CLASSICAL MECHANICS SOLUTIONS

GATE- 2010

Q1. For the set of all Lorentz transformations with velocities along the x-axis consider the two statements given below:

P: If L is a Lorentz transformation, then, L^{-1} is also a Lorentz transformation.

Q: If L_1 and L_2 are Lorentz transformations, then L_1L_2 is necessarily a Lorentz transformation.

Choose the correct option

(a) P is true and Q is false (b) Both P and Q are true

(c) Both P and Q are false (d) P is false and Q is true

Ans: (b)

Q2. A particle is placed in a region with the potential $V(x)=\frac{1}{2}kx^2-\frac{\lambda}{3}x^3$, where $k,\lambda>0$.

Then,

(a) $x=0$ and $x=\frac{k}{\lambda}$ are points of stable equilibrium

(b) $x=0$ is a point of stable equilibrium and $x=\frac{k}{\lambda}$ is a point of unstable equilibrium

(c) $x=0$ and $x=\frac{k}{\lambda}$ are points of unstable equilibrium

(d) There are no points of stable or unstable equilibrium

Ans: (b)

Solution: $V=\frac{1}{2}kx^2-\frac{\lambda x^3}{3}\Rightarrow\frac{\partial V}{\partial x}=kx-\lambda x^2=0\Rightarrow x=0,x=\frac{k}{\lambda}$.

$\Rightarrow\frac{\partial^2 V}{\partial x^2}=k-2\lambda x$

$\Rightarrow At\ x=0,\frac{\partial^2 V}{\partial x^2}=+ve$ (Stable) and at $x=\frac{k}{\lambda},\frac{\partial^2 V}{\partial x^2}=-ve$ (unstable)

Q3. A π^0 meson at rest decays into two photons, which moves along the x-axis. They are both detected simultaneously after a time, $t = 10\,s$. In an inertial frame moving with a velocity $v = 0.6c$ in the direction of one of the photons, the time interval between the two detections is

(a) $15c$ (b) $0\,s$ (c) $10\,s$ (d) $20\,s$

Ans: (a)

Solution:

$$t_1 = t_0\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} = 10\sqrt{\frac{1+0.6}{1-0.6}} = 10\times 2 = 20\sec,\ t_2 = t_0\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} = 10\sqrt{\frac{1-0.6}{1+0.6}} = 10\times\frac{1}{2} = 5\sec$$

$$\Rightarrow \mathrm{t}_1 - \mathrm{t}_2 = 15\sec$$

Statement for Linked Answer Questions 4 and 5:

The Lagrangian for a simple pendulum is given by $L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

Q4. Hamilton's equations are then given by

(a) $\dot{p}_\theta = -mgl\sin\theta;\quad \dot{\theta} = \frac{p_\theta}{ml^2}$ (b) $\dot{p}_\theta = mgl\sin\theta;\quad \dot{\theta} = \frac{p_\theta}{ml^2}$

(c) $\dot{p}_\theta = -m\ddot{\theta};\quad \dot{\theta} = \frac{p_\theta}{m}$ (d) $\dot{p}_\theta = -\left(\frac{g}{l}\right)\theta;\quad \dot{\theta} = \frac{p_\theta}{ml}$

Ans: (a)

Solution: $H = \frac{P_\theta^2}{2ml^2} + mgl(1-\cos\theta) \Rightarrow \frac{\partial H}{\partial\theta} = -\dot{P}_\theta \Rightarrow \dot{P}_\theta = -mgl\sin\theta;\ \frac{\partial H}{\partial P_\theta} = \dot{\theta} \Rightarrow \dot{\theta} = \frac{P_\theta}{ml^2}$.

Q5. The Poisson bracket between θ and $\dot{\theta}$ is

(a) $\{\theta,\dot{\theta}\} = 1$ (b) $\{\theta,\dot{\theta}\} = \frac{1}{ml^2}$

(c) $\{\theta,\dot{\theta}\} = \frac{1}{m}$ (d) $\{\theta,\dot{\theta}\} = \frac{g}{l}$

Ans: (b)

Solution: $\{\theta,\dot{\theta}\} = \left\{\theta, \frac{P_\theta}{ml^2}\right\}$ where $\dot{\theta} = \frac{P_\theta}{ml^2} \Rightarrow \frac{1}{ml^2}\left(\frac{\partial\theta}{\partial\theta}\frac{\partial\theta}{\partial P_\theta} - \frac{\partial\theta}{\partial P_\theta}\frac{\partial P_\theta}{\partial\theta}\right) = 1\cdot\frac{1}{ml^2} - 0 = \frac{1}{ml^2}$.

GATE- 2011

Q6. A particle is moving under the action of a generalized potential $V(q,\dot{q})=\frac{1+\dot{q}}{q^2}$. The magnitude of the generalized force is

(a) $\frac{2(1+\dot{q})}{q^3}$ (b) $\frac{2(1-\dot{q})}{q^3}$ (c) $\frac{2}{q^3}$ (d) $\frac{\dot{q}}{q^3}$

Ans: (c)

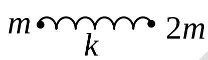
Solution: $\frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}}\right)-\frac{\partial V}{\partial q}=F_q \Rightarrow F_q=\frac{2}{q^3}$

Q7. Two bodies of mass m and $2m$ are connected by a spring constant k. The frequency of the normal mode is

(a) $\sqrt{3k/2m}$ (b) $\sqrt{k/m}$ (c) $\sqrt{2k/3m}$ (d) $\sqrt{k/2m}$

Ans: (a)

Solution:



$$\omega=\sqrt{\frac{k}{\mu}}=\sqrt{\frac{k}{\frac{2m}{3}}}=\sqrt{\frac{3k}{2m}}$$ where reduce mass $\mu=\frac{2mm}{2m+m}=\frac{2m}{3}$.

Q8. Let (p,q) and (P,Q) be two pairs of canonical variables. The transformation

$$Q=q^{\alpha}\cos(\beta p),\ P=q^{\alpha}\sin(\beta p)$$

is canonical for

(a) $\alpha=2,\beta=\frac{1}{2}$ (b) $\alpha=2,\beta=2$ (c) $\alpha=1,\beta=1$ (d) $\alpha=\frac{1}{2},\beta=2$

Ans: (d)

Solution: $\frac{\partial Q}{\partial q}\cdot\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\cdot\frac{\partial P}{\partial q}=1$

$$\Rightarrow \alpha q^{\alpha-1}\cos(\beta p)\times q^{\alpha}\beta\cos(\beta p)-q^{\alpha}\beta(-\sin(\beta p))\times\alpha q^{\alpha-1}\sin(\beta p)=1$$

$$\alpha q^{2\alpha-1}\beta(\cos^2\beta p+\sin^2\beta p)=1\Rightarrow \alpha\beta q^{2\alpha-1}=1 \Rightarrow \alpha=\frac{1}{2},\beta=2.$$

Q9. Two particles each of rest mass m collide head-on and stick together. Before collision, the speed of each mass was 0.6 times the speed of light in free space. The mass of the final entity is

(a) $5m/4$ (b) $2m$ (c) $5m/2$ (d) $25m/8$

Ans: (c)

Solution: From conservation of energy

$$\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}+\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}=m_1c^2 \Rightarrow \frac{2mc^2}{\sqrt{1-\frac{v^2}{c^2}}}=m_1c^2$$

Since $v=0.6c \Rightarrow m_1=5m/2$

GATE- 2012

Q10. In a central force field, the trajectory of a particle of mass m and angular momentum L in plane polar coordinates is given by,

$$\frac{1}{r}=\frac{m}{l^2}(1+\varepsilon\cos\theta)$$

where, ε is the eccentricity of the particle's motion. Which one of the following choice for ε gives rise to a parabolic trajectory?

(a) $\varepsilon=0$ (b) $\varepsilon=1$ (c) $0<\varepsilon<1$ (d) $\varepsilon>1$

Ans: (b)

Solution: $\frac{1}{r}=\frac{m}{l^2}(1+\varepsilon\cos\theta)$

For parabolic trajectory $\varepsilon=1$.

Q11. A particle of unit mass moves along the x-axis under the influence of a potential, $V(x)=x(x-2)^2$. The particle is found to be in stable equilibrium at the point $x=2$. The time period of oscillation of the particle is

(a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

Ans: (b)

Solution: $V(x)=x(x-2)^2 \Rightarrow \frac{\partial V}{\partial x}=(x-2)^2+2x(x-2)=0 \Rightarrow x=2, x=\frac{2}{3}$

$$\frac{\partial^2 V}{\partial x^2} = 2(x-2)+2(x-2)+2x \Rightarrow \left.\frac{\partial^2 V}{\partial x^2}\right|_{x=2} = 2\times 2 = 4$$

$$\Rightarrow \omega = \sqrt{\left.\frac{\partial^2 V}{\partial x^2}\right|_{x=2}} \Rightarrow \omega = \frac{2\pi}{T} = 2 \Rightarrow T = \pi$$

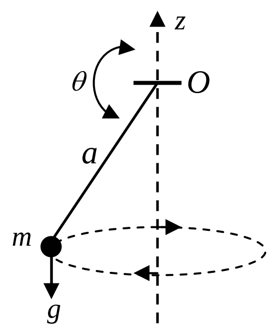
Q12. A rod of proper length l_0 oriented parallel to the x-axis moves with speed $2c/3$ along the x-axis in the S-frame, where c is the speed of light in free space. The observer is also moving along the x-axis with speed $c/2$ with respect to the S-frame. The length of the rod as measured by the observer is

(a) $0.35 l_0$ (b) $0.48 l_0$ (c) $0.87 l_0$ (d) $0.97 l_0$

Ans: (d)

Solution: $l = l_0\sqrt{1-\frac{u^2{}_x}{c^2}} = 0.97\ l_0$

Q13. A particle of mass m is attached to a fixed point O by a weightless inextensible string of length a. It is rotating under the gravity as shown in the figure. The Lagrangian of the particle is



$L(\theta,\phi) = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mga\cos\theta$ where θ and ϕ are the polar angles. The Hamiltonian of the particles is

(a) $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) - mga\cos\theta$ (b) $H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + mga\cos\theta$

(c) $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) - mga\cos\theta$ (d) $H = \frac{1}{2ma^2}\left(p_\theta^2 + p_\phi^2\right) + mga\cos\theta$

Ans: (b)

Solution: $H = P_\theta\dot{\theta} + P_\phi\dot{\phi} - L = P_\theta\dot{\theta} + P_\phi\dot{\phi} - \frac{1}{2}ma^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) + mga\cos\theta$

$$\frac{\partial L}{\partial\dot{\theta}} = P_\theta \Rightarrow ma^2\dot{\theta} = P_\theta \Rightarrow \dot{\theta} = \frac{P_\theta}{ma^2} \text{ and } P_\phi = \frac{\partial L}{\partial\dot{\phi}} = ma^2\sin^2\theta\dot{\phi} \Rightarrow \dot{\phi} = \frac{P_\phi}{ma^2\sin^2\theta}$$

Put the value of $\dot{\theta}$ and $\dot{\phi}$

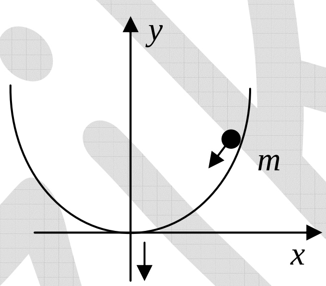
$$H = P_\theta \times \frac{P_\theta}{ma^2} + P_\phi \times \frac{P_\phi}{ma^2 \sin^2\theta} - \frac{1}{2}ma^2\left(\left(\frac{P_\theta}{ma^2}\right)^2 + \sin^2\theta\left(\frac{P_\phi}{ma^2\sin^2\theta}\right)^2\right) + mga\cos\theta$$

$$H = \frac{P_\theta^2}{ma^2} - \frac{P_\theta^2}{2ma^2} + \frac{P_\phi^2}{ma^2\sin^2\theta} - \frac{P_\phi^2}{2ma^2\sin^2\theta} + mga\cos\theta$$

$$H = \frac{1}{2ma^2}\left(P_\theta^2 + \frac{P_\phi^2}{\sin^2\theta}\right) + mga\cos\theta$$

Statement for Linked Answer Questions 14 and 15:

Q14. A particle of mass m slides under the gravity without friction along the parabolic path $y = ax^2$, as shown in the figure. Here a is a constant.



The Lagrangian for this particle is given by

(a) $L = \frac{1}{2}m\dot{x}^2 - mgax^2$

(b) $L = \frac{1}{2}m(1+4a^2x^2)\dot{x}^2 - mgax^2$

(c) $L = \frac{1}{2}m\dot{x}^2 + mgax^2$

(d) $L = \frac{1}{2}m(1+4a^2x^2)\dot{x}^2 + mgax^2$

Ans: (b)

Solution: Equation of constrain is given by $y = ax^2$, $K.E., T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$

$$\dot{y} = 2ax\dot{x} \Rightarrow T = \frac{1}{2}m\left(\dot{x}^2 + 4a^2x^2\dot{x}^2\right) = \frac{1}{2}m\dot{x}^2\left(1 + 4ax^2\right)$$

$$V = mgy = mgax^2.$$

$$\because\ L = T - V \Rightarrow L = \frac{1}{2}m\left(1 + 4a^2x^2\right)\dot{x}^2 - mgax^2$$

Q15. The Lagrange's equation of motion of the particle for above question is given by

(a) $\ddot{x} = 2gax$ (b) $m\left(1+4a^2x^2\right)\ddot{x} = -2mgax - 4ma^2x\dot{x}^2$

(c) $m\left(1+4a^2x^2\right)\ddot{x} = 2mgax + 4ma^2x\dot{x}^2$ (d) $\ddot{x} = -2gax$

Ans: (b)

Solution: $\frac{d}{dt}\left(\frac{dL}{d\dot{x}}\right) = \frac{dL}{dx} \Rightarrow m\ddot{x}\left(1+4a^2x^2\right) = -4ma^2x\dot{x}^2 - 2mgax$

GATE- 2013

Q16. In the most general case, which one of the following quantities is NOT a second order tensor?

(a) Stress (b) Strain

(c) Moment of inertia (d) Pressure

Ans: (b)

Solution: Strain is not a tensor.

Q17. An electron is moving with a velocity of $0.85c$ in the same direction as that of a moving photon. The relative velocity of the electron with respect to photon is

(a) c (b) $-c$ (c) $0.15c$ (d) $-0.15c$

Ans: (b)

Q18. The Lagrangian of a system with one degree of freedom q is given by $L = \alpha\dot{q}^2 + \beta q^2$, where α and β are non-zero constants. If p_q denotes the canonical momentum conjugate to q then which one of the following statements is CORRECT?

(a) $p_q = 2\beta q$ and it is a conserved quantity.

(b) $p_q = 2\beta q$ and it is not a conserved quantity.

(c) $p_q = 2\alpha\dot{q}$ and it is a conserved quantity.

(d) $p_q = 2\alpha\dot{q}$ and it is not a conserved quantity.

Ans: (d)

Solution: As, $\frac{\partial L}{\partial \dot{q}} = p_q$ but $\frac{\partial L}{\partial q} \neq 0$. Thus, it is not a conserved quantity.

Q19. The relativistic form of Newton's second law of motion is

(a) $F = \frac{mc}{\sqrt{c^2 - v^2}} \frac{dv}{dt}$

(b) $F = \frac{m\sqrt{c^2 - v^2}}{c} \frac{dv}{dt}$

(c) $F = \frac{mc^2}{c^2 - v^2} \frac{dv}{dt}$

(d) $F = m \frac{c^2 - v^2}{c^2} \frac{dv}{dt}$

Ans: (c)

Solution: $P = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow F = \frac{dP}{dt} = m \frac{dv}{dt} \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} + mv\left(-\frac{1}{2}\right) \cdot \frac{1}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \cdot \frac{-2v}{c^2} \frac{dv}{dt}$

$$\Rightarrow F = m \frac{dv}{dt} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(1 + \frac{1}{2} \frac{v^2 / c^2}{\left(1 - \frac{v^2}{c^2}\right)} \right) = m \frac{dv}{dt} \left(\frac{1 - v^2 / 2c^2}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \right)$$

$$= m \frac{dv}{dt} \left[\frac{\left(1 - v^2 / c^2\right)^{1/2}}{\left(1 - v^2 / c^2\right)\left(1 - v^2 / c^2\right)^{1/2}} \right] = \frac{mc^2}{\left(c^2 - v^2\right)} \frac{dv}{dt}$$

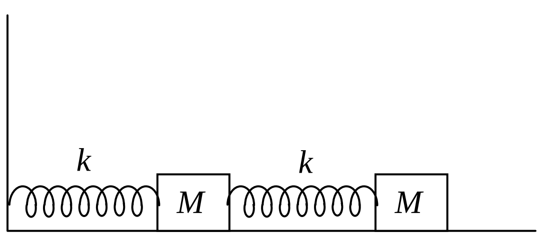
Q20. Consider two small blocks, each of mass M, attached to two identical springs. One of the springs is attached to the wall, as shown in the figure. The spring constant of each spring is k. The masses slide along the surface and the friction is negligible. The frequency of one of the normal modes of the system is,

(a) $\sqrt{\frac{3 + \sqrt{2}}{2}} \sqrt{\frac{k}{M}}$

(b) $\sqrt{\frac{3 + \sqrt{3}}{2}} \sqrt{\frac{k}{M}}$

(c) $\sqrt{\frac{3 + \sqrt{5}}{2}} \sqrt{\frac{k}{M}}$

(d) $\sqrt{\frac{3 + \sqrt{6}}{2}} \sqrt{\frac{k}{M}}$



Ans: (c)

Solution: $T=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}m\dot{x}_2^2$,

$$V=\frac{1}{2}kx_1^2+\frac{1}{2}k\left(x_2-x_1\right)^2=\frac{1}{2}kx_1^2+\frac{1}{2}k\left(x_2^2+x_1^2-2x_2x_1\right)=\frac{1}{2}k\left(2x_1^2+x_2^2-2x_2x_1\right)$$

$$T=\begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix};\ V=\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$$

$$\begin{vmatrix} 2k-\omega^2 m & -k \\ -k & k-\omega^2 m \end{vmatrix}=0\Rightarrow\left(2k-\omega^2 m\right)\left(k-\omega^2 m\right)-k^2=0\Rightarrow\omega=\sqrt{\frac{3+\sqrt{5}}{2}}\sqrt{\frac{k}{m}}$$

GATE- 2014

Q21. If the half-life of an elementary particle moving with speed 0.9c in the laboratory frame is $5\times10^{-8}\,s$, then the proper half-life is ________________ $\times10^{-8}\,s.\left(c=3\times10^{8}\,m/s\right)$

Ans: 2.18

Solution: $t=\frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$, $t_0=t\times\sqrt{1-\frac{v^2}{c^2}}=5\times10^{-8}\times\sqrt{0.19}=2.18\times10^{-8}s$

Q22. Two masses m and $3m$ are attached to the two ends of a massless spring with force constant K. If $m=100g$ and $K=0.3N/m$, then the natural angular frequency of oscillation is _________ Hz.

Ans: 0.318

Solution: $f=\frac{1}{2\pi}\sqrt{\frac{k}{\mu}}$, $\mu=\frac{m_1.m_2}{m_1+m_2}=\frac{3m.m}{4m}=\frac{3m}{4}$, $\omega=\sqrt{\frac{4k}{3m}}=2\Rightarrow f=0.318\ Hz$

Q23. The Hamilton's canonical equation of motion in terms of Poisson Brackets are

(a) $\dot{q}=\{q,H\};\dot{p}=\{p,H\}$

(b) $\dot{q}=\{H,q\};\dot{p}=\{H,p\}$

(c) $\dot{q}=\{H,p\};\dot{p}=\{H,p\}$

(d) $\dot{q}=\{p,H\};\dot{p}=\{q,H\}$

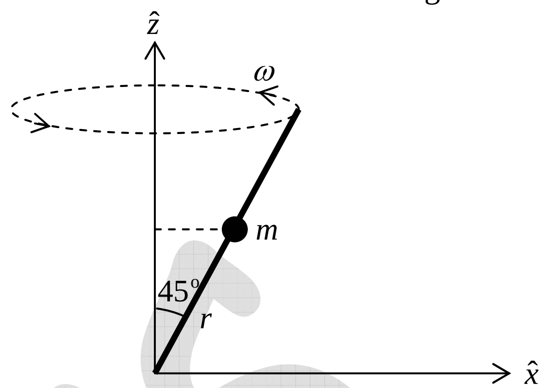
Ans: (a)

Solution: $\frac{df}{dt}=\frac{\partial f}{\partial q}.\frac{\partial q}{\partial t}+\frac{\partial f}{\partial p}.\frac{\partial p}{\partial t}+\frac{\partial f}{\partial t}$

$$\frac{df}{dt}=\frac{\partial f}{\partial q}.\frac{\partial H}{\partial p}-\frac{\partial f}{\partial p}.\frac{\partial H}{\partial q}+\frac{\partial f}{\partial t}\Rightarrow\frac{df}{dt}=\{f,H\}+\frac{\partial f}{\partial t}$$

$$\frac{dq}{dt}=\{q,H\}\ \text{and}\ \frac{dp}{dt}=\{p,H\}$$

Q24. A bead of mass m can slide without friction along a massless rod kept at 45° with the vertical as shown in the figure. The rod is rotating about the vertical axis with a constant angular speed ω. At any instant r is the distance of the bead from the origin. The momentum conjugate to r is



(a) $m\dot{r}$

(b) $\frac{1}{\sqrt{2}}m\dot{r}$

(c) $\frac{1}{2}m\dot{r}$

(d) $\sqrt{2}m\dot{r}$

Ans: (a)

Solution: $L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2)-mgr\cos\theta$

Equation of constrain is $\theta=\frac{\pi}{4}$ and it is given $\dot{\phi}=\omega$

$$L=\frac{1}{2}m(\dot{r}^2+\frac{1}{2}r^2\omega^2)-\frac{1}{\sqrt{2}}mgr$$

Thus the momentum conjugate to r is $p_r=\frac{\partial L}{\partial \dot{r}}\Rightarrow p_r=m\dot{r}$

Q25. A particle of mass m is in a potential given by

$$V(r)=-\frac{a}{r}+\frac{ar_0^2}{3r^3}$$

where a and r_0 are positive constants. When disturbed slightly from its stable equilibrium position it undergoes a simple harmonic oscillation. The time period of oscillation is

(a) $2\pi\sqrt{\frac{mr_0^3}{2a}}$ (b) $2\pi\sqrt{\frac{mr_0^3}{a}}$ (c) $2\pi\sqrt{\frac{2mr_0^3}{a}}$ (d) $4\pi\sqrt{\frac{mr_0^3}{a}}$

Ans: (a)

Solution: $V(r)=-\frac{a}{r}+\frac{ar_0^2}{3r^3}$,

For equilibrium $\frac{\partial V}{\partial r}=\frac{a}{r^2}-\frac{3ar_0^2}{3r^4}=0,\quad r=\pm r_0$

$$\frac{\partial^2 V}{\partial r^2}=-\frac{2a}{r^3}+\frac{4ar_0^2}{r^5}\Bigg|_{r_0}=-\frac{2a}{r_0^3}+\frac{4ar_0^2}{r_0^5}=\frac{2a}{r_0^3}$$

$$\omega = \sqrt{\frac{\left.\frac{\partial^2 V}{\partial r^2}\right|_{r_0}}{m}} \Rightarrow T = 2\pi\sqrt{\frac{mr_0^3}{2a}}$$

Q26. A planet of mass m moves in a circular orbit of radius r_0 in the gravitational potential $V(r) = -\frac{k}{r}$, where k is a positive constant. The orbit angular momentum of the planet is

(a) $2r_0 km$ (b) $\sqrt{2r_0 km}$ (c) $r_0 km$ (d) $\sqrt{r_0 km}$

Ans: (d)

Solution: $V_{effctive} = \frac{J^2}{2mr^2} - \frac{k}{r} \Rightarrow \frac{dV_{effect}}{dr} = -\frac{J^2}{mr^3} + \frac{k}{r^2} = 0$ at $r = r_0$

so $J = \sqrt{r_0 km}$

Q27. Given that the linear transformation of a generalized coordinate q and the corresponding momentum p, $Q = q + 4ap$, $P = q + 2p$ is canonical, the value of the constant a is ____

Ans: 0.25

Solution: $\frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} = 1 \Rightarrow 1\times 2 - 4a \times 1 = 1 \Rightarrow a = 0.25$

Q28. The Hamiltonian of particle of mass m is given by $H = \frac{p^2}{2m} - \frac{\alpha q^2}{2}$. Which one of the following figure describes the motion of the particle in phase space?

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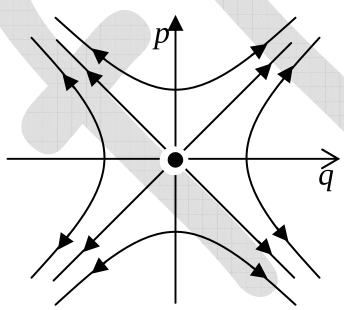

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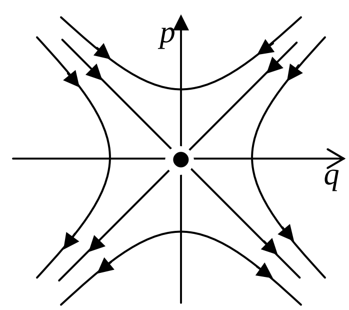

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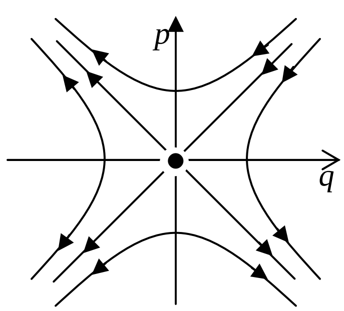

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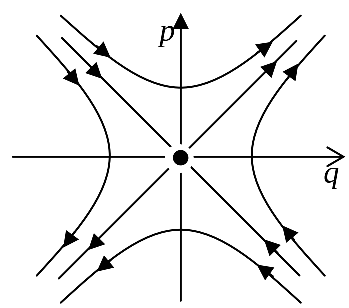


Ans: (d)

GATE- 2015

Q29. A satellite is moving in a circular orbit around the Earth. If T, V and E are its average kinetic, average potential and total energies, respectively, then which one of the following options is correct?

(a) $V = -2T; E = -T$

(b) $V = -T; E = 0$

(c) $V = -\frac{T}{2}; E = \frac{T}{2}$

(d) $V = \frac{-3T}{2}; E = \frac{-T}{2}$

Ans.: (a)

Solution: From Virial theorem $\langle T \rangle = \frac{n+1}{2}\langle V \rangle$ where $V \propto r^{n+1}$

$$\because V = \frac{-k}{r} \Rightarrow V \propto \frac{1}{r} \Rightarrow n = -2 \Rightarrow \langle V \rangle = -2\langle T \rangle$$

Q30. In an inertial frame S, two events A and B take place at $(ct_A = 0, \vec{r}_A = 0)$ and $(ct_B = 0, \vec{r}_B = 2\hat{y})$, respectively. The times at which these events take place in a frame S' moving with a velocity $0.6c\,\hat{y}$ with respect to S are given by

(a) $ct'_A = 0;\ ct'_B = -\frac{3}{2}$

(b) $ct'_A = 0;\ ct'_B = 0$

(c) $ct'_A = 0;\ ct'_B = \frac{3}{2}$

(d) $ct'_A = 0;\ ct'_B = \frac{1}{2}$

Ans.: (a)

Solution: Velocity of S' with respect to S is $v = 0.6c$

$$t'_A = \frac{t_A - \frac{v}{c^2}y}{\sqrt{1-\frac{v^2}{c^2}}}$$

For event A, $t_A = 0, y = 0$. So $ct'_A = 0$

$$t'_B = \frac{t_B - \frac{v}{c^2}y}{\sqrt{1-\frac{v^2}{c^2}}}$$

For event B, $t_B = 0, y = 2$. So $ct_B' = -\frac{3}{2}$

Q31. The Lagrangian for a particle of mass m at a position $\vec{r}$ moving with a velocity $\vec{v}$ is given by $L = \frac{m}{2}\vec{v}^2 + C\vec{r}.\vec{v} - V(r)$, where $V(r)$ is a potential and C is a constant. If $\vec{p}_c$ is the canonical momentum, then its Hamiltonian is given by

(a) $\frac{1}{2m}(\vec{p}_c + C\vec{r})^2 + V(r)$ (b) $\frac{1}{2m}(\vec{p}_c - C\vec{r})^2 + V(r)$

(c) $\frac{p_c^2}{2m} + V(r)$ (d) $\frac{1}{2m}p_c^2 + C^2r^2 + V(r)$

Ans.: (b)

Solution: $L = \frac{m}{2}\vec{v}^2 + C\vec{r}.\vec{v} - V(r)$ where $v = \dot{r}$

$$H = \sum \dot{r}p_c - L = \dot{r}p_c - L$$

$$\Rightarrow \frac{\partial L}{\partial \dot{r}} = p_c = (m\dot{r} + Cr) \Rightarrow \dot{r} = \frac{p_c - Cr}{m}$$

$$\Rightarrow H = \left(\frac{p_c - Cr}{m}\right)p_c - \frac{m}{2}\left(\frac{p_c - Cr}{m}\right)^2 - cr\left(\frac{p_c - Cr}{m}\right) + V(r)$$

$$\Rightarrow H = \left(\frac{p_c - Cr}{m}\right)(p_c - Cr) - \frac{m}{2}\left(\frac{p_c - Cr}{m}\right)^2 + V(r)$$

$$\Rightarrow H = \frac{(p_c - Cr)^2}{m} - \frac{(p_c - Cr)^2}{2m} + V(r) \quad \Rightarrow H = \frac{1}{2m}(p_c - Cr)^2 + V(r)$$

Q32. The Hamiltonian for a system of two particles of masses m_1 and m_2 at $\vec{r}_1$ and $\vec{r}_2$ having velocities $\vec{v}_1$ and $\vec{v}_2$ is given by $H = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{C}{(\vec{r}_1 - \vec{r}_2)^2}\hat{z}\cdot(\vec{r}_1 \times \vec{r}_2)$, where C is constant. Which one of the following statements is correct?

(a) The total energy and total momentum are conserved

(b) Only the total energy is conserved

(c) The total energy and the z - component of the total angular momentum are conserved

(d) The total energy and total angular momentum are conserved

Ans.: (c)

Solution: Lagrangian is not a function of time, so energy is conserved and component of $(\vec{r}_1 \times \vec{r}_2)$ is only in $\hat{z}$ direction means potential is symmetric under ϕ, so L_z is conserved.

Q33. A particle of mass 0.01 kg falls freely in the earth's gravitational field with an initial velocity $(0) = 10ms^{-1}$. If the air exerts a frictional force of the form, $f = -kv$, then for $k = 0.05\, Nm^{-1}s$, the velocity (in ms^{-1}) at time $t = 0.2\, s$ is __________ (upto two decimal places). (use $g = 10\, ms^{-2}$ and $e = 2.72$)

Ans.: 4.94

Solution: $m\frac{dv}{dt} = mg - kv \Rightarrow \frac{dv}{dt} = g - \frac{k}{m}v \Rightarrow \frac{dv}{g - \frac{k}{m}v} = dt \Rightarrow \int_{10}^{u} \frac{dv}{g - \frac{k}{m}v} = \int_0^{0.2} dt$

$$\Rightarrow -\frac{m}{k}\left[\ln\left[g - \frac{k}{m}v\right]\right]_{10}^{u} = [t]_0^{0.2} \Rightarrow -\frac{m}{k}\left\{\left[\ln\left(g - \frac{k}{m}u\right)\right] - \ln\left(g - \frac{10k}{m}\right)\right\} = 0.2$$

$$\Rightarrow -\frac{m}{k}\left\{\ln\left(10 - \frac{0.05}{0.01}u\right) - \ln\left(10 - 10 \times \frac{.05}{.01}\right)\right\} = 0.2$$

$$\Rightarrow -\frac{m}{k}\{\ln(10 - 5u) - \ln(-40)\} = 0.2$$

$$\ln\left(\frac{8}{u-2}\right) = \frac{0.2k}{m} \Rightarrow \ln\left(\frac{8}{u-2}\right) = \frac{0.2k}{m} \Rightarrow \frac{8}{u-2} = e$$

$$\Rightarrow u = \frac{8}{e} + 2 = 4.94\ m/s$$

Q34. Consider the motion of the Sun with respect to the rotation of the Earth about its axis. If $\vec{F}_c$ and $\vec{F}_{Co}$ denote the centrifugal and the Coriolis forces, respectively, acting on the Sun, then

(a) $\vec{F}_c$ is radially outward and $\vec{F}_{Co} = \vec{F}_c$

(b) $\vec{F}_c$ is radially inward and $\vec{F}_{Co} = -2\vec{F}_c$

(c) $\vec{F}_c$ is radially outward and $\vec{F}_{Co} = -2\vec{F}_c$

(d) $\vec{F}_c$ is radially outward and $\vec{F}_{Co} = 2\vec{F}_c$

Ans.: (b)

Q35. A particle with rest mass M is at rest and decays into two particles of equal rest masses $\frac{3}{10}M$ which move along the z axis. Their velocities are given by

(a) $\vec{v}_1 = \vec{v}_2 = (0.8c)\hat{z}$ (b) $\vec{v}_1 = -\vec{v}_2 = (0.8c)\hat{z}$

(c) $\vec{v}_1 = -\vec{v}_2 = (0.6c)\hat{z}$ (d) $\vec{v}_1 = (0.6c)\hat{z};\ \vec{v}_2 = (-0.8c)\hat{z}$

Ans.: (b)

Solution: $M \rightarrow \frac{3}{10}M + \frac{3}{10}M$

From momentum conservation

$$0 = \vec{P}_1 + \vec{P}_2 \Rightarrow \vec{P}_1 = -\vec{P}_2 \Rightarrow |P_1| = |P_2|$$

From energy conservation

$$E = E_1 + E_2$$

$$\Rightarrow Mc^2 = \frac{3}{10}\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{3}{10}\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \quad \Rightarrow Mc^2 = \frac{3}{5}\frac{Mc^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$\left(1-\frac{v^2}{c^2}\right) = \frac{9}{25} \Rightarrow \frac{v^2}{c^2} = \frac{16}{25} \Rightarrow v = 0.8c$$

GATE-2016

Q36. The kinetic energy of a particle of rest mass m_0 is equal to its rest mass energy. Its momentum in units of $m_0 c$, where c is the speed of light in vacuum, is _______.

(Give your answer upto two decimal places)

Ans. : 1.73

Solution: $m_0c^2 = E - m_0c^2 \Rightarrow E = 2m_0c^2$

$$\Rightarrow \frac{m_0c^2}{\sqrt{1-\frac{v^2}{c^2}}} = 2m_0c^2 \Rightarrow v = \frac{\sqrt{3}}{2}c$$

$$\because E^2 = p^2c^2 + m_0^2c^4 \Rightarrow 4m_0^2c^4 - m_0^2c^4 = p^2c^2 \Rightarrow p = \sqrt{3}m_0c = 1.732m_0c$$

Q37. In an inertial frame of reference S, an observer finds two events occurring at the same time at coordinates $x_1 = 0$ and $x_2 = d$. A different inertial frame S' moves with velocity v with respect to S along the positive x-axis. An observer in S' also notices these two events and finds them to occur at times t_1' and t_2' and at positions x_1' and x_2' respectively.

If $\Delta t' = t_2' - t_1', \Delta x' = x_2' - x_1'$ and $\gamma = \dfrac{1}{\sqrt{1-\dfrac{v^2}{c^2}}}$, which of the following statements is true?

(a) $\Delta t' = 0, \Delta x' = \gamma d$ (b) $\Delta t' = 0, \Delta x' = \dfrac{d}{\gamma}$

(c) $\Delta t' = \dfrac{-\gamma v d}{c^2}, \Delta x' = \gamma d$ (d) $\Delta t' = \dfrac{-\gamma v d}{c^2}, \Delta x' = \dfrac{d}{\gamma}$

Ans.: (c)

Solution: $t_2' - t_1' = \left(\dfrac{t_2 - \dfrac{vx_2}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) - \left(\dfrac{t_1 - \dfrac{vx_1}{c^2}}{\sqrt{1-\dfrac{v^2}{c^2}}}\right) \Rightarrow \Delta t' = \gamma \Delta t - \dfrac{\gamma v \Delta x}{c^2}$

It is given, $\Delta t = 0, \Delta x = d$

$$\Rightarrow \Delta t' = -\frac{\gamma v \Delta x}{c^2} = -\frac{\gamma v d}{c^2}$$

$$x_2' - x_1' = \left(\frac{x_2 - vt_2}{\sqrt{1-\frac{v^2}{c^2}}}\right) - \left(\frac{x_1 - vt_1}{\sqrt{1-\frac{v^2}{c^2}}}\right) \Rightarrow \Delta x' = \gamma\left(\Delta x - v\Delta t\right)$$

$\Rightarrow \Delta x' = \gamma d$.

Q38. The Lagrangian of a system is given by

$L = \dfrac{1}{2} m l^2 \left[\dot{\theta}^2 + \sin^2\theta \dot{\varphi}^2\right] - mgl\cos\theta$, where m, l and g are constants.

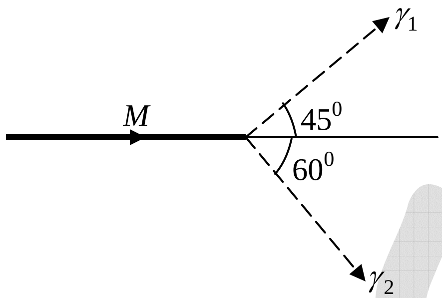
Which of the following is conserved?

(a) $\dot{\varphi}\sin^2\theta$ (b) $\dot{\varphi}\sin\theta$ (c) $\dfrac{\dot{\varphi}}{\sin\theta}$ (d) $\dfrac{\dot{\varphi}}{\sin^2\theta}$

Ans.: (a)

Solution: As φ is cyclic coordinate, so $\dfrac{\partial L}{\partial \dot{\varphi}} = p_\varphi = ml^2 \sin^2 \dot{\varphi}$, is a constant since m, l and g are constants. Thus $\dot{\varphi}\sin^2\theta$ is conserved.

Q39. A particle of rest mass M is moving along the positive x-direction. It decays into two photons γ_1 and γ_2 as shown in the figure. The energy of γ_1 is $1\,GeV$ and the energy of γ_2 is $0.82\,GeV$. The value of M (in units of $\frac{GeV}{c^2}$) is ________. (Give your answer upto two decimal places)



Ans.: 1.44

Solution: $\sqrt{p^2c^2+M^2c^4}=E_1+E_2=1.82GeV$

$$p=\frac{E_1}{c}\cos\theta_1+\frac{E_2}{c}\cos\theta_2=\frac{1GeV}{c}\frac{1}{\sqrt{2}}+\frac{0.82GeV}{c}\frac{1}{2}=\frac{1.11GeV}{c}$$

$$\Rightarrow p^2c^2+m^2c^4=3.312\Rightarrow m^2c^4=3.312-1.23=2.08$$

$$\Rightarrow m=\sqrt{2.076}=1.44$$

GATE- 2017

Q40. If the Lagrangian $L_0=\frac{1}{2}m\left(\frac{dq}{dt}\right)^2-\frac{1}{2}m\omega^2q^2$ is modified to $L=L_0+\alpha q\left(\frac{dq}{dt}\right)$, which one of the following is TRUE?

(a) Both the canonical momentum and equation of motion do not change

(b) Canonical momentum changes, equation of motion does not change

(c) Canonical momentum does not change, equation of motion changes

(d) Both the canonical momentum and equation of motion change

Ans. : (b)

Solution: For Lagrangian $L_0=\frac{1}{2}m\left(\frac{dq}{dt}\right)^2-\frac{1}{2}m\omega^2q^2$ canonical momentum is $p=m\dot{q}$ and equation of motion is given by $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}}\right)-\left(\frac{\partial L}{\partial q}\right)=0\Rightarrow m\ddot{q}+m\omega^2q=0$

For Lagrangian $L = L_0 + \alpha q\left(\frac{dq}{dt}\right) \Rightarrow L = \frac{1}{2}m\left(\frac{dq}{dt}\right)^2 - \frac{1}{2}m\omega^2 q^2 + \alpha q\dot{q}$

Canonical momentum is $p = m\dot{q} + \alpha q$

Equation of motion is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \left(\frac{\partial L}{\partial q}\right) = 0 \Rightarrow m\ddot{q} + m\omega^2 q = 0$$

Q41. Two identical masses of $10\,gm$ each are connected by a massless spring of spring constant $1\,N/m$. The non-zero angular eigenfrequency of the system is………..rad/s. (up to two decimal places)

Ans. : 14.14

Solution: $\omega = \sqrt{\frac{k}{\mu}}$, where $\mu = \frac{m}{2} = \frac{10}{2\times 1000} = \frac{1}{200}$ and $k = 1N/m$, $\omega = 14.14$

Q42. The phase space trajectory of an otherwise free particle bouncing between two hard walls elastically in one dimension is a

(a) straight line (b) parabola (c) rectangle (d) circle

Ans. : (c)

Solution: $E = \frac{p^2}{2m}$, $p = \pm\sqrt{2mE}$

Q43. The Poisson bracket $\left[x, xp_y + yp_x\right]$ is equal to

(a) $-x$ (b) y (c) $2p_x$ (d) p_y

Ans. : (b)

Solution: $\left[x, xp_y + yp_x\right] = \left[x, xp_y\right] + \left[x, yp_x\right] = 0 + y\left[x, p_x\right] = y$

Q44. An object travels along the x-direction with velocity $\frac{c}{2}$ in a frame O. An observer in a frame O' sees the same object travelling with velocity $\frac{c}{4}$. The relative velocity of O' with respect to O in units of c is…………….. (up to two decimal places).

Ans. : 0.28

Solution: $u_x^{'} = \frac{c}{2}, v = \frac{c}{4}$, $u_x = \frac{u_x^{'} - v}{1 - \frac{u_x^{'} v}{c^2}} = \frac{\frac{c}{2} - \frac{c}{4}}{1 - \frac{c}{2} \cdot \frac{c}{4} \cdot \frac{1}{c^2}} = \frac{2c}{7} = 0.28c$

Q45. A uniform solid cylinder is released on a horizontal surface with speed $5\, m/s$ without any rotation (slipping without rolling). The cylinder eventually starts rolling without slipping. If the mass and radius of the cylinder are $10\, gm$ and $1\, cm$ respectively, the final linear velocity of the cylinder is…………… m/s. (up to two decimal places).

Ans. : 3.33

Solution: $mvr = mv_{cm}r + I_{cm}\omega = mv_{cm}r + \frac{1}{2}mr^2 \frac{v_{cm}}{r} \Rightarrow v = \frac{3}{2}v_{cm} \Rightarrow v_{cm} = \frac{2}{3}v = \frac{10}{3} = 3.33 m/\sec$

Q46. A person weighs w_p at Earth's north pole and w_e at the equator. Treating the Earth as a perfect sphere of radius 6400 km, the value $100 \times \frac{(w_p - w_e)}{w_p}$ is………….. (up to two decimal places). (Take $g = 10\, ms^{-2}$).

Ans. : 0.33

Solution: $g_p = g$, $g_e = g - \omega^2 R \Rightarrow 100 \times \frac{w_p - w_e}{w_p} = \frac{\omega^2 R}{g}$

Now, $g = 10\, m/\sec^2$ and $R = 6400 \times 10^3\, m$

$\Rightarrow \omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600}$

Then $100 \times \frac{w_p - w_e}{w_p} = 0.33$

GATE - 2018

Q47. In the context of small oscillations, which one of the following does NOT apply to the normal coordinates?

(a) Each normal coordinate has an eigen-frequency associated with it

(b) The normal coordinates are orthogonal to one another

(c) The normal coordinates are all independent

(d) The potential energy of the system is a sum of squares of the normal coordinates with constant coefficients

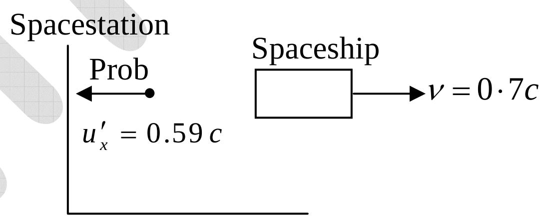
Ans. : (b)

Solution: Normal co-ordinate must be independent. It is not necessary that it should orthogonal.

Q48. A spaceship is travelling with a velocity of $0.7c$ away from a space station. The spaceship ejects a probe with a velocity $0.59c$ opposite to its own velocity. A person in the space station would see the probe moving at a speed Xc, where the value of X is ____________ (up to three decimal places).

Ans.: $0.187c$

Solution: $v = 0\cdot 7c$, $u'_x = -0\cdot 59c$,



$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c_2}}$$

$$u_x = \frac{-0\cdot 59c + 0\cdot 7c}{1 - 0\cdot 7 \times 0\cdot 59} = \frac{0.11c}{1 - 0.413} = \frac{0.11c}{0.587} = 0.187\,c$$

Q49. An interstellar object has speed v at the point of its shortest distance R from a star of much larger mass M. Given $v^2 = 2GM/R$, the trajectory of the object is

(a) circle (b) ellipse (c) parabola (d) hyperbola

Ans. : (c)

Solution: At shortest distance $E = \frac{J^2}{2mR^2} - \frac{GMm}{R}$

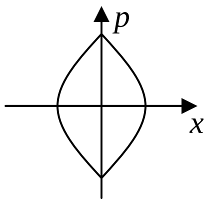
Since, $mvR = J \Rightarrow J^2 = m^2 v^2 R^2$

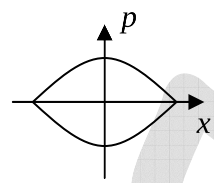
Now, $J^2 = m^2 2GMR = 2GMm^2 R$ (Given that $v^2 = \frac{2GM}{R}$)

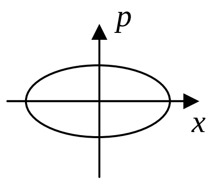
$$E = \frac{2GMm^2R}{2mR^2} - \frac{GMm}{R} = \frac{GMm}{R} - \frac{GMm}{R} = 0$$

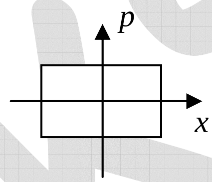
For Kepler's potential, if energy is zero, then the shape is parabola.

Q50. A particle moves in one dimension under a potential $V(x) = \alpha|x|$ with some non-zero total energy. Which one of the following best describes the particle trajectory in the phase space?

(a) 

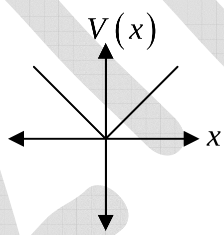
(b) 

(c) 

(d) 

Ans.: (a)

Solution: $E = \dfrac{p^2}{2m} + \alpha|x|$

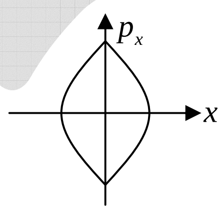


For $x > 0$, $E = \dfrac{p^2}{2m} + \alpha x$

$\Rightarrow p^2 = 2m(E - \alpha x)$

For $x < 0$, $E = \dfrac{p^2}{2m} - \alpha x$

$\Rightarrow p^2 = 2m(E + \alpha x)$



Q51. If H is the Hamiltonian for a free particle with mass m, the commutator $[x,[x,H]]$ is

(a) $\hbar^2/m$ (b) $-\hbar^2/m$ (c) $-\hbar^2/(2m)$ (d) $\hbar^2/(2m)$

Ans. : (b)

Solution: For free particle, potential is zero.

$$\Rightarrow H = \frac{P_x^2}{2m}$$

Now, $[x,H]=\left[x,\frac{P_x^2}{2m}\right]=\frac{2i\hbar}{2m}P_x$

$$[x,[x,H]]=\frac{2i\hbar}{2m}[x,P_x]=\frac{i\hbar}{m}(i\hbar)=-\frac{\hbar^2}{m}$$

Q52. For the transformation

$$Q=\sqrt{2q}\,e^{-1+2\alpha}\cos p, P=\sqrt{2q}\,e^{-\alpha-1}\sin p$$

(where α is a constant) to be canonical, the value of α is __________.

Ans. : 2

Solution: $Q=\sqrt{2q}e^{-1+2\alpha}\cos p, P=\sqrt{2q}e^{-\alpha-1}.\sin p$

Since, $[Q,P]=1$

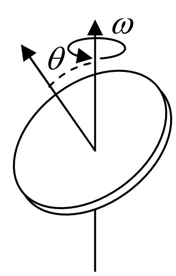
$$\Rightarrow \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}=1$$

$$\Rightarrow\left(\frac{1}{2}\sqrt{2}q^{-\frac{1}{2}}.e^{-1+2\alpha}\cos p\right)\left(\sqrt{2q}e^{-\alpha-1}\cos p\right)-\sqrt{2q}e^{-1+2\alpha}(-\sin p).\frac{\sqrt{2}}{2}q^{-\frac{1}{2}}e^{-\alpha.1}\sin p=1$$

$$\Rightarrow e^{\alpha-2}.\left[\cos^2 p+\sin^2 p\right]=1=e^0$$

$$\Rightarrow \alpha=2$$

Q53. A uniform circular disc of mass m and radius R is rotating with angular speed ω about an axis passing through its centre and making an angle $\theta=30^0$ with the axis of the disc. If the kinetic energy of the disc is $\alpha m\omega^2R^2$, the value of α is___________ (up to two decimal places).



Ans. : 0.21

Solution: The kinetic energy of the disc is,

$$T=\frac{1}{2}\vec{L}\cdot\vec{\omega}$$

Where $\vec{L}$ is angular momentum and ω is angular velocity

$$T=\frac{1}{2}|\vec{L}||\vec{\omega}|\cos 30^0=\frac{1}{2}I\omega\cdot\omega\frac{\sqrt{3}}{2}=\frac{1}{2}\left(\frac{mR^2}{2}\right)\omega^2\times\frac{\sqrt{3}}{2}$$

$$T=\frac{\sqrt{3}}{8}m\omega^2R^2=0.21m\omega^2R^2\Rightarrow \alpha m\omega^2R^2=0.21m\omega^2R^2$$

Hence, $\alpha=0.21$

GATE-2019

Q54. Consider a transformation from one set of generalized coordinate and momentum (q, p) to another set (Q, P) denoted by,

$$Q = pq^s; \qquad P = q^r$$

where s and r are constants. The transformation is canonical if

(a) $s = 0$ and $r = 1$ (b) $s = 2$ and $r = -1$

(c) $s = 0$ and $r = -1$ (d) $s = 2$ and $r = 1$

Ans. : (b)

Solution: $\frac{\partial Q}{\partial q}.\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}.\frac{\partial P}{\partial q} = 1 \Rightarrow 0 - q^s rq^{r-1} = 1$

$-rq^{r+s-1} = 1 \Rightarrow s = 2$ and $r = -1$

Q55. The Hamiltonian for a particle of mass m is $H = \frac{p^2}{2m} + kqt$ where q and p are the generalized coordinate and momentum, respectively, t is time and k is a constant. For the initial condition, $q = 0$ and $p = 0$ at $t = 0, q(t) \propto t^\alpha$. The value of α is ________

Ans. : 3

Solution: $\frac{\partial H}{\partial p} = \dot{q} = \frac{p}{m}$ ....(1)

$\frac{\partial H}{\partial q} = -\dot{p} = kt \Rightarrow p = -\frac{kt^2}{2}$ ....(2)

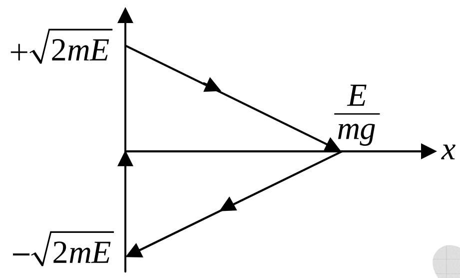
$\frac{dq}{dt} = -\frac{kt^2}{2} \Rightarrow q = -\frac{kt^3}{6}$ $q \propto t^3$ so $\alpha = 3$

Q56. A ball bouncing of a rigid floor is described by the potential energy function

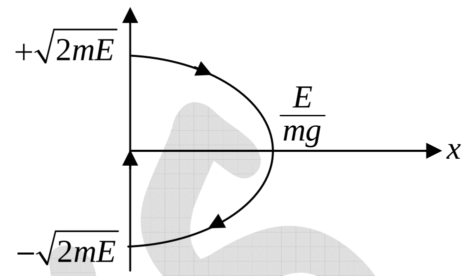
$$V(x)=\begin{cases} mgx & \text{for} \quad x>0 \\ \infty & \text{for} \quad x\le 0 \end{cases}$$

Which of the following schematic diagrams best represents the phase space plot of the ball?

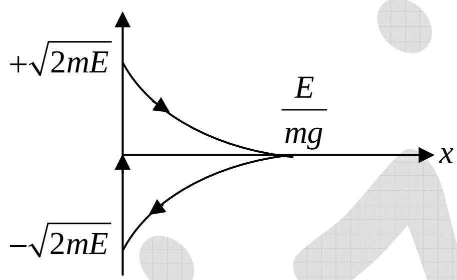
(a)



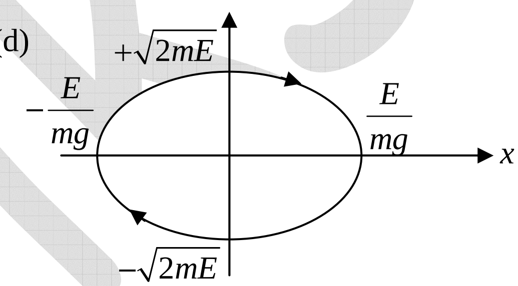
(b)



(c)



(d)



Ans. : (b)

Solution: $E=\frac{p^2}{2m}+mgx \Rightarrow p^2=2m(E-mgx)$ which is equation of parabola

Q57. Consider the Hamiltonian $H(q,p)=\frac{ap^2q^4}{2}+\frac{\beta}{q^2}$, where α and β are parameters with appropriate dimensions, and q and p are the generalized coordinate and momentum, respectively. The corresponding Lagrangian $L(q,\dot{q})$ is

(a) $\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4}-\frac{\beta}{q^2}$ (b) $\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4}+\frac{\beta}{q^2}$ (c) $\frac{1}{\alpha}\frac{\dot{q}^2}{q^4}+\frac{\beta}{q^2}$ (d) $-\frac{1}{2\alpha}\frac{\dot{q}^2}{q^4}+\frac{\beta}{q^2}$

Ans. : (a)

Solution: $L=p\dot{q}-H \Rightarrow p\dot{q}-\frac{ap^2q^4}{2}-\frac{\beta}{q^2}$ from Hamiltonian equation of motion

$$\frac{\partial H}{\partial p}=\dot{q} \Rightarrow p=\frac{\dot{q}}{aq^4}$$

$$L = \frac{1}{2\alpha}\frac{\dot{q}^2}{q^4} - \frac{\beta}{q^2}$$

Q58. A projectile of mass $1kg$ is launched at an angle of 30^0 from the horizontal direction at $t=0$ and takes time T before hitting the ground. If its initial speed is 10 ms^{-1}, the value of the action integral for the entire flight in the units of kgm^2s^{-1} (round off to one decimal place) is____________. [Take $g = 10\,ms^{-2}$]

Ans. : 33.3

Solution: $T = \dfrac{2v\sin\theta}{g} = 1\sec$

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$

$$\dot{x} = v\cos\theta = 5\sqrt{3}ms^{-1} \quad \dot{y} = v\sin\theta - gt = 5 - 10t$$

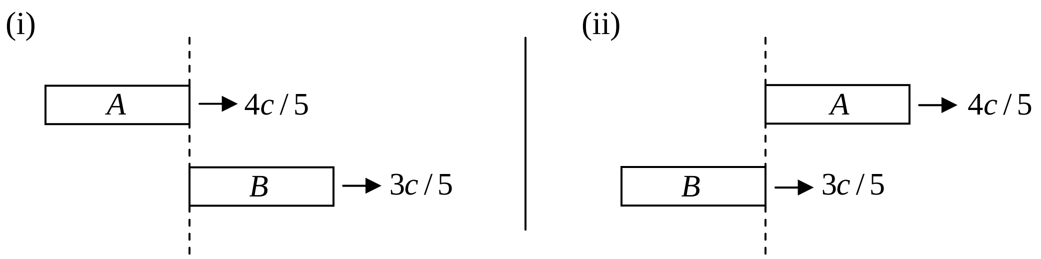
$$y = ut - \frac{1}{2}gt^2 = v\sin\theta t - \frac{1}{2}gt^2 = 10.\frac{1}{2}t - \frac{1}{2}10t^2 = 5t - 5t^2$$

$$L = \frac{1}{2}1\left(\left(5\sqrt{3}\right)^2 + \left(5-10t\right)^2\right) - 1\times 10\times\left(5t - 5t^2\right)$$

$$L = 100t^2 - 100t + 50$$

$$A = \int_0^T Ldt = \int_0^1 \left(100t^2 - 100t + 50\right)dt = 33.3$$

Q59. Two spaceships A and B, each of the same rest length L, are moving in the same direction with speeds $\frac{4c}{5}$ and $\frac{3c}{5}$, respectively, where c is the speed of light. As measured by B, the time taken by A to completely overtake B [see figure below] in units of L/c (to the nearest integer) is ______________



Ans. : 5

Solution: $u_{A,B}=\dfrac{\frac{4}{5}c-\frac{3}{5}c}{1-\frac{4}{5}c.\frac{3}{5}c.\frac{1}{c^2}}=\dfrac{\frac{c}{5}}{\frac{13}{25}}=\dfrac{5}{13}c$

Kinematic equation is given by

$$\frac{5}{13}c\times t=L\sqrt{1-\frac{25}{169}}+L\Rightarrow t=\frac{5L}{c}\Rightarrow \alpha=5$$

Q60. Two events, one on the earth and the other one on the Sun, occur simultaneously in the earth's frame. The time difference between the two events as seen by an observer in a spaceship moving with velocity $0.5c$ in the earth's frame along the line joining the earth to the Sun is Δt, where c is the speed of light. Given that light travels from the Sun to the earth in 8.3 minutes in the earth's frame, the value of $|\Delta t|$ in minutes (rounded off to two decimal places) is____________

(Take the earth's frame to be inertial and neglect the relative motion between the earth and the sun)

Ans. : 4.77

Solution: $t_2'-t_1'=0 \quad x_2'-x_1'=8.3\times 3\times 10^8\times 60 \quad v=0.5c$

$$\Delta t=t_2-t_1=\left(\frac{t_2'+\frac{vx_2'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)-\left(\frac{t_1'+\frac{vx_1'}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\right)=\left(\frac{t_2'-t_1'}{\sqrt{1-\frac{v^2}{c^2}}}\right)+\frac{v}{c^2}\frac{\left(x_2'-x_1'\right)}{\sqrt{1-\frac{v^2}{c^2}}}=4.77\,\text{min}$$