MATHEMATICAL PHYSICS SOLUTIONS

GATE-2010

Q1. Consider an anti-symmetric tensor P_{ij} with indices i and j running from 1 to 5. The number of independent components of the tensor is

(a) 3 (b) 10 (c) 9 (d) 6

Ans: (b)

Solution: The number of independent components of the tensor

$$= \frac{1}{2}\left(N^2 - N\right) = \frac{1}{2}(25-5) = 10 \qquad (\therefore\ N = 5)$$

Q2. The value of the integral $\oint_C \frac{e^z \sin(z)}{z^2} dz$, where the contour C is the unit circle: $|z-2| = 1$, is

(a) $2\pi i$ (b) $4\pi i$ (c) πi (d) 0

Ans: (d)

Solution: $\because$ $|z-2| = 1 \Rightarrow 1 < z < 3$ i.e. the pole $z = 0$ does not lie inside the contour.

$$\therefore \quad \oint_C \frac{e^z \sin z}{z^2} dz = 2\pi i \times 0 = 0.$$

Q3. The eigenvalues of the matrix $\begin{pmatrix} 2 & 3 & 0 \\ 3 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are

(a) 5, 2, -2 (b) -5, -1, -1 (c) 5, 1, -1 (d) -5, 1, 1

Ans: (c)

Solution: The characteristic equation of the matrix A, $|A - \lambda I| = 0$

$$\Rightarrow |A-\lambda I| = \begin{vmatrix} 2-\lambda & 3 & 0 \\ 3 & 2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 9\right] = 0 \Rightarrow \lambda = 1,\ 2-\lambda = \pm 3$$

$$\Rightarrow \lambda = 5, 1, -1$$

Q4. If $f(x) = \begin{cases} 0 & \text{for } x < 3, \\ x-3 & \text{for } x \geq 3 \end{cases}$ then the Laplace transform of $f(x)$ is

(a) $s^{-2}e^{3s}$ (b) $s^2 e^{3s}$ (c) s^{-2} (d) $s^{-2}e^{-3s}$

Ans: (d)

Solution: $L\{f(x)\}=\int_0^\infty e^{-sx}f(x)dx=\int_0^3 e^{-sx}f(x)dx+\int_3^\infty e^{-sx}f(x)dx=\int_3^\infty (x-3)e^{-sx}dx$

$$L\{f(x)\}=(x-3)\left.\frac{e^{-sx}}{-s}\right|_3^\infty-\int_3^\infty 1\cdot\left(\frac{e^{-sx}}{-s}\right)\ dx=0+\frac{1}{s}\int_3^\infty e^{-sx}\ \ dx=\frac{1}{s}\left[\frac{e^{-sx}}{-s}\right]_3^\infty=s^{-2}e^{-3s}$$

Q5. The solution of the differential equation for $y(t)$: $\frac{d^2y}{dt^2}-y=2\cosh(t)$, subject to the initial conditions $y(0)=0$ and $\left.\frac{dy}{dt}\right|_{t=0}=0$, is

(a) $\frac{1}{2}\cosh(t)+t\sinh(t)$ (b) $-\sinh(t)+t\cosh(t)$

(c) $t\cosh(t)$ (d) $t\sinh(t)$

Ans: (d)

Solution: For C.F $(D^2-1)y=0\Rightarrow m=\pm1\Rightarrow C.F.=C_1e^t+C_2e^{-t}$

$$P.I.=\frac{1}{D^2-1}(2\cosh t)=\frac{1}{D^2-1}2\left(\frac{e^t+e^{-t}}{2}\right)=\frac{1}{D^2-1}(e^t)+\frac{1}{D^2-1}(e^{-t})=\frac{t}{2}e^t+\frac{t}{2}(-e^{-t})$$

$$\Rightarrow y=C_1e^t+C_2e^{-t}+\frac{t}{2}e^t-\frac{t}{2}e^{-t}$$

As, $y(0)=0\Rightarrow C_1+C_2=0$............(1)

$$\frac{dy}{dt}=C_1e^t-C_2e^{-t}+\frac{t}{2}e^t+\frac{1}{2}e^t+\frac{t}{2}e^{-t}-\frac{1}{2}e^{-t}$$

Also, $\left.\frac{dy}{dt}\right|_{t=0}=0\Rightarrow C_1-C_2+0+\frac{1}{2}+0-\frac{1}{2}=0\Rightarrow C_1-C_2=0$...........(2)

From equation (1) and (2),

$$C_1=0, C_2=0.$$

Thus $y=\frac{t}{2}e^t-\frac{t}{2}e^{-t}\Rightarrow y=t\sinh t$

GATE-2011

Q6. Two matrices A and B are said to be similar if $B = P^{-1}AP$ for some invertible matrix P. Which of the following statements is NOT TRUE?

(a) Det A = Det B (b) Trace of A = Trace of B

(c) A and B have the same eigenvectors (d) A and B have the same eigenvalues

Ans: (c)

Solution: If A and B be square matrices of the same type and if P be invertible matrix, then matrices A and $B = P^{-1}AP$ have the same characteristic roots.

Then, $B-\lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A-\lambda I)P$ where I is identity matrix.

$$|B-\lambda I| = |P^{-1}(A-\lambda I)P| = |P^{-1}||A-\lambda I||P| = |A-\lambda I||P^{-1}||P| = |A-\lambda I||PP^{-1}| = |A-\lambda I|$$

Thus, the matrices A and B (= $P^{-1}AP$) have the same characteristic equation and hence same characteristic roots or eigen values. Since, the sum of the eigen values of a matrix and product of eigen values of a matrix is equal to the determinant of matrix, hence third alternative is incorrect.

Q7. If a force $\vec{F}$ is derivable from a potential function $V(r)$, where r is the distance from the origin of the coordinate system, it follows that

(a) $\vec{\nabla}\times\vec{F}=0$ (b) $\vec{\nabla}\cdot\vec{F}=0$ (c) $\vec{\nabla}V=0$ (d) $\nabla^2 V=0$

Ans: (a)

Solution: Since, $\vec{F}$ is derivative of potential $V(r)$ and $\vec{F} = -\vec{\nabla}V(r)$

$$\Rightarrow \vec{\nabla}\times\vec{F} = -\vec{\nabla}\times\left(\vec{\nabla}V\right) = 0.$$

Q8. A 3×3 matrix has elements such that its trace is 11 and its determinant is 36. The eigenvalues of the matrix are all known to be positive integers. The largest eigenvalues of the matrix is

(a) 18 (b) 12 (c) 9 (d) 6

Ans: (d)

Solution: We know that for any matrix

1. The product of eigenvalues is equals to the determinant of that matrix.

2. $\lambda_1+\lambda_2+\lambda_3+\ldots\ldots =$ Trace of matrix

$\lambda_1+\lambda_2+\lambda_3 = 11$ and $\lambda_1\lambda_2\lambda_3 = 36$. Hence, the largest eigen value of the matrix is 6.

Q9. The unit vector normal to the surface $x^2 + y^2 - z = 1$ at the point $P(1, 1, 1)$ is

(a) $\frac{\hat{i}+\hat{j}-\hat{k}}{\sqrt{3}}$ (b) $\frac{2\hat{i}+\hat{j}-\hat{k}}{\sqrt{6}}$ (c) $\frac{\hat{i}+2\hat{j}-\hat{k}}{\sqrt{6}}$ (d) $\frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$

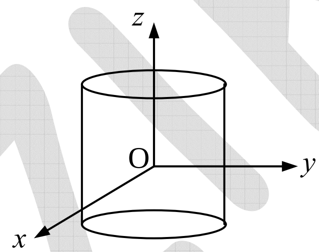
Ans: (d)

Solution: The equation of the system is $f(x,y,z) \equiv (x^2+y^2-z-1) = 0$

$$\vec{\nabla}f = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)(x^2+y^2-z-1) = 2x\hat{i} + 2y\hat{j} - \hat{k}$$

Hence, unit normal vector at (1, 1, 1) $= \frac{\vec{\nabla}f}{|\vec{\nabla}f|} = \frac{2\hat{i}+2\hat{j}-\hat{k}}{3}$.

Q10. Consider a cylinder of height h and radius a, closed at both ends, centered at the origin. Let $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ be the position vector and $\hat{n}$ be a unit vector normal to the surface. The surface integral $\int_S \vec{r}\cdot\hat{n}\,ds$ over the closed surface of the cylinder is



(a) $2\pi a^2(a+h)$ (b) $3\pi a^2 h$ (c) $2\pi a^2 h$ (d) zero

Ans: (b)

Solution: $\oint_S \vec{r}.\hat{n}ds = \int_V (\vec{\nabla}.\vec{r})d\tau = 3\int_V d\tau = 3\pi a^2 h$

Q11. The solutions to the differential equation $\frac{dy}{dx} = -\frac{x}{y+1}$ are a family of

(a) circles with different radii

(b) circles with different centres

(c) straight lines with different slopes

(d) straight lines with different intercepts on the y-axis

Ans: (a)

Solution: $\frac{dy}{dx} = -\frac{x}{y+1} \Rightarrow xdx + ydy + dy = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + y = C_1 \Rightarrow x^2 + y^2 + 2y = 2C_1$

$$\Rightarrow (x-0)^2 + (y+1)^2 = 2C_1 + 1 = C$$

which is a family of circles with different radii.

Q12. Which of the following statements is **TRUE** for the function $f(z) = \dfrac{z\sin z}{(z-\pi)^2}$?

(a) $f(z)$ is analytic everywhere in the complex plane

(b) $f(z)$ has a zero at $z = \pi$

(c) $f(z)$ has a pole of order 2 at $z = \pi$

(d) $f(z)$ has a simple pole at $z = \pi$

Ans: (c)

Solution: $f(z) = \dfrac{z\sin z}{(z-\pi)^2}$ has a pole of order 2 at $z = \pi$

Q13. Consider a counterclockwise circular contour $|z| = 1$ about the origin. Let $f(z) = \dfrac{z\sin z}{(z-\pi)^2}$,

then the integral $\oint f(z)dz$ over this contour is

(a) $-i\pi$ (b) zero (c) $i\pi$ (d) $2i\pi$

Ans: (b)

Solution: Since, pole $z = \pi$ does not lie inside the contour, hence

$$\oint f(z)dz = 0$$

GATE-2012

Q14. Identify the correct statement for the following vectors $\vec{a} = 3\hat{i} + 2\hat{j}$ and $\vec{b} = \hat{i} + 2\hat{j}$

(a) The vectors $\vec{a}$ and $\vec{b}$ are linearly independent

(b) The vectors $\vec{a}$ and $\vec{b}$ are linearly dependent

(c) The vectors $\vec{a}$ and $\vec{b}$ are orthogonal

(d) The vectors $\vec{a}$ and $\vec{b}$ are normalized

Ans: (a)

Solution: If $\vec{a} = 3\hat{i} + 2\hat{j}$, $\vec{b} = \hat{i} + 2\hat{j}$ are linearly dependent, then

$\vec{a}+m\vec{b}=0$, for some values of m but here,

$3+m=0$ and $2+2m=0$, do not have any solution. So, they are linearly independent.

$\vec{a}\cdot\vec{b}\neq 0$ (Not orthogonal); $\vec{a}\times\vec{b}\neq 0$ (Not normalized)

Q15. The number of independent components of the symmetric tensor A_{ij} with indices $i,j=1,2,3$ is

(a) 1 (b) 3 (c) 6 (d) 9

Ans: (c)

Solution: For symmetric tensor, $A_{ij}=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33}\end{bmatrix}$

$\because A_{12}=A_{21},\quad A_{23}=A_{32},\quad A_{13}=A_{31}$, hence there are six independent components.

Q16. The eigenvalues of the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0\end{pmatrix}$ are

(a) 0, 1, 1 (b) $0,-\sqrt{2},\sqrt{2}$

(c) $\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0$ (d) $\sqrt{2},\sqrt{2},0$

Ans: (b)

Solution: $|A-\lambda I|=0\Rightarrow\begin{vmatrix} -\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda\end{vmatrix}=0\Rightarrow-\lambda\left(\lambda^2-1\right)+\lambda=0\Rightarrow\lambda=0,+\sqrt{2},-\sqrt{2}$

GATE-2013

Q17. If $\vec{A}$ and $\vec{B}$ are constant vectors, then $\vec{\nabla}\left(\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right)$ is

(a) $\vec{A}\cdot\vec{B}$ (b) $\vec{A}\times\vec{B}$ (c) $\vec{r}$ (d) zero

Ans: (d)

Solution: Let $\vec{A}=A_0\left(\hat{x}+\hat{y}+\hat{z}\right)$, $\vec{B}=B_0\left(\hat{x}+\hat{y}+\hat{z}\right)$ and $\vec{r}=x\hat{x}+y\hat{y}+z\hat{z}$.

$\vec{B}\times\vec{r}=\hat{x}(z-y)B_0-\hat{y}(z-x)B_0+\hat{z}(y-x)B_0\Rightarrow\vec{\nabla}\left(\vec{A}\cdot\left(\vec{B}\times\vec{r}\right)\right)=0$.

Q18. For the function $f(z)=\frac{16z}{(z+3)(z-1)^2}$, the residue at the pole $z=1$ is (your answer should be an integer) ____________.

Ans: 3

Solution: At $z=1$, pole is of order 2. So, residue is $\frac{1}{\lfloor 2-1}\frac{d^{2-1}}{dz^{2-1}}\left[\frac{(z-1)^2 16z}{(z+3)(z-1)^2}\right]_{z=1}=3$.

Q19. The degenerate eigenvalue of the matrix $\begin{bmatrix}4 & -1 & -1\\ -1 & 4 & -1\\ -1 & -1 & 4\end{bmatrix}$ is (your answer should be an integer) ____________

Ans: 2,5,5

$$\begin{bmatrix}4-\lambda & -1 & -1\\ -1 & 4-\lambda & -1\\ -1 & -1 & 4-\lambda\end{bmatrix}=0\Rightarrow(2-\lambda)\begin{bmatrix}1 & -1 & -1\\ 0 & 5-\lambda & 0\\ 0 & 0 & 5-\lambda\end{bmatrix}=(2-\lambda)(5-\lambda)^2=0\Rightarrow\lambda=2,5,5.$$

Q20. The number of distinct ways of placing four indistinguishable balls into five distinguishable boxes is ____________.

Ans: 120

Solution: $\lfloor 4\times C_4^5$=120 ways

GATE-2014

Q21. The unit vector perpendicular to the surface $x^2+y^2+z^2=3$ at the point (1, 1, 1) is

(a) $\frac{\hat{x}+\hat{y}-\hat{z}}{\sqrt{3}}$ (b) $\frac{\hat{x}-\hat{y}-\hat{z}}{\sqrt{3}}$ (c) $\frac{\hat{x}-\hat{y}+\hat{z}}{\sqrt{3}}$ (d) $\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}$

Ans: (d)

Solution: Let, $f=x^2+y^2+z^2-3=0 \Rightarrow \vec{\nabla}f=2x\hat{x}+2y\hat{y}+2z\hat{z}$

$$\Rightarrow \hat{n}=\frac{\vec{\nabla}f}{|\vec{\nabla}f|}\,at\,(1,1,1)=\frac{2\hat{x}+2\hat{y}+2\hat{z}}{\sqrt{12}}=\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}$$

Q22. The matrix

$$A=\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 1+i\\ 1-i & -1\end{bmatrix} \text{ is}$$

(a) orthogonal (b) symmetric (c) anti-symmetric (d) Unitary

Ans. : (d)

Solution: Unitary $A^{\dagger}A=I$

Q23. The value of the integral

$$\oint_C \frac{z^2}{e^z+1}dz$$

where C is the circle $|z|=4$, is

(a) $2\pi i$ (b) $2\pi^2 i$ (c) $4\pi^3 i$ (d) $4\pi^2 i$

Ans. : (c)

Solution: Pole $e^z=-1 \Rightarrow e^z=e^{i(2m+1)\pi}$ where $m=0,1,2,3.....$

For $z=i\pi$, $\text{Res}=\lim\limits_{z=i\pi}\frac{\phi(z)}{\phi'(z)}=-\frac{\pi^2}{e^{i\pi}}=\pi^2$

Similarly, for $z=-i\pi, \text{Res}=\pi^2$

$\therefore I=2\pi i\left(\pi^2+\pi^2\right)=4\pi^3 i$

Q24. The solution of the differential equation $\frac{d^2y}{dt^2}-y=0$, subject to the boundary conditions $y(0)=1$ and $y(\infty)=0$ is

(a) $\cos t+\sin t$ (b) $\cosh t+\sinh t$

(c) $\cos t-\sin t$ (d) $\cosh t-\sinh t$

Ans: (d)

Soluiton:

$D^2-1=0 \Rightarrow D=\pm 1 \Rightarrow y(t)=c_1e^t+c_2e^{-t}$

Applying boundary condition,

$y(0)=1 \Rightarrow 1=c_1+c_2$ and $y(\infty)=0 \Rightarrow 0=c_1e^{\infty}+c_2e^{-\infty} \Rightarrow c_1=0, c_2=1$

$\Rightarrow y(t)=e^{-t} \Rightarrow y(t)=\cosh t-\sinh t$

GATE-2015

Q25. Consider a complex function $f(z)=\dfrac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$. Which one of the following statements is correct?

(a) $f(z)$ has simple poles at $z=0$ and $z=-\frac{1}{2}$

(b) $f(z)$ has second order pole at $z=-\frac{1}{2}$

(c) $f(z)$ has infinite number of second order poles

(d) $f(z)$ has all simple poles

Ans.: (a)

Solution: $f(z)=\dfrac{1}{z\left(z+\frac{1}{2}\right)\cos(z\pi)}$

For n^{th} order pole, $\text{Res.}=\lim\limits_{z\to a}(z-a)^n f(z)=\text{finite}$

At $z=0$, $\lim\limits_{z\to 0} zf(z)=finite \Rightarrow z=0$ is a simple pole.

At $z=-\frac{1}{2}$, $\lim\limits_{z\to -\frac{1}{2}}\dfrac{\left(z+\frac{1}{2}\right)^2}{z\left(z+\frac{1}{2}\right)\cos z\pi}=\lim\limits_{z\to -\frac{1}{2}}\dfrac{\left(z+\frac{1}{2}\right)}{z\cos z\pi}=\lim\limits_{z\to -\frac{1}{2}}\dfrac{1}{1.\cos z\pi+z.\pi(-\sin z\pi)}$

$=\lim\limits_{z\to -\frac{1}{2}}\dfrac{1}{\cos z\pi-z\pi\sin z\pi}=\dfrac{1}{-\frac{\pi}{2}}=-\dfrac{2}{\pi}=finite$

$\Rightarrow f(z)$ has second order pole at $z=-\frac{1}{2}$

Q26. The value of $\int_0^3 t^2\delta(3t-6)dt$ is_______________ (upto one decimal place)

Ans.: 1.33

Solution: $\int_0^3 t^2\delta(3t-6)\,dt=\int_0^3 t^2\delta[3(t-2)]\,dt=\frac{1}{3}\int_0^3 t^2\delta(t-2)\,dt=\frac{4}{3}$

Q27. If $f(x)=e^{-x^2}$ and $g(x)=|x|e^{-x^2}$, then

(a) f and g are differentiable everywhere

(b) f is differentiable everywhere but g is not

(c) g is differentiable everywhere but f is not

(d) g is discontinuous at $x=0$

Ans. (b)

Solution: $f(x)=e^{-x^2}$ is differentiable but $g(x)=|x|e^{-x^2}$ is not differentiable.

$$g(x)=\begin{cases}-xe^{-x^2}; & x<0\\ xe^{-x^2}; & x>0\end{cases}$$

Left hand Limit $\lim\limits_{h\to 0} g(x-h)=-(x-h)e^{-(x-h)^2}$

Right hand Limit $\lim\limits_{h\to 0} g(x+h)=(x+h)e^{-(x+h)^2}$

$\Rightarrow \lim\limits_{h\to 0} g(x-h)\neq \lim\limits_{h\to 0} g(x+h)$

Q28. Consider $w=f(z)=u(x,y)+iv(x,y)$ to be an analytic function in a domain D. Which one of the following options is NOT correct?

(a) $u(x,y)$ satisfies Laplace equation in D

(b) $v(x,y)$ satisfies Laplace equation in D

(c) $\int_{z_1}^{z_2} f(z)dz$ is dependent on the choice of the contour between z_1 and z_2 in D

(d) $f(z)$ can be Taylor expended in D

Ans.: (c)

Solution: $w=f(z)=u(x,y)+iv(x,y)$ to be an analytic function in a domain D, $\int_{z_1}^{z_2} f(z)dz$ is independent of the choice of the contour between z_1 and z_2 in D.

Q29. The Heaviside function is defined as $H(t)=\begin{cases}+1, & \text{for} \quad t>0 \\ -1, & \text{for} \quad t<0\end{cases}$ and its Fourier transform is given by $-2i/\omega$. The Fourier transform of $\frac{1}{2}[H(t+1/2)-H(t-1/2)]$ is

(a) $\dfrac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$ (b) $\dfrac{\cos\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$ (c) $\sin\left(\frac{\omega}{2}\right)$ (d) 0

Ans.: (a)

Solution: $H(f)=\int_{-\infty}^{\infty} H(t)e^{-i2\pi ft}dt$, for a function $H(t)$, $H(f)=-\dfrac{2i}{\omega}$

For $H(t-t_0)$, Fourier Transform is $e^{-i2\pi ft_0}H(f)$

Shifting Theorem

For $\frac{1}{2}\left[H\left(t+\frac{1}{2}\right)-H\left(t-\frac{1}{2}\right)\right]=\frac{1}{2}\left[e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}\right]\frac{-2i}{\omega}=\frac{1}{2i}\left[e^{i\frac{\omega}{2}}-e^{-i\frac{\omega}{2}}\right]\frac{-2i}{\omega}\times i$

The Fourier transform of $\frac{1}{2}\left[H(t+1/2)-H(t-1/2)\right]=\dfrac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}}$.

Q30. A function $y(z)$ satisfies the ordinary differential equation $y''+\frac{1}{z}y'-\frac{m^2}{z^2}y=0,$ where $m=0,\ 1,\ 2,\ 3,.....$ Consider the four statements P, Q, R, S as given below.

P: z^m and z^{-m} are linearly independent solutions for all values of m

Q: z^m and z^{-m} are linearly independent solutions for all values of $m>0$

R: $\ln z$ and 1 are linearly independent solutions for $m=0$

S: z^m and $\ln z$ are linearly independent solutions for all values of m

The correct option for the combination of valid statements is

(a) P, R and S only (b) P and R only (c) Q and R only (d) R and S only

Ans.: (c)

Solution: $y''+\frac{1}{z}y'-\frac{m^2}{z^2}y=0 \Rightarrow z^2y''+zy'-m^2y=0$, $m=0,1,2,3,...,$ $z=e^x$, $D=\dfrac{d}{dx}$

If $m=0$; $z^2y''+zy'=0$, $\left[D\left(D-1\right)+D\right]y=0 \Rightarrow \left[D^2-D+D\right]y=0$

$D^2y=0 \Rightarrow y=c_1+c_2x \Rightarrow y=c_1+c_2\ln z$ (R is correct)

And if $m\neq 0,\ m>0$, then $m\neq 0$, then $\left(D^2-m^2\right)y=0 \Rightarrow D=\pm m$

$y=c_1e^{mx}+c_2e^{-mx}=c_1e^{m\log z}+c_2e^{-m\log z}=c_1z^m+c_2z^{-m}$

or if $m\neq 0,\ m>0$, then

$y=c_1\cosh\left(m\log\left(z\right)\right)+ic_2\sinh\left(m\log\left(x\right)\right),\quad m>0$

GATE-2016

Q31. Consider the linear differential equation $\frac{dy}{dx}=xy$. If $y=2$ at $x=0$, then the value of y at $x=2$ is given by

(a) e^{-2} (b) $2e^{-2}$ (c) e^2 (d) $2e^2$

Ans.: (d)

Solution: $\frac{dy}{dx}=xy \Rightarrow \frac{1}{y}dy=xdx \Rightarrow \ln y=\frac{x^2}{2}+\ln c \Rightarrow y=ce^{x^2/2}$

If $y=2$ at $x=0 \Rightarrow c=2 \Rightarrow y=2e^{x^2/2}$.

The value of y at $x=2$ is given by $y=2e^2$

Q32. Which of the following is an analytic function of z everywhere in the complex plane?

(a) z^2 (b) $\left(z^*\right)^2$ (c) $|z|^2$ (d) $\sqrt{z}$

Ans.: (a)

Solution: $z^2=\left(x+iy\right)^2=x^2-y^2+i\left(2xy\right) \Rightarrow u=x^2-y^2$ and $v=2xy$

Cauchy Riemann equations $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=2x,\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}=2y$ satisfies.

Q33. The direction of $\vec{\nabla}f$ for a scalar field $f\left(x,y,z\right)=\frac{1}{2}x^2-xy+\frac{1}{2}z^2$ at the point $P(1,1,2)$ is

(a) $\frac{\left(-\hat{j}-2\hat{k}\right)}{\sqrt{5}}$ (b) $\frac{\left(-\hat{j}+2\hat{k}\right)}{\sqrt{5}}$ (c) $\frac{\left(\hat{j}-2\hat{k}\right)}{\sqrt{5}}$ (d) $\frac{\left(\hat{j}+2\hat{k}\right)}{\sqrt{5}}$

Ans.: (b)

Solution: $\vec{\nabla} f=(x-y)\hat{i}-x\hat{j}+z\hat{k} \Rightarrow \hat{n}=\left(\frac{\vec{\nabla} f}{|\vec{\nabla} f|}\right)_{1,1,2}=\frac{-\hat{j}+2\hat{k}}{\sqrt{5}}$

Q34. A periodic function $f(x)$ of period 2π is defined in the interval $(-\pi<x<\pi)$

$$f(x)=\begin{cases}-1, & -\pi<x<0 \\ 1, & 0<x<\pi\end{cases}$$

The appropriate Fourier series expansion for $f(x)$ is

(a) $f(x)=\left(\frac{4}{\pi}\right)[\sin x+(\sin 3x)/3+(\sin 5x)/5+\ldots]$

(b) $f(x)=\left(\frac{4}{\pi}\right)[\sin x-(\sin 3x)/3+(\sin 5x)/5-..]$

(c) $f(x)=\left(\frac{4}{\pi}\right)[\cos x+(\cos 3x)/3+(\cos 5x)/5+\ldots]$

(d) $f(x)=\left(\frac{4}{\pi}\right)[\cos x-(\cos 3x)/3+(\cos 5x)/5-\ldots]$

Ans.: (a)

Solution: $f(x)=\begin{cases}-1, & -\pi<x<0 \\ 1, & 0<x<\pi\end{cases}$

Let $f(x)=a_0+\sum_{n=1}^{\infty}(a_n \cos nx+b_n \sin nx)$

$\because \quad a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$

$\Rightarrow a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx=\frac{1}{2\pi}\left[\int_{-\pi}^{0}(-1)\,dx+\int_{0}^{\pi}(1)\,dx\right]=\frac{1}{2\pi}\left[[-x]_{-\pi}^{0}+[x]_{0}^{\pi}\right]=0$

This can also be seen without integration, since the area under the curve of $f(x)$ between $-\pi$ to π is zero.

$\because \quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$\Rightarrow a_n=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\cos nx dx+\int_{0}^{\pi}(1)\cos nx dx\right]=\frac{1}{\pi}\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{0}+\left\{\frac{\sin nx}{n}\right\}_{0}^{\pi}\right]=0$

$\because \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$\Rightarrow b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\sin nx dx + \int_{0}^{\pi}(1)\sin nx dx\right]$

$\Rightarrow b_n = \frac{1}{\pi}\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{0} - \left\{\frac{\cos nx}{n}\right\}_{0}^{\pi}\right] = \frac{1}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right] = \frac{1}{\pi}\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$

$\Rightarrow b_n = \begin{cases} 0; & n = even \\ \frac{4}{n\pi}; & n = odd \end{cases}$

Thus, Fourier series is $f(x) = \frac{4}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + ...\right]$

GATE-2017

Q35. The contour integral $\oint \frac{dz}{1+z^2}$ evaluated along a contour going from $-\infty$ to $+\infty$ along the real axis and closed in the lower half-plane circle is equal to………….. (up to two decimal places).

Ans. : π

Solution: $\oint_C \frac{1}{1+z^2}dz = \int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx + \oint_C \frac{1}{1+z^2}dz$

Poles, $1+z^2 = 0 \Rightarrow z = \pm i$, $z = -i$ is inside C

$\therefore$ $\text{Res}(z=-i) = \lim_{z\to -i}(z+i)\frac{1}{(z-i)(z+i)} = \frac{1}{-i-i} = \frac{1}{-2i}$

$\therefore \int_{-\infty}^{+\infty}\frac{1}{1+x^2}dx = -\frac{1}{2i}\times -2\pi i = \pi$

(Since, here we use lower half plane i.e., we traversed in clockwise direction, hence we have to take $-2\pi i$)

Q36. The coefficient of e^{ikx} in the Fourier expansion of $u(x) = A\sin^2(\alpha x)$ for $k = -2\alpha$ is

(a) $\frac{A}{4}$ (b) $\frac{-A}{4}$ (c) $\frac{A}{2}$ (d) $\frac{-A}{2}$

Ans.: (b)

Solution: Since, $\sin(\alpha x)=\dfrac{e^{i\alpha x}-e^{-i\alpha x}}{2i}\Rightarrow \sin^2(\alpha x)=\dfrac{e^{i2\alpha x}-2+e^{-2i\alpha x}}{(-4)}$

Since, $2\alpha=-k$, hence $\sin^2(\alpha x)=\dfrac{e^{-ikx}-2+e^{ikx}}{(-4)}$

Hence, $c_k=\dfrac{A}{2\pi}\int_{-\pi}^{\pi}\sin^2(\alpha x)dx=-\dfrac{A}{8\pi}\left[\int_{-\pi}^{\pi}e^{-ikx}e^{-ikx}dx-2\int_{-\pi}^{\pi}e^{-ikx}dx+\int_{-\pi}^{\pi}e^{-ikx}e^{ikx}dx\right]$

$$=-\frac{A}{8\pi}\left[\int_{-\pi}^{\pi}e^{-2ikx}dx-2\int_{-\pi}^{\pi}e^{-ikx}dx+\int_{-\pi}^{\pi}dx\right]$$

The first two integrals are zero and the third integral has the value 2π.

Thus,

$$c_k=-\frac{A}{8\pi}(2\pi)=-\frac{A}{4}$$

Q37. The imaginary part of an analytic complex function is $v(x,y)=2xy+3y$. The real part of the function is zero at the origin. The value of the real part of the function at $1+i$ is .... (up to two decimal places)

Ans. : 3

Solution: The imaginary part of the given analytic function is $v(x,y)=2xy+3y$. From the Cauchy – Riemann condition

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=2x+3$$

Integrating partially gives

$$u(x,y)=x^2+3x+g(y)$$

From the second Cauchy – Riemann condition

$\dfrac{\partial u}{\partial y}=-\dfrac{\partial v}{\partial x}$, we obtain $\dfrac{\partial u}{\partial y}=-2y,\mu(x,y)=-y^2+g(x)$

$$\frac{dg(y)}{dy}=-2y\Rightarrow g(y)=-y^2+c$$

Hence, $u(x,y)=x^2+3x-y^2+c$

Since, the real part of the analytic function is zero at the origin.

Hence, $0 = 0 + 0 - 0 + c \Rightarrow c = 0$

Thus, $u(x, y) = x^2 + 3x - y^2$

$\therefore f(z) = (x^2 + 3x - y^2) + i(2xy + 3y)$

Thus, the value of real part when

$z = 1 + i$, i.e. $x = 1$ and $y = 1$ is $u(x, y) = (1)^2 + 3(1) - 1 = 3$.

Q38. Let X be a column vector of dimension $n > 1$ with at least one non-zero entry. The number of non-zero eigenvalues of the matrix $M = XX^T$ is

(a) 0 (b) n (c) 1 (d) $n-1$

Ans. : (c)

Solution: Let $X = \begin{bmatrix} 0 \\ 0 \\ a \\ 0 \\ 0 \\ 0 \end{bmatrix}$, then $X^T = [0 \;\; 0 \;\; a \ldots \;\; 0]$

Here, X is an $n \times 1$ column vector with the entry in the ith row equal to a. X^T is a row vector having entry in the ith column equal to a. Then, XX^T is an $n \times 1$ matrix having the entry in the ith row and ith column equal to a^2.

Hence,

$$XX^T = \begin{array}{c c} \begin{bmatrix} 0 \; 0 \; 0 \ldots 0 \ldots 0 \; 0 \\ 0 \; 0 \; 0 \ldots 0 \ldots 0 \; 0 \\ 0 \; 0 \; 0 \ldots 0 \ldots 0 \; 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0 \; 0 \; 0 \ldots 0 \ldots 0 \; 0 \end{bmatrix} & \begin{array}{l} \\ \\ \rightarrow i\text{th row} \\ \\ \\ \\ \end{array} \\ \downarrow & \\ i\text{th row} & \end{array}$$

Since this matrix is diagonal, its eigenvalues are $a^2, 0, 0.....0$. Hence, the number of non zero eigenvalues of the matrix XX^T is 1.

Q39. Consider the differential equation $\frac{dy}{dx} + y\tan(x) = \cos(x)$. If $y(0) = 0, y\left(\frac{\pi}{3}\right)$ is ………….. (up to two decimal places)

Ans.: 0.52

Solution: The given differential equation is a linear differential equation of the form

$$\frac{dy}{dx}+p(x)y=\cos x$$

Integrating factor $=e^{\int p(x)dx}$

Thus integrating factor $=e^{\int \tan x\,dx}$

$$\Rightarrow I\cdot F=e^{\ln\sec x}=\sec x$$

Thus the general solution of the given differential equation is

$$y\cdot\sec x=\int \sec x\cdot\cos x dx+c$$

$$\Rightarrow\ y\sec x=x+c \qquad \text{-(i)}$$

It is given that $y(0)=0\Rightarrow 0\cdot\sec 0=0+c\Rightarrow c=0$

Thus the solution satisfying the given condition is

$$y\sec x=x\Rightarrow y=\frac{x}{\sec x}$$

Thus the value of $y\left(\frac{\pi}{3}\right)$ is

$$y=\frac{\pi/3}{\sec\pi/3}=\frac{\pi/3}{2}=\frac{\pi}{6}=0\cdot 52$$

GATE-2018

Q40. The eigenvalues of a Hermitian matrix are all

(a) real (b) imaginary (c) of modulus one (d) real and positive

Ans. : (a)

Solution: Eigenvalue of Hermitian matrix must be real.

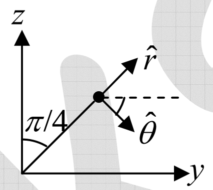
Q41. In spherical polar coordinates (r,θ,ϕ), the unit vector $\hat{\theta}$ at $(10,\pi/4,\ \pi/2)$ is

(a) $\hat{k}$ (b) $\frac{1}{\sqrt{2}}\left(\hat{j}+\hat{k}\right)$ (c) $\frac{1}{\sqrt{2}}\left(-\hat{j}+\hat{k}\right)$ (d) $\frac{1}{\sqrt{2}}\left(\hat{j}-\hat{k}\right)$

Ans. : (d)

Solution: $\hat{\theta}=\cos 45^0\,\hat{j}-\sin 45^0\,\hat{k}$

$$\Rightarrow \hat{\theta}=\frac{1}{\sqrt{2}}\left(\hat{j}-\hat{k}\right)$$



Q42. The scale factors corresponding to the covariant metric tensor g_{ij} in spherical polar coordinates are

(a) $1, r^2, r^2\sin^2\theta$ (b) $1, r^2, \sin^2\theta$ (c) 1,1,1 (d) $1, r, r\sin\theta$

Ans. : (d)

Q43. Given $\vec{V}_1=\hat{i}-\hat{j}$ and $\vec{V}_2=-2\hat{i}+3\hat{j}+2\hat{k}$, which one of the following $\vec{V}_3$ makes $\left(\vec{V}_1,\vec{V}_2,\vec{V}_3\right)$ a complete set for a three dimensional real linear vector space?

(a) $\vec{V}_3=\hat{i}+\hat{j}+4\hat{k}$ (b) $\vec{V}_3=2\hat{i}-\hat{j}+2\hat{k}$

(c) $\vec{V}_3=\hat{i}+2\hat{j}+6\hat{k}$ (d) $\vec{V}_3=2\hat{i}+\hat{j}+4\hat{k}$

Ans. : (d)

Solution: Let A be the matrix formed by taking $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ as column matrix i.e.,

$$A=\left[V_1\quad V_2\quad V_3\right]=\begin{bmatrix}1 & -2 & 2\\ -1 & 3 & 1\\ 0 & 2 & 4\end{bmatrix}\Rightarrow |A|=-2\text{. Here } V_3=\left(2\hat{i}+\hat{j}+4\hat{k}\right)$$

Since, $|A|\neq 0$, hence, $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$ form a three dimensional real vector space.

Hence, option (d) is correct.

Q44. Given

$$\frac{d^2 f(x)}{dx^2} - 2\frac{df(x)}{dx} + f(x) = 0,$$

and boundary conditions $f(0)=1$ and $f(1)=0$, the value of $f(0.5)$ is __________ (up to two decimal places) .

Ans. : 0.81

Solution: $\frac{d^2 f(x)}{dx^2} - 2\frac{df(x)}{dx} + f(x) = 0$

Auxiliary equation is,

$$(m^2 - 2m + 1) = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1,1$$

Hence, the solution is

$$f(x) = (c_1 + c_2 x)e^x$$

using boundary condition,

$$f(0) = c_1 e^0 \Rightarrow c_1 = 1 \qquad \text{(i)}$$

$$f(1) = (c_1 + c_2)e = 0 \qquad \text{(ii)}$$

From (i) and (ii), $c_2 = -1$

Hence, $f(x) = (1-x)e^x \Rightarrow f(0.5) = (1-0.5)e^{0.5} = 0.81$

Q45. The absolute value of the integral

$$\int \frac{5z^3 + 3z^2}{z^2 - 4} dz,$$

over the circle $|z - 1.5| = 1$ in complex plane, is __________ (up to two decimal places).

Ans. : 81.64

Solution: $f(z) = \frac{5z^3 + 3z^2}{(z-2)(z+2)}$

Pole, $z = 2, -2$

$z = -2$ is outside the center

$|-2 - 1.5| > 1$ So, will not be considered

Now, $\operatorname{Re}s(2) = \lim_{z\to 2}(z-2)\frac{(5z^3+3z^2)}{(z-2)(z+2)} = \frac{52^3+32^2}{4} = \frac{40+12}{4} = 13$

$I = 2\pi i \times residue = 2\pi i \times 13 = 26\times 3.14 \Rightarrow I = 81.64$

GATE-2019

Q46. For the differential equation $\frac{d^2y}{dx^2} - n(n+1)\frac{y}{x^2} = 0$, where n is a constant, the product of its two independent solutions is

(a) $\frac{1}{x}$ (b) x (c) x^n (d) $\frac{1}{x^{n+1}}$

Ans. : (b)

Q47. During a rotation, vectors along the axis of rotation remain unchanged. For the rotation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$, the vector along the axis of rotation is

(a) $\frac{1}{3}(2\hat{i} - \hat{j} + 2\hat{k})$ (b) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} - \hat{k})$

(c) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$ (d) $\frac{1}{3}(2\hat{i} + 2\hat{j} - \hat{k})$

Ans. : (b)

Q48. The pole of the function $f(z) = \cot z$ at $z = 0$ is

(a) a removable pole (b) an essential singularity

(c) a simple pole (d) a second order pole

Ans. : (c)

Solution: $f(z) = \cot z$ at $z = 0$

$f(z) = \frac{1}{\tan z}$ $z = 0$ is a simple pole $f(z) = \frac{1}{z}\left[1 - \frac{1}{3}z^2 + ...\right]$

Q49. The value of the integral $\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2+a^2}dx$, where $k > 0$ and $a > 0$, is

(a) $\frac{\pi}{a}e^{-ka}$ (b) $\frac{2\pi}{a}e^{-ka}$ (c) $\frac{\pi}{2a}e^{-ka}$ (d) $\frac{3\pi}{2a}e^{-ka}$

Ans. : (a)

Solution: $\int_{-\infty}^{\infty} \frac{\cos kx}{x^2+a^2} dx$

$$f(z) = \frac{e^{ikx}}{z^2+a^2} = \frac{e^{ikz}}{(z+ia)(z-ia)}$$

$$I = \text{Re}.2\pi i \times \frac{e^{ik(ia)}}{2ia} = \frac{\pi e^{-ka}}{a}$$

Q50. Let θ be a variable in the range $-\pi \le \theta < \pi$. Now consider a function

$$\psi(\theta) = \begin{cases} 1 & \text{for } \frac{-\pi}{2} \le \theta < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

if its Fourier-series is written as $\psi(\theta) = \sum_m^{\infty} = -\infty C_m e^{-im\theta}$, then the value of $|C_3|^2$ (rounded off to three decimal places) is__________