omega}{2}\Rightarrow\frac{b^2\hbar^2}{2m}$

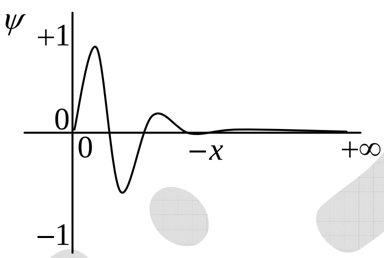
Q22. A quantum particle of mass m is in a one dimensional potential of the form

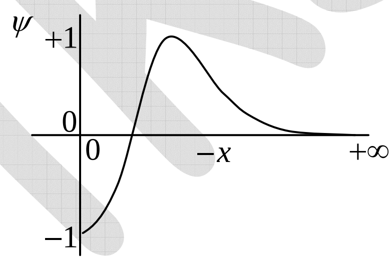
$$V(x)=\begin{cases}\frac{1}{2}m\omega^2x^2, & \text{if } x>0\\ \infty & \text{if } x\le 0\end{cases}$$

where ω is a constant. Which one of the following represents the possible ground state wave function of the particle?

(a)


(b)


(c)


(d)


Ans. : (b)

Q23. Consider a 2×2 matrix $A=\begin{pmatrix}1 & 13\\ 0 & 1\end{pmatrix}$ what is A^{27}?

(a) $\begin{pmatrix}1 & 13\\ 0 & 1\end{pmatrix}$ (b) $\begin{pmatrix}1 & 13^{27}\\ 0 & 1\end{pmatrix}$ (c) $\begin{pmatrix}1 & 27\\ 0 & 1\end{pmatrix}$ (d) $\begin{pmatrix}1 & 351\\ 0 & 1\end{pmatrix}$

Ans. : (d)

Solution: Given $A=\begin{pmatrix}1 & 13\\ 0 & 1\end{pmatrix}$, it can be easily proved (by mathematical induction) that

$$A^n=\begin{pmatrix}1 & 13n\\ 0 & 1\end{pmatrix}$$

For $n=27$,

$$A^{27}=\begin{pmatrix}1 & 13.27\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 351\\ 0 & 1\end{pmatrix}$$

Q24. Consider a grand ensemble of a system of one dimensional non-interacting classical harmonic oscillators (each of frequency ω). Which one of the following equations is correct? Here the angular bracket $\langle \cdot \rangle$ indicate the ensemble average. N, E and T represent the number of particles, energy and temperature, respectively. k_B is the Boltzmann constant.

(a) $\langle E \rangle = N\frac{k_BT}{2}$ (b) $\langle E \rangle = \langle N \rangle \frac{k_BT}{2}$

(c) $\langle E \rangle = Nk_BT$ (d) $\langle E \rangle = \langle N \rangle k_BT$

Ans. : (d)

Solution: $E = K.E. + P.E.$

$$E = \frac{P_x^2}{2m} + \frac{1}{2}kx^2 \, (1D)$$

$$E = \frac{1}{2}k_BT + \frac{1}{2}k_BT = k_BT \qquad \text{(Equiportion)}$$

$$\langle E \rangle = \langle N \rangle k_BT$$

Q25. A bullet with initial speed v_0 is fired at a log of wood. The resistive force by wood on the bullet is given by ηv^α, where $\alpha < 1$. What is the time taken to stop the bullet inside the wood log?

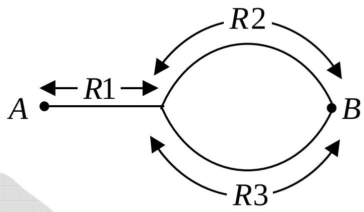
(a) $\frac{m}{\eta}\frac{v_0^{\alpha-1}}{1-\alpha}$ (b) $\frac{m}{\eta}\frac{v_0^{\alpha+1}}{\alpha+1}$ (c) $\frac{m}{\eta}\frac{v_0^{1-\alpha}}{1-\alpha}$ (d) $\frac{\eta}{m}\frac{v_0^{1-\alpha}}{1-\alpha}$

Ans. : (c)

Solution: $m\frac{dv}{dt} = -\eta v^\alpha \Rightarrow \int_0^t dt = -\frac{m}{\eta}\int_{v_0}^{0}\frac{dv}{v^\alpha} = \frac{m}{\eta}\frac{v_0^{1-\alpha}}{1-\alpha}$

PART- B: 3 Mark-Questions

Q1. A person plans to go from town A to town B by taking either the route $(R1+R2)$ with probability $\frac{1}{2}$ or the route $(R1+R3)$ with probability $\frac{1}{2}$ (see figure). Further, there is a probability $\frac{1}{3}$ that $R1$ is blocked, a probability $\frac{1}{3}$ that $R2$ is blocked, and a probability $\frac{1}{3}$ that $R3$ is blocked. What is the probability that he/she would reach town B?



(a) $\frac{8}{9}$ (b) $\frac{1}{3}$ (c) $\frac{4}{9}$ (d) $\frac{2}{3}$

Ans. : (c)

Solution: Given that probability of $R1$ blocked =1/3

Probability of $R1$ not blocked $=1-\frac{1}{3}=\frac{2}{3}$

Probability from A to B without restriction $=\frac{1}{2}$

Route $R2$ probability $=\frac{1}{2}\times\frac{2}{3}$ not blocked

Route $R3$ $=\frac{1}{2}\times\frac{2}{3}$

Total probability $(A\rightarrow B)=\frac{2}{3}\left[\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{2}{3}\right]=\frac{4}{9}$

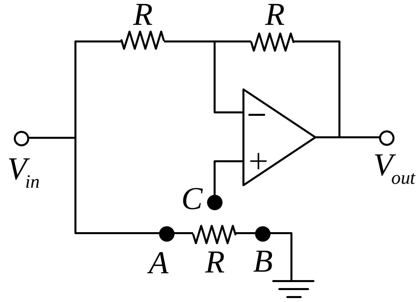
Q2. What is the change in the kinetic energy of rotation of the earth if its radius shrinks by 1%? Assume that the mass remains the same and the density is uniform.

(a) increases by 1% (b) increases by 2%

(c) decreases by 1% (d) decreases by 2%

Ans. : (b)

Q3. Analyse the ideal op-amp circuit in the figure. Which one of the following statements is true about the output voltage V_{out}, when terminal 'C' is connected to point 'A' and then to point 'B'?



(a) $V_{\text{out}} = V_{\text{in}}$ and $V_{\text{out}} = -V_{in}$ when 'C' is connected to 'A' and 'B', respectively

(b) $V_{\text{out}} = -V_{in}$ and $V_{\text{out}} = V_{\text{in}}$ when 'C' is connected to 'A' and 'B', respectively

(c) $V_{\text{out}} = -V_{in}$ when 'C' is connected to either 'A' or 'B'

(d) $V_{\text{out}} = V_{\text{in}}$ when 'C' is connected to either 'A' or 'B'

Ans. : (a)

Solution: When terminal 'C' is connected to point 'A'

$$V_{\text{out}} = \left(1+\frac{1}{1}\right)V_{\text{in}} - \frac{1}{1}V_{\text{in}} = V_{\text{in}}$$

When terminal 'C' is connected to point 'B'

$$V_{\text{out}} = -\frac{1}{1}V_{\text{in}} = -V_{\text{in}}$$

Q4. White light of intensity I_0 is incident normally on a filter plate of thickness d. The plate has a wavelength (λ) dependent absorption coefficient $\alpha(\lambda) = \alpha_0\left(1-\frac{\lambda}{\lambda_0}\right)$ per unit length. The band pass edge of the filter is defined as the wavelength at which the intensity, after passig through the filter, is $I = \frac{I_0}{\rho}, \alpha_0, \lambda_0$ and ρ are constants. The reflection coefficient of the plate may be assumed to be independent of λ. Which one of the following statements is true about the bandwidth of the filter?

(a) The bandwidth is linear dependent on λ_0

(b) The bandwidth is independent of the plate thickness d

(c) The bandwidth is linearly dependent on α_0

(d) The bandwidth is dependent on the ratio α_0 / d

Ans. : (a)

Solution: For example C-band and L-band in fiber optics communication, the central wavelength λ_c of band pass is $\lambda_c = \lambda_0\sqrt{1-\frac{\sin^2\theta}{n^{*2}}}$

Where $\lambda_0 =$ central wavelength at normal incidence

$n^* =$ filter effective index of refraction

$\theta =$ angle of incidence

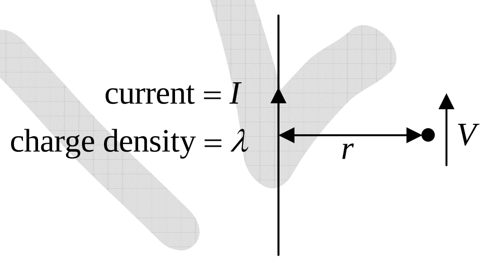
Here this result is applicable only for very low absorption.

Q5. Consider two concentric spherical metal shells of radii r_1 and $r_2\left(r_2 > r_1\right)$. The outer shell has a charge q and the inner shell is grounded. What is the charge on the inner shell?

(a) $\frac{r_1}{r_2}q$ (b) $\frac{r_1}{r_2}q$ (c) 0 (d) $\frac{r_2}{r_1}q$

Ans. : (a)

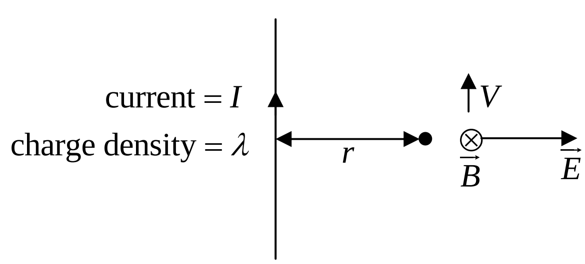
Q6. A wire with uniform line charge density λ per unit length carries a current I as shown in the figure. Take the permittivity and permeability of the medium to be $\varepsilon_0 = \mu_0 = 1$. A particle of charge q is at a distance r and is travelling along a trajectory parallel to the wire. What is the speed of the charge?



(a) $\frac{\lambda}{I}$ (b) $\frac{\lambda}{2I}$ (c) $\frac{\lambda}{3I}$ (d) $\frac{4\lambda}{I}$

Ans.: (a)

Solution: $E = \frac{\lambda}{2\pi\varepsilon_0 r}$ and $B = \frac{\mu_0 I}{2\pi r}$



Directions are shown in the figure.

Net force on charge q is zero i.e. $\vec{F} = 0$.

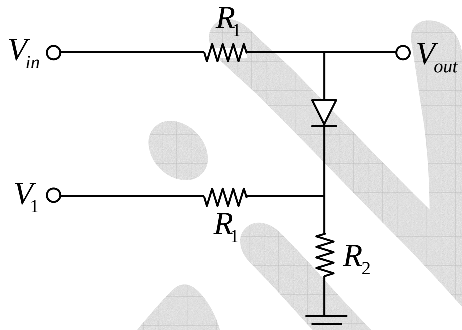
$$\Rightarrow q\left[\vec{E} + \left(\vec{v}\times\vec{B}\right)\right] = 0 \Rightarrow E = vB \Rightarrow \frac{\lambda}{2\pi\varepsilon_0 r} = v\frac{\mu_0 I}{2\pi r} \Rightarrow v = \frac{\lambda}{I} \qquad \because \varepsilon_0 = \mu_0 = 1$$

Q7. Consider a non-relativistic two-dimensional gas of N electrons with the Fermi energy E_F. What is the average energy per particle at temperature $T = 0$?

(a) $\frac{3}{5}E_F$ (b) $\frac{2}{5}E_F$ (c) $\frac{1}{2}E_F$ (d) E_F

Ans. : (c)

Q8. The circuit given below is fed by a sinusoidal voltage $V_{in} = V_0 \sin \omega t$. Assume that the cut-in voltage of the diode is 0.7 volts and V_1 is a positive dc voltage smaller than V_0. Which one of the following statements is true about V_{out}?



(a) Positive part of V_{out} is restricted to a maximum voltage of $0.7 + \frac{R_2}{R_1 + R_2}V_1$

(b) Negative part of V_{out} is restricted to a maximum voltage of $0.7 + \frac{R_2}{R_1 + R_2}V_1$

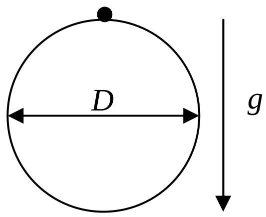
(c) Positive part of V_{out} is restricted to a maximum voltage of $0.7 + \frac{R_1}{R_1 + R_2}V_1$

(d) Negative part of V_{out} is restricted to a maximum voltage of $0.7 + \frac{R_1}{R_1 + R_2}V_1$

Ans: (a)

Solution: Reference voltage $V_R = \frac{R_2}{R_1 + R_2}V_1$ and diode will be ON when $V_{in} > \left(0.7 + \frac{R_2}{R_1 + R_2}V_1\right)$.

Q9. A hoop of diameter D is pivoted at the topmost point on the circumference as shown in the figure. The acceleration due to gravity g is acting downwards. What is the time period of small oscillations in the plane of the hoop?



(a) $2\pi\sqrt{\frac{D}{2g}}$ (b) $2\pi\sqrt{\frac{5D}{6g}}$

(c) $2\pi\sqrt{\frac{D}{2g}}$ (d) $2\pi\sqrt{\frac{2D}{g}}$

Ans. : (c)

Q10. For a spin $\frac{1}{2}$ particle placed in a magnetic field B, the Hamiltonian is $H = -\gamma B S_y = -\omega S_y$, where S_y is the y-component of the spin operator. The state of the system at time $t=0$ is $\left|\psi(t=0)\right\rangle = |+\rangle$, where $S_z|\pm\rangle = \pm\frac{\hbar}{2}|\pm\rangle$. At a later time t, if S_z measured then what is the probability to get a value $-\frac{\hbar}{2}$?

(a) $\cos^2(\omega t)$ (b) $\sin^2(\omega t)$ (c) 0 (d) $\sin^2\left(\frac{\omega t}{2}\right)$

Ans. : (d)

Solution: $H = -\gamma B S_y = -\omega S_y$ Eigen value is $\frac{-\omega\hbar}{2}, \frac{\omega\hbar}{2}$ with eigen vector $|\phi_1\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right]$ and $|\phi_2\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right]$ respectively.

$$\left|\psi(t=0)\right\rangle = |+\rangle \Rightarrow I|+\rangle \Rightarrow |\phi_1\rangle\langle\phi_1|+\rangle + |\phi_2\rangle\langle\phi_2|+\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$$

$$\left|\psi(t=t)\right\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle \exp\left(\frac{i\omega t}{2}\right) + \frac{1}{\sqrt{2}}|\phi_2\rangle \exp\left(-\frac{i\omega t}{2}\right)$$

If S_z is measured on $\left|\psi(t)\right\rangle$ then probability to find $-\frac{\hbar}{2}$ is

$$P\left(-\frac{\hbar}{2}\right) = \frac{\left|\langle -|\psi(t)\rangle\right|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{1}{4}\left|\left(\exp\left(\frac{i\omega t}{2}\right) - \exp\left(-\frac{i\omega t}{2}\right)\right)\right|^2 = \sin^2\frac{\omega t}{2}$$

Q11. In a fixed target elastic scattering experiment, a projectile of mass m, having initial velocity v_0, and impact parameter b, approaches the scatterer. It experiences a central repulsive force $f(r)=\frac{k}{r^2}(k>0)$. What is the distance of the closest approach d?

(a) $d=\left(b^2+\frac{k}{mv_0^2}\right)^{\frac{1}{2}}$ (b) $d=\left(b^2-\frac{k}{mv_0^2}\right)^{\frac{1}{2}}$

(c) $d=b$ (d) $d=\sqrt{\frac{k}{mv_0^2}}$

Ans. : (a)

Solution: $f(r)=\frac{k}{r^2}(k>0)$ so potential is $V(r)=\frac{k}{r}$

Conservation of angular momentum $mv_0b=md^2\dot{\theta}\Rightarrow\dot{\theta}=\frac{v_0b}{d^2}$

Conservation of energy is given by $\frac{mv_0^2}{2}=\frac{md^2\dot{\theta}^2}{2}+\frac{k}{d}$ $\quad\dot{\theta}=\frac{v_0b}{d^2}$

$$d=\left(b^2+\frac{k}{mv_0^2}\right)^{\frac{1}{2}}$$

Q12. Consider a quantum particle in a one-dimensional box of length L. The coordinates of the leftmost wall of the box is at $x=0$ and that of the rightmost wall is at $x=L$. The particle is in the ground state at $t=0$. At $t=0$, we suddenly change the length of the box to $3L$ by moving the right wall. What is the probability that the particle is in the ground state of the new system immediately after the change?

(a) 0.36 (b) $\frac{9}{8\pi}$ (c) $\frac{81}{64\pi^2}$ (d) $\frac{0.5}{\pi}L$

Ans. : (c)

Solution: $|\phi_1\rangle=\begin{cases}\sqrt{\frac{2}{3a}}\sin\frac{\pi x}{3a}. & 0<x<3a\\ 0, & otherwise\end{cases}$ $\quad|\psi\rangle=\begin{cases}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}. & 0<x<a\\ 0, & otherwise\end{cases}$

$$P\left(\frac{\pi^2\hbar^2}{2m(3a)^2}\right)=\frac{|\langle\phi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle}=\int_0^a\sqrt{\frac{2}{3a}}\sin\frac{\pi x}{3a}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}dx=\frac{81}{64\pi^2}$$

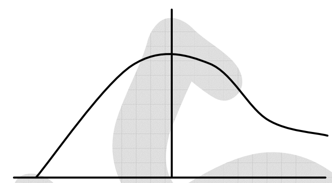
Q13. Consider a function $f(x) = P_k(x)e^{-(x^4+2x^2)}$ in the domain $x \in (-\infty, \infty)$, where P_k is any polynomial of degree k. What is the maximum possible number of extrema of the function?

(a) $k+3$ (b) $k-3$ (c) $k+2$ (d) $k+1$

Ans. : (a)

Solution: $f(x) = p_k(x)e^{-(x^4+2x^2)}$

Let $k=0$, $f(x) = \rho_0(x)e^{-(x^4+2x^2)}$



Number of extrema

$P_0(x) = 1, k = 0$

Number of extrema $=1$

$k+1 = 0+1 = 1$

Q14. The energy spectrum of a particle consists of four states with energies $0, \in, 2\in, 3\in$. Let $Z_B(T), Z_F(T)$ and $Z_C(T)$ denote the canonical partition functions for four non-interacting particles at temperature T. The subscripts B, F and C corresponds to bosons, fermions and distinguishable classical particles, respectively. Let $y = \exp\left(-\frac{\in}{k_B T}\right)$. Which one of the following statements is true about $Z_B(T), Z_F(T)$ and $Z_C(T)$?

(a) They are polynomials in y of degree $12, 6$ and 12, respectively.

(b) They are polynomials in y of degree $16, 10$ and 16, respectively

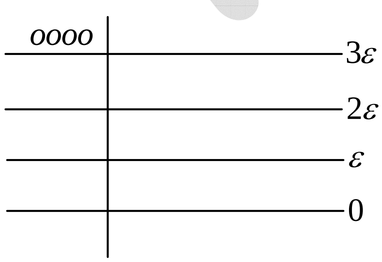
(c) They are polynomials in y of degree $9, 6$ and 12, respectively.

(d) They are polynomials in y of degree $12, 10$ and 16, respectively.

Ans. : (a)

Solution:

Bose



$y = e^{-\varepsilon/k_B T}$

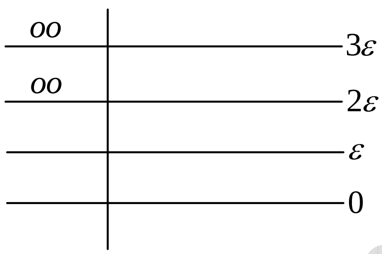
Number of particle $N = 4$ $\qquad \omega = \prod_i \frac{(n_i + g_i)!}{n_i!\, g_i!}$

Maximum energy $= 12\varepsilon$

$$Z_B = e^{-12\varepsilon / k_B T} + \cdots$$

$$= y^{12} + \cdots \qquad \text{degree } = 12$$

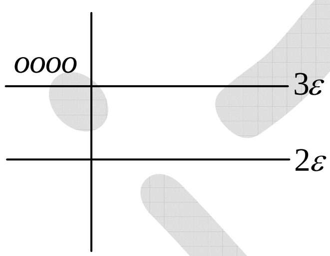
Fermions



Maximum energy $e^{-6\varepsilon / k_B T - 4\varepsilon / k_B T} + e + \cdots\cdots$

$Z_F = y^6 + \cdots$ degree $= 6$

Classical



$Z_C = y^{12} + \ldots..$

Q15. Consider a quantum particle of mass m and a charge e moving in a two dimensional potential given as:

$$V(x, y) = \frac{k}{2}(x - y)^2 + k(x + y)^2$$

The particle is also subject to an external electric field $\vec{E} = \lambda(\hat{i} - \hat{j})$, where λ is a constant $\hat{i}$ and $\hat{j}$ corresponds to unit vectors along x and y directions, respectively. Let E_1 and E_0 be the energies of the first excited state and ground state, respectively. What is the value of $E_1 - E_0$?

(a) $\hbar\sqrt{\frac{2k}{m}}$ (b) $\hbar\sqrt{\frac{2k}{m}} + e\lambda^2$ (c) $3\hbar\sqrt{\frac{2k}{m}}$ (d) $3\hbar\sqrt{\frac{2k}{m}} + e\lambda^2$

Ans. : (a)

Solution: For constant electric field we know there is not any change in frequency and and energy of each level is changed by constant value.

The total potential is

$$V(x,y)=\frac{k}{2}(x-y)^2+k(x+y)^2-\lambda x+\lambda y \Rightarrow V(x,y)=\frac{3}{2}kx^2+\frac{3}{2}ky^2+kxy-\lambda x+\lambda y$$

$$T=\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad V=\begin{pmatrix} 3k & k \\ k & 3k \end{pmatrix}$$

Secular equation is given by $\left|V-\omega^2 m\right|=0 \Rightarrow \left(3k-\omega^2 m\right)^2-k^2=0 \Rightarrow \omega_x=\sqrt{\frac{4k}{m}}, \omega_y=\sqrt{\frac{2k}{m}}$

The equivalent quantum mechanical energy is $E_{n_x,n_y}=\left(n_x+\frac{1}{2}\right)\hbar\omega_x+\left(n_y+\frac{1}{2}\right)\hbar\omega_y+V_0$

Where $n_x=0,1,2,3\ldots$ and $n_y=0,1,2,3\ldots$

The ground state energy $E_0=E_{0.0}=\frac{\hbar}{2}\sqrt{\frac{4k}{m}}+\frac{\hbar}{2}\sqrt{\frac{2k}{m}}$

The first excited state energy $E_1=E_{0.1}=\frac{\hbar}{2}\sqrt{\frac{4k}{m}}+\frac{3\hbar}{2}\sqrt{\frac{2k}{m}}$

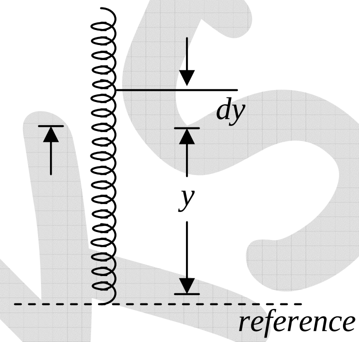
$$E_1-E_0=\hbar\sqrt{\frac{2k}{m}}$$

PART C: 3-Mark Numerical Questions

Q1. A thin uniform steel chain is $10\,m$ long with a linear mass density of $2\,kg\,m^{-1}$. The chain hangs vertically with one end attached to a horizontal axle, having a negligibly small radius compared to its length. What is the work done (in $N-m$) to slowly wind up the chain on to the axle? The acceleration due to gravity is $g = 9.81\,ms^{-1}$.

Ans. : 981

Solution: $l = 10\,m$



Mass to be pulled

Mass of small elementary $\frac{m}{l}\times dy$

PE of mass $= -\frac{m}{l}\times dy\times y\times g$

So work required in pulling

$$W = -\int_l dU = -\int_0 -\frac{m}{l}\, y\,dy\times g$$

$$= \frac{m}{l}\times\frac{l_2}{2}\times g = \frac{mgl}{2} = \frac{2\times 10\times 9.81\times 10}{2} = 981\,J$$

Q2. A one-dimensional harmonic oscillator is in the state

$$|\psi\rangle = \sum_{n=0}^{\infty}\frac{1}{\sqrt{n!}}|n\rangle$$

where $|n\rangle$ is the normalized energy eigenstate with eigenvalue $\left(n+\frac{1}{2}\right)\hbar\omega$. Let the expectation value of the Hamiltonian in the state $|\psi\rangle$ be expressed as $\frac{1}{2}\alpha\hbar\omega$. What is the value of α ?

Ans. : 3

Solution: $\langle H\rangle = \sum_{n=0}^{\infty}\frac{\left(n+\frac{1}{2}\right)\hbar\omega}{\lfloor n} = \frac{1}{2}\hbar\omega + \hbar\omega\sum_{n=1}^{\infty}\frac{n}{\lfloor n} = \left[\frac{1}{2}+e\right]\hbar\omega = 3.2\hbar\omega$

Q3. The Euler polynomials are defined by

$$\frac{2e^{xs}}{e^x+1}=\sum_{n=0}^{\infty}E_n(s)\frac{x^n}{n!}$$

What is the value of $E_5(2)+E_5(3)$?

Ans. : 64

Solution: $\frac{2e^{xs}}{e^x+1}=\sum_{n=0}^{\infty}E_n(s)\frac{x^n}{n!}$

$E_n(x+1)+E_n(x)=2x^n$

$E_5(x+1)+E_5(x)=2x^5$

$x=2=2\times 2^5=64$

Q4. What is the angle (in degrees) between the surfaces $y^2+z^2=2$ and $y^2-x^2=0$ at the point $(1,-1,1)$

Ans. : 60

Solution: The equations of two surfaces are

$$f(x,y,z)=2 \text{ and } g(x,y,z)=0$$

where $f(x.y,z)=y^2+z^2$ and $g(x,y,z)=y^2=x^2$

The normal to the first surfaces is

$$\overrightarrow{\nabla f}=\frac{\partial f}{\partial x}\hat{i}+\frac{\partial f}{\partial y}\hat{j}+\frac{\partial f}{\partial z}\hat{k}\Rightarrow\overrightarrow{\nabla f}=2y\hat{j}+2z\hat{k}$$

$$\overrightarrow{\nabla g}=\frac{\partial g}{\partial x}\hat{i}+\frac{\partial g}{\partial y}+\hat{j}+\frac{\partial g}{\partial z}\hat{k}\Rightarrow\overrightarrow{\nabla g}=-2x\hat{i}+2y\hat{j}$$

At point $(1,-1,1)$, $\overrightarrow{\nabla f}=-2\hat{j}+2\hat{k}$ and $\overrightarrow{\nabla g}=-2\hat{i}-\hat{j}$

Hence the angle between the two surfaces is

$$\theta=\cos^{-1}\frac{\overrightarrow{\nabla f}\cdot\overrightarrow{\nabla g}}{\left|\overrightarrow{\nabla f}\right|\left|\overrightarrow{\nabla g}\right|}=\cos^{-1}\frac{\left(-2\hat{j}+2\hat{k}\right)\cdot\left(-2\hat{i}-2\hat{j}\right)}{\sqrt{8}\sqrt{8}}$$

or $$\theta=\cos^{-1}\frac{4}{8}=\cos^{-1/2}=60^0$$

Q5. Consider a system of 15 non-interacting spin-polarized electrons. They are trapped in a two dimensional isotropic harmonic oscillator potential $V(x,y)=\frac{1}{2}m\omega^2\left(x^2+y^2\right)$. The angular frequency ω is such that $\hbar\omega=1$ in some chosen unit. What is the ground state energy of the system in the same units?

Ans. : 55

Solution: Non-interacting spin-polarized electrons means direction of spin is fixed

$1\times\hbar\omega+2\times2\hbar\omega+3\times3\hbar\omega+4\times4\hbar\omega+5\times5\hbar\omega=55\hbar\omega$

Q6. Consider the motion of a particle in two dimensions given by the Lagrangian

$$L=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{\lambda}{4}(x+y)^2$$

where $\lambda>0$. The initial conditions are given as $y(0)=0, x(0)=42$ meters, $\dot{x}(0)=\dot{y}(0)=0$. What is the value of $x(t)-y(t)$ at $t=25$ seconds in meters?

Ans. : 42

Solution: $L=\frac{m}{2}\left(\dot{x}^2+\dot{y}^2\right)-\frac{\lambda}{4}(x+y)^2$

The equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\left(\frac{\partial L}{\partial x}\right)=0\Rightarrow m\ddot{x}+\frac{\lambda}{2}x+\frac{\lambda}{2}y=0 \quad \text{....(1)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)-\left(\frac{\partial L}{\partial y}\right)=0\Rightarrow m\ddot{y}+\frac{\lambda}{2}y+\frac{\lambda}{2}x=0 \quad \text{....(2)}$$

Subtracting equation (2) from (1) gives $m(\ddot{x}-\ddot{y})=0\Rightarrow\ddot{x}-\ddot{y}=0$

Integrating both sides with t gives

$\dot{x}-\dot{y}=c_1$

From the equation $\dot{x}(0)=\dot{y}(0)=0$, there $c_1=0$

Hence, $\dot{x}-\dot{y}=0$ ....(3)

Integrating both sides of this equation with t gives

$x-y=c_2$

Putting $x(0)=42, y(0)=0$ gives

$$42-0=c_2 \Rightarrow 42$$

Therefore, $x-y=42$

The value of $x-y$ is independent of t.

Therefore, at $t=25s$

$$x(t)-y(t)=42$$

Q7. A diatomic ideal gas at room temperature, is expanded at a constant pressure P_0. If the heat absorbed by the gas is $Q=14$ Joules, what is the maximum work in Joules that can be extracted from the system?

Ans. : 4

Solution: Diatomic gas has $C_v=\frac{5}{2}R,\ C_p=\frac{7}{2}R$

$$Q=C_p\Delta T \Rightarrow 14=\frac{7}{2}R\Delta T$$

(Constant pressure process)

$$\Rightarrow \Delta T=\frac{14\times 2}{7\times 8.314}=0.481\,^0c \text{ and } \Delta U=C_v\Delta T=\frac{5}{2}R\times\Delta T$$

$$=\frac{5}{2}\times 8.314\times 0.481=9.99\,J \text{ and } W_{\max}=Q-\Delta U$$

$$W_{\max}=14-9.99=4\,J$$

Q8. An optical line of wavelength $5000\,\overset{0}{A}$ in the spectrum of light from a star is found to be red-shifted by an amount of $2\,\overset{0}{A}$. Let v be the velocity at which the star is receding. Ignoring relativistic effects, what is the value of $\frac{c}{v}$?

Ans. : 2500

Solution: $\frac{c}{v}=\frac{\lambda_0}{\Delta\lambda}=\frac{5000}{2}=2500$.

Q9. In the Young's double slit experiment (screen distance $D = 50\,cm$ and $d = 0.1\,cm$), a thin mica sheet of refractive index $n = 1.5$ is introduced in the path of one of the beams. If the central fringe gets shifted by 0.2, what is the thickness (in micrometer) of the mica sheet?

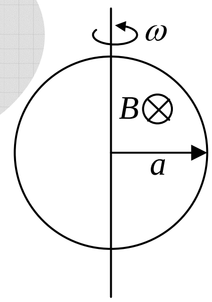
Ans. : 8

Solution: $x_0 = \frac{D}{d}(\mu - 1)t$

$$0.2 = \frac{50}{0.1}(1.5-1)t$$

$$t = \frac{0.2\times 0.1}{50\times 0.5}cm = 8\times 10^{-4}\,cm = 8\,\mu m.$$

Q10. A circular metal loop of radius $a = 1m$ spins with a constant angular velocity $\omega = 20\pi$ rad/s in a magnetic field $B = 3$ Tesla, as shown in the figure. The resistance of the loop is 10 ohms. Let P be the power dissipated in one complete cycle. What is the value of $\frac{P}{\pi^4}$ in Watts?



Ans. : 18

Solution: Magnetic flux through the loop is $\phi_m = \int_S \vec{B}.d\vec{a} = B\times \pi a^2 \times \cos\omega t$

Induced e.m.f $\varepsilon = -\frac{d\phi_m}{dt} = \omega B \times \pi a^2 \times \sin\omega t$.

Power dissipated $p = \frac{\varepsilon^2}{R} = \frac{\omega^2 B^2 \pi^2 a^4 \sin^2 \omega t}{R}$

Power dissipated in one complete cycle $P = \langle p \rangle = \frac{\omega^2 B^2 \pi^2 a^4}{2R}$ $\quad \because \langle \sin^2 \omega t \rangle = \frac{1}{2}$

$$\frac{P}{\pi^4} = \frac{\omega^2 B^2 a^4}{2\pi^2 R} \Rightarrow P = \frac{(20\pi)^2 (3)^2 (1)^4}{2(10)(10)} = 18$$