QUANTUM MECHANICS SOLUTIONS

GATE- 2010

Q1. Which of the following is an allowed wavefunction for a particle in a bound state? N is a constant and $\alpha, \beta > 0$.

(a) $\psi = N\dfrac{e^{-\alpha r}}{r^3}$ (b) $\psi = N\left(1-e^{-\alpha r}\right)$

(c) $\psi = Ne^{-\alpha x}e^{-\beta\left(x^2+y^2+z^2\right)}$ (d) $\psi = \begin{cases}\text{non - zero constant} & \text{if } r<R \\ 0 & \text{if } r>R\end{cases}$

Ans: (c)

Q2. A particle of mass m is confined in the potential

$$V(x)=\begin{cases}\frac{1}{2}m\omega^2x^2, & \text{for } x>0 \\ \infty \quad , & \text{for } x\le 0\end{cases}$$

Let the wavefunction of the particle be given by

$$\psi(x)=-\frac{1}{\sqrt{5}}\psi_0+\frac{2}{\sqrt{5}}\psi_1,$$

where ψ_0 and ψ_1 are the eigenfunctions of the ground state and the first excited state respectively. The expectation value of the energy is

(a) $\frac{31}{10}\hbar\omega$ (b) $\frac{25}{10}\hbar\omega$ (c) $\frac{13}{10}\hbar\omega$ (d) $\frac{11}{10}\hbar\omega$

Ans: (a)

Solution: For half parabolic potential

$$E_0=\frac{3}{2}\hbar\omega, E_1=\frac{7}{2}\hbar\omega \Rightarrow \langle E\rangle=\left(\frac{1}{5}\times\frac{3}{2}+\frac{4}{5}\times\frac{7}{2}\right)\hbar\omega=\frac{31}{10}\hbar\omega.$$

Q3. For a spin-s particle, in the eigen basis of $\vec{S}^2$, S_x the expectation value $\langle sm|S_x^2|sm\rangle$ is

(a) $\dfrac{\hbar^2\left\{s(s+1)-m^2\right\}}{2}$ (b) $\hbar^2\left\{s(s+1)-2m^2\right\}$

(c) $\hbar^2\left\{s(s+1)-m^2\right\}$ (d) $\hbar^2m^2$

Ans: (a)

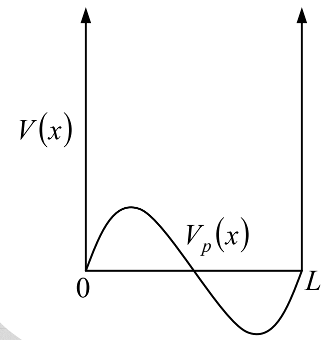
Solution: $\langle sm|S_x^2|sm\rangle=\frac{1}{4}\langle sm|\left(S_++S_-\right)^2|sm\rangle=\frac{1}{4}\langle sm|S_+^2+S_-^2+S_+S_-+S_-S_+|sm\rangle$

$$=\frac{1}{4}\langle sm|S_+S_-+S_-S_+|sm\rangle=\frac{\hbar^2}{2}\left[s(s+1)-m^2\right] \qquad \left[\because S_+S_-+S_-S_+=2\left(S^2-S_z^2\right)\right]$$

Q4. A particle of mass m is confined in an infinite potential well:

$$V(x)=\begin{cases}0, & \text{if } 0<x<L,\\ \infty, & \text{otherwise.}\end{cases}$$

It is subjected to a perturbing potential $V_p(x)=V_o\sin\left(\frac{2\pi x}{L}\right)$ within the well. Let $E^{(1)}$ and $E^{(2)}$ be corrections to the ground state energy in the first and second order in V_0, respectively. Which of the following are true?



(a) $E^{(1)}=0; E^{(2)}<0$ (b) $E^{(1)}>0; E^{(2)}=0$

(c) $E^{(1)}=0; E^{(2)}$ depends on the sign of V_0 (d) $E^{(1)}<0; E^{(2)}<0$

Ans: (a)

Solution: $E_1^1=\frac{2}{L}\int_0^L V_0\sin\frac{2\pi x}{L}dx \; =0$; $E_1^2=\sum_{m\neq 1}\frac{\left|\langle\psi_m|V_P|\psi_1\rangle\right|^2}{E_1-E_m}$ $\because E_1<E_m$ so $E_1^2=-ve$.

GATE- 2011

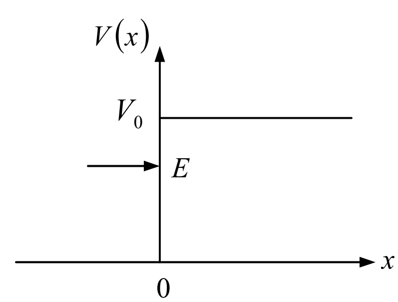
Q5. The quantum mechanical operator for the momentum of a particle moving in one dimension is given by

(a) $i\hbar\frac{d}{dx}$ (b) $-i\hbar\frac{d}{dx}$ (c) $i\hbar\frac{\partial}{\partial t}$ (d) $-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$

Ans: (b)

Q6. An electron with energy E is incident from left on a potential barrier, given by

$$V(x)=\begin{cases}0, & \text{for } x<0\\ V_0, & \text{for } x>0\end{cases}$$



as shown in the figure. For $E<V_0$, the space part of the wavefunction for $x>0$ is of the form

(a) e^{ax} (b) e^{-ax} (c) e^{iax} (d) e^{-iax}

Ans: (b)

Solution: $\because E<V_0$, so there is decaying wave function.

Q7. If L_x, L_y and L_z are respectively the x, y and z components of angular momentum operator L. The commutator $[L_x L_y, L_z]$ is equal to

(a) $i\hbar\left(L_x^2+L_y^2\right)$ (b) $2i\hbar L_z$ (c) $i\hbar\left(L_x^2-L_y^2\right)$ (d) 0

Ans: (c)

Solution: $\left[L_x L_y, L_z\right]=L_x\left[L_y L_z\right]+\left[L_x, L_z\right]L_y=i\hbar\left(L_x^2-L_y^2\right)$

Q8. The normalized ground state wavefunciton of a hydrogen atom is given by $\psi(r)=\frac{1}{\sqrt{4\pi}}\frac{2}{a^{3/2}}e^{-r/a}$, where a is the Bohr radius and r is the distance of the electron from the nucleus, located at the origin. The expectation value $\left\langle\frac{1}{r^2}\right\rangle$ is

(a) $\frac{8\pi}{a^2}$ (b) $\frac{4\pi}{a^2}$ (c) $\frac{4}{a^2}$ (d) $\frac{2}{a^2}$

Ans: (d)

Solution: $\left\langle\frac{1}{r^2}\right\rangle=\frac{4}{4\pi a^3}\int_0^{\infty}\frac{1}{r^2}r^2 e^{-\frac{2r}{a}}dr\int_0^{\pi}\int_0^{2\pi}\sin\theta\ d\theta\ d\phi=\frac{2}{a^2}$

Q9. The normalized eigenstates of a particle in a one-dimensional potential well

$$V(x)=\begin{cases}0 & \text{if } 0\le x\le a\\ \infty & \text{otherwise}\end{cases}$$

are given by $\psi_n(x)=\sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$, where n = 1, 2, 3,….

The particle is subjected to a perturbation

$$V'(x)=\begin{cases}V_0\cos\left(\frac{\pi x}{a}\right), & \text{for } 0\le x\le\frac{a}{2}\\ 0\ , & \text{otherwise}\end{cases}$$

The shift in the ground state energy due to the perturbation, in the first order perturbation theory,

(a) $\frac{2V_o}{3\pi}$ (b) $\frac{V_o}{3\pi}$ (c) $-\frac{V_o}{3\pi}$ (d) $-\frac{2V_o}{3\pi}$

Ans: (a)

Solution: $E_1^1=\int_0^{a/2}\psi_1^* V'(x)\psi_1 dx=\frac{2}{a}\int_0^{a/2}\sin^2\left(\frac{\pi x}{a}\right)V_0\cos\left(\frac{\pi x}{a}\right)dx=\frac{2}{a}V_0\left.\frac{\sin^3\frac{\pi x}{a}}{3\frac{\pi}{a}}\right|_0^{a/2}=\frac{2V_0}{3\pi}$

Common data questions Q10 and Q11

In a one-dimensional harmonic oscillator, φ_0, φ_1 and φ_2 are respectively the ground, first and the second excited states. These three states are normalized and are orthogonal to one another ψ_1 and ψ_2 are two states defined by

$$\psi_1 = \varphi_0 - 2\varphi_1 + 3\varphi_2,\ \psi_2 = \varphi_0 - \varphi_1 + \alpha\varphi_2,\ \psi_2 = \varphi_0 - \varphi_1 + \alpha\varphi_2$$

where α is a constant

Q10. The value of α which ψ_2 is orthogonal to ψ_1 is

(a) 2 (b) 1 (c) – 1 (d) – 2

Ans: (c)

Solution: For orthogonal condition scalar product $(\psi_2, \psi_1) = 0$, so $1 + 2 + 3\alpha = 0 \Rightarrow \alpha = -1$

Q11. For the value of α determined in Q10, the expectation value of energy of the oscillator in the state ψ_2 is

(a) $\hbar\omega$ (b) $3\hbar\omega/2$ (c) $3\hbar\omega$ (d) $9\hbar\omega/2$

Ans: (b)

Solution: $\psi_2 = \phi_0 - \phi_1 + \alpha\phi_2$ put $\alpha = -1, \langle H\rangle = \dfrac{\langle\psi_2|H|\psi_2\rangle}{\langle\psi_2|\psi_2\rangle} = \dfrac{\frac{\hbar\omega}{2} + \frac{3\hbar\omega}{2} + \frac{5\hbar\omega}{2}}{3} = \dfrac{3}{2}\hbar\omega$

GATE- 2012

Q12. A particle of mass m is confined in a two dimensional square well potential of dimension a. This potential $V(x, y)$ is given by

$$V(x, y) = 0 \text{ for } -a < x < a \text{ and } -a < y < a$$
$$= \infty \text{ elsewhere}$$

The energy of the first excited state for this particle is given by,

(a) $\dfrac{\pi^2\hbar^2}{ma^2}$ (b) $\dfrac{2\pi^2\hbar^2}{ma^2}$ (c) $\dfrac{5\pi^2\hbar^2}{8ma^2}$ (d) $\dfrac{4\pi^2\hbar^2}{ma^2}$

Ans: (c)

Solution: $E = \left(n_x^2 + n_y^2\right)\dfrac{\pi^2\hbar^2}{2m(2a)^2} = \left(n_x^2 + n_y^2\right)\dfrac{\pi^2\hbar^2}{8ma^2} = \dfrac{5\pi^2\hbar^2}{8ma^2}$ $\quad \because n_x = 1, n_y = 2$.

Q13. Consider the wavefunction $\psi = \psi(\vec{r}_1, \vec{r}_2)\chi_s$ for a fermionic system consisting of two spin-half particles. The spatial part of the wavefunction is given by

$$\psi(\vec{r}_1, \vec{r}_2) = \frac{1}{\sqrt{2}}[\phi_1(\vec{r}_1)\phi_2(\vec{r}_2) + \phi_2(\vec{r}_1)\phi_1(\vec{r}_2)]$$

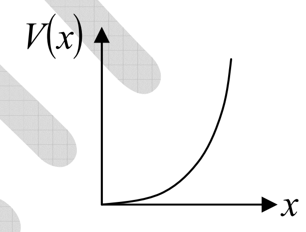
where ϕ_1 and ϕ_2 are single particle states. The spin part χ_s of the wavefunction with spin states $\alpha(+1/2)$ and $\beta(-1/2)$ should be

(a) $\frac{1}{\sqrt{2}}(\alpha\beta + \beta\alpha)$ (b) $\frac{1}{\sqrt{2}}(\alpha\beta - \beta\alpha)$ (c) $\alpha\alpha$ (d) $\beta\beta$

Ans: (b)

Solution: Since $\psi(r_1, r_2)$ is symmetric the total wavefunction must be antisymmetric for fermions so spin part must be antisymmetric.

Q14. A particle is constrained to move in a truncated harmonic potential well ($x > 0$) as shown in the figure. Which one of the following statements is CORRECT?



(a) The parity of the first excited state is even

(b) The parity of the ground state is even

(c) the ground state energy is $\frac{1}{2}\hbar\omega$

(d) The first excited state energy is $\frac{7}{2}\hbar\omega$

Ans: (d)

Solution: There is only odd parity. Ground state is $\frac{3}{2}\hbar\omega$ and first excited $= \frac{7}{2}\hbar\omega$

Q15. Consider a system in the unperturbed state described by the Hamiltonian, $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The system is subjected to a perturbation of the form $H' = \begin{pmatrix} \delta & \delta \\ \delta & \delta \end{pmatrix}$, where $\delta << 1$. The energy eigenvalues of the perturbed system using the first order perturbation approximation are

(a) 1 and $(1 + 2\delta)$ (b) $(1 + \delta)$ and $(1 - \delta)$

(c) $(1 + 2\delta)$ and $(1 - 2\delta)$ (d) $(1 + \delta)$ and $(1 - 2\delta)$

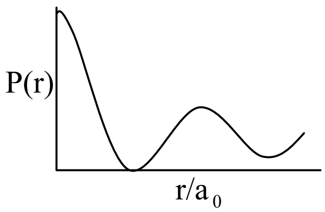
Ans: (a)

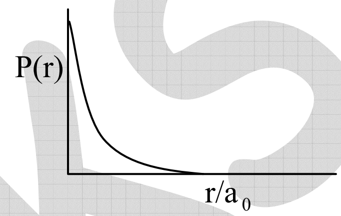
Solution: $H_0 + H'$, H_0 is degenerate so after using degenerate perturbation through diagonalized

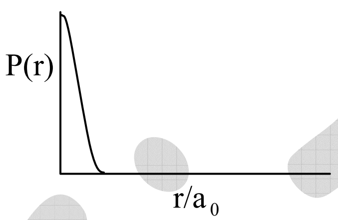
H' one will get $H' = \delta\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$.

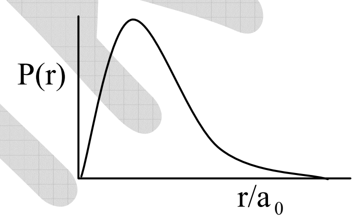
So $E = 1 + 2\delta$ and $1 + 0\delta$.

Q16. The ground state wavefunction for the hydrogen atom is given by $\psi_{100} = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$, where a_0 is the Bohr radius. The plot of the radial probability density, $P(r)$ for the hydrogen atom in the ground state is

(a) 

(b) 

(c) 

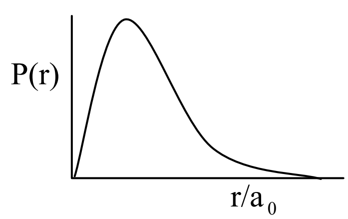
(d) 

Ans: (d)

Solution: The ground state is given by $\psi_{100} = \frac{1}{\sqrt{4\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$

Radial probability function

$$P(r) = |\psi|^2 r^2 = \frac{1}{4\pi}\frac{1}{a_0^3} r^2 e^{-2r/a_0}$$



Common Data for Questions 17–18

The wavefunction of particle moving in free space is given by, $\psi = \left(e^{ikx} + 2e^{-ikx}\right)$

Q17. The energy of the particle is

(a) $\frac{5\hbar^2 k^2}{2m}$ (b) $\frac{3\hbar^2 k^2}{4m}$ (c) $\frac{\hbar^2 k^2}{2m}$ (d) $\frac{\hbar^2 k^2}{m}$

Ans: (c)

Solution: $H\psi = E\psi$, $H\psi = \frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{-\hbar^2}{2m}\left[(ik)(ik)e^{ikx} + 2(-ik)(-ik)e^{-ikx}\right]$

$$\Rightarrow H\psi = \frac{\hbar^2 k^2}{2m}(e^{ikx} + 2e^{-2ikx}) = \frac{\hbar^2 k^2}{2m}\psi$$

Q18. The probability current density for the real part of the wavefunction is

(a) 1 (b) $\frac{\hbar k}{m}$ (c) $\frac{\hbar k}{2m}$ (d) 0

Ans: (d)

Solution: The real part of the wave function $\psi_{real} = \cos kx + 2\cos kx$

Current density for real part of wave function = 0

GATE- 2013

Q19. Which one of the following commutation relations is **NOT CORRECT**? Here, symbols have their usual meanings.

(a) $[L^2, L_z] = 0$ (b) $[L_x, L_y] = i\hbar L_z$

(c) $[L_z, L_+] = \hbar L_+$ (d) $[L_z, L_-] = \hbar L_-$

Ans: (d)

Q20. A proton is confined to a cubic box, whose sides have length $10^{-12} m$. What is the minimum kinetic energy of the proton? The mass of proton is $1.67\times10^{-27} kg$ and Planck's constant is $6.63\times10^{-34} Js$.

(a) $1.1\times10^{-17} J$ (b) $3.3\times10^{-17} J$ (c) $9.9\times10^{-17} J$ (d) $6.6\times10^{-17} J$

Ans: (c)

Solution: $\frac{3\pi^2\hbar^2}{2ma^2} = 9.9\times10^{-17}$

Q21. A spin-half particle is in a linear superposition $0.8|\uparrow\rangle + 0.6|\downarrow\rangle$ of its spin-up and spin-down states. If $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of σ_z , then what is the expectation value up to one decimal place, of the operator $10\sigma_z + 5\sigma_x$? Here, symbols have their usual meanings. _______________

Ans: 7.6

Solution: $|\psi\rangle = .8|\uparrow\rangle + .6|\downarrow\rangle = 0.8\begin{pmatrix}1\\0\end{pmatrix} + 0.6\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0.8\\0.6\end{pmatrix}$

Operator $A = 10\sigma_z + 5\sigma_x = 10\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} + 5\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \Rightarrow A = \begin{pmatrix}10 & 5\\5 & -10\end{pmatrix}$

$$\langle A\rangle = \langle\psi|A|\psi\rangle = \begin{pmatrix}0.8 & 0.6\end{pmatrix}\begin{pmatrix}10 & 5\\5 & -10\end{pmatrix}\begin{pmatrix}0.8\\0.6\end{pmatrix} = (8.8-1.2) = 7.6$$

Q22. Consider the wave function $Ae^{ikr}(r_0/r)$, where A is the normalization constant. For $r = 2r_0$, the magnitude of probability current density up to two decimal places, in units of $(A^2\hbar k/m)$ is ______________

Ans: 0.25

Solution: $\vec{J} = |\psi|^2 \frac{\hbar k}{m} = |A|^2 \left|\frac{r_0}{r}\right|^2 \frac{\hbar k}{m} \Rightarrow J = |A|^2 \left|\frac{r_0}{2r_0}\right|^2 \frac{\hbar k}{m} \Rightarrow J = |A|^2 \frac{\hbar k}{4m} = (0.25)|A|^2 \frac{\hbar k}{m}$

Common data questions 23 and 24

To the given unperturbed Hamiltonian $\begin{bmatrix}5 & 2 & 0\\2 & 5 & 0\\0 & 0 & 2\end{bmatrix}$

we add a small perturbation given by $\varepsilon\begin{bmatrix}1 & 1 & 1\\1 & 1 & -1\\1 & -1 & 1\end{bmatrix}$ where ε is small quantity.

Q23. The ground state eigenvector of the unperturbed Hamiltonian is

(a) $(1/\sqrt{2}, 1\sqrt{2}, 0)$

(b) $(1/\sqrt{2}, -1/\sqrt{2}, 0)$

(c) $(0,0,1)$

(d) $(1,0,0)$

Ans: (c)

$$H_0 = \begin{bmatrix}5 & 2 & 0\\2 & 5 & 0\\0 & 0 & 2\end{bmatrix},\ H_P = \varepsilon\begin{pmatrix}1 & 1 & 1\\1 & 1 & -1\\1 & -1 & 1\end{pmatrix}$$

Eigen value of H_0 is $E_1 = 2,\ E_2 = 3,\ E_3 = 7$ and the Eigen vector corresponds to $|\phi_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $|\phi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Q24. A pair of eigenvalues of the perturbed Hamiltonian, using first order perturbation theory, is

(a) $3+2\varepsilon, 7+2\varepsilon$ (b) $3+2\varepsilon, +2+\varepsilon$ (c) $3, 7+2\varepsilon$ (d) $3, 2+2\varepsilon$

Ans: (c)

Solution: $E_1' = \langle \phi_1 | H_P | \phi_1 \rangle = 1\varepsilon \Rightarrow E_1 = 2 + 1\varepsilon$

$$E_2' = \langle \phi_2 | H_P | \phi_2 \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}.\varepsilon\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \in\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0$$

$$E_3' = \langle \phi_3 | H_P | \phi_3 \rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}.\varepsilon\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}.\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \varepsilon.\frac{1}{2}\begin{pmatrix} 2 & 2 & 0 \end{pmatrix}.\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow E_3' = \frac{1}{2}(4)\varepsilon = 2\varepsilon$$

$E_1 = 2 + 1\varepsilon$, $E_2 = 3 + 0\varepsilon$, $E_3 = 7 + 2\varepsilon$.

GATE- 2014

Q25. The recoil momentum of an atom is p_A when it emits an infrared photon of wavelength $1500\,nm$, and it is p_B when it emits a photon of visible wavelength $500\,nm$. The ratio $\frac{p_A}{p_B}$ is

(a) 1 : 1 (b) $1:\sqrt{3}$ (c) 1 : 3 (d) 3 : 2

Ans: (c)

Solution: $p = \frac{h}{\lambda}$, $\frac{p_A}{p_B} = \frac{\lambda_B}{\lambda_A}$, $\frac{\lambda_B}{\lambda_A} = \frac{500}{1500}$ =1 : 3

Q26. The ground state and first excited state wave function of a one dimensional infinite potential well are ψ_1 and ψ_2 respectively. When two spin-up electrons are placed in this potential which one of the following with x_1 and x_2 denoting the position of the two electrons correctly represents the space part of the ground state wave function of the system?

(a) $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_1)-\psi_1(x_2)\psi_2(x_2)]$ (b) $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2)+\psi_1(x_2)\psi_2(x_1)]$

(c) $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_1)+\psi_1(x_2)\psi_2(x_2)]$ (d) $\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2)-\psi_1(x_2)\psi_2(x_1)]$

Ans: (d)

Solution: From the given information only possible spin configuration is symmetric in nature so space part will anti symmetric

$$\frac{1}{\sqrt{2}}[\psi_1(x_1)\psi_2(x_2)-\psi_1(x_2)\psi_2(x_1)]$$

Q27. If $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the spin angular momentum, then $\vec{L}.\vec{S}$ does not commute with

(a) S_z (b) L^2 (c) S^2 (d) $\left(\vec{L}+\vec{S}\right)^2$

Ans: (d)

Q28. An electron in the ground state of the hydrogen atom has the wave function $\psi(\vec{r})=\frac{1}{\sqrt{\pi a_0^3}}e^{-\left(\frac{r}{a_0}\right)}$, where a_0 is constant. The expectation value of the operator $\hat{Q}=z^2-r^2$, where $z=r\cos\theta$ is (Hint: $\int_0^\infty e^{-ar}r^n dr=\frac{\Gamma n}{a^{n+1}}=\frac{(n-1)!}{a^{n+1}}$)

(a) $\frac{-a_0^2}{2}$ (b) $-a_0^2$ (c) $\frac{-3a_0^2}{2}$ (d) $-2a_0^2$

Ans: (d)

Solution: $\langle\hat{Q}\rangle=\langle z^2\rangle-\langle r^2\rangle \quad \Rightarrow a_0^2-3a_0^2=-2a_0^2$

Q29. A particle of mass m is subjected to a potential

$$V(x,y)=\frac{1}{2}m\omega^2\left(x^2+y^2\right),-\infty\le x\le\infty,-\infty\le y\le\infty$$

The state with energy $4\hbar\omega$ is $g-$fold degenerate. The value of g is ______

Ans: 4

Solution: This is two isotropic dimensional harmonic oscillator the energy eigen value for nth state is $E_n = (n+1)\hbar\omega$ with degeneracy $g_n = (n+1)$ so degeneracy for $4\hbar\omega$ is 4.

Q30. A hydrogen atom is in the state

$$\psi = \sqrt{\frac{8}{21}}\psi_{200} - \sqrt{\frac{3}{7}}\psi_{310} + \sqrt{\frac{4}{21}}\psi_{321},$$

where n, l, m in ψ_{nlm} denote the principal, orbital and magnetic quantum numbers, respectively. If $\vec{L}$ is the angular momentum operator, then the average value of L^2 is_______$\hbar^2$

Ans: 2

Solution: If L^2 will measure on state ψ the measurement is $0\hbar^2$, $2\hbar^2$ and $6\hbar^2$ with probability $\frac{8}{21}, \frac{3}{7}, \frac{4}{21}$ so, $\langle L^2 \rangle = 2\hbar^2 \times \frac{3}{7} + 6\hbar^2 \times \frac{4}{21} = 2\hbar^2$

Q31. ψ_1 and ψ_2 are two orthogonal states of a spin $\frac{1}{2}$ system. It is given that

$\psi_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{2}{3}}\begin{pmatrix}0\\1\end{pmatrix}$, where $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ represent the spin-up and spin-down states, respectively. When the system is in the state ψ_2 its probability to be in the spin-up state is_______

Ans: $\frac{2}{3}$

Solution: If is $\psi_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{2}{3}}\begin{pmatrix}0\\1\end{pmatrix}$, then $\psi_2 = \sqrt{\frac{2}{3}}\begin{pmatrix}1\\0\end{pmatrix} + \sqrt{\frac{1}{3}}\begin{pmatrix}0\\1\end{pmatrix}$,

so probability that ψ_2 is in up state is $\frac{2}{3}$

Q32. A particle is confined to a one dimensional potential box, with the potential

$$V(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & \text{otherwise} \end{cases}$$

If particle is subjected to a perturbation within the box. $W = \beta x$. Where β is small constant, the first order correction to the ground state energy is

(a) 0 (b) $a\beta/4$ (c) $a\beta/2$ (d) $a\beta$

Ans: (c)

Solution: First order energy correction is $\langle W \rangle = \beta \langle x \rangle$. The average value of position in ground state is $\langle x \rangle = \frac{a}{2}$ so answer is $a\beta/2$

Q33. A one dimensional harmonic oscillator is in the superposition of number state $|n\rangle$ given by $|\psi\rangle = \frac{1}{2}|2\rangle + \frac{\sqrt{3}}{2}|3\rangle$.

The average energy of the oscillator in the given state is______$\hbar\omega$.

Ans: 3.25

Solution: Average energy will $\dfrac{\frac{1}{4}\cdot\frac{5\hbar\omega}{2}+\frac{3}{4}\cdot\frac{7\hbar\omega}{2}}{\frac{1}{4}+\frac{3}{4}} = 3.25\hbar\omega$

Q34. If L_+ and L_- are the angular momentum ladder operators then the expectation value of $(L_+L_- + L_-L_+)$ in the state $|l=1, m=1\rangle$ of an atom is _____$\hbar^2$

Ans: 2

Solution: $(L_+L_- + L_-L_+) = 2(L^2 - L_z^2) = 2(l.(l+1) - m^2)\hbar^2 = 2\hbar^2$

GATE- 2015

Q35. An operator for a spin $\frac{1}{2}$ particle is given by $\hat{A}=\lambda\vec{\sigma}\cdot\vec{B}$, where $\vec{B}=\frac{B}{\sqrt{2}}(\hat{x}+\hat{y})$, $\vec{\sigma}$ denotes Pauli matrices and λ is a constant. The eigenvalues of $\hat{A}$ are

(a) $\pm\frac{\lambda B}{\sqrt{2}}$ (b) $\pm\lambda B$ (c) $0,\lambda B$ (d) $0,-\lambda B$

Ans.: (b)

Solution: $\hat{A}=\lambda\vec{\sigma}\cdot\vec{\beta}$, $\vec{B}=\frac{B}{\sqrt{2}}(\hat{x}+\hat{y})$

$$\hat{A}=\lambda\left(\sigma_x B_x+\sigma_y B_y+\sigma_z B_z\right)\Rightarrow \hat{A}=\lambda\left[\sigma_x B_x+\sigma_y B_y\right]$$

$$\hat{A}=\lambda\left[\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\frac{B}{\sqrt{2}}+\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\frac{B}{\sqrt{2}}\right]\Rightarrow \hat{A}=\frac{\lambda B}{\sqrt{2}}\begin{bmatrix}0 & 1-i\\ 1+i & 0\end{bmatrix}$$

$$\left|A-\lambda' I\right|=0\Rightarrow \frac{\lambda B}{\sqrt{2}}\begin{bmatrix}-\lambda & 1-i\\ 1+i & -\lambda\end{bmatrix}=0\Rightarrow \lambda'=\pm\lambda B$$

Q36. The Pauli matrices for three spin $-\frac{1}{2}$ particles are $\vec{\sigma}_1,\vec{\sigma}_2$ and $\vec{\sigma}_3$, respectively. The dimension of the Hilbert space required to define an operator $\hat{O}=\vec{\sigma}_1\cdot\vec{\sigma}_2\times\vec{\sigma}_3$ is________

Ans.: 8

Solution: $\sigma_2\times\sigma_3$ has dimension of 4 and $\sigma_1.\ \sigma_2\times\sigma_3$ has dimension of $2\times 4=8$

Q37. Let $\vec{L}$ and $\vec{p}$ be the angular and linear momentum operators, respectively, for a a particle. The commutator $\left[L_x, p_y\right]$ gives

(a) $-i\hbar\, p_z$ (b) 0 (c) $i\hbar\, p_x$ (d) $i\hbar\, p_z$

Ans.: (d)

Solution: $\left[L_x,p_y\right]=\left[yp_z-zp_y,p_y\right]=\left[yp_z,p_y\right]-\left[zp_y,p_y\right]=\left[y,p_y\right]p_z$

$\because\left[p_y,p_y\right]=0$ and $\left[z,p_y\right]=0\Rightarrow\left[L_x,p_y\right]=i\hbar p_z$ $\quad\because\left[y,p_y\right]=i\hbar$

Q38. Consider a system of eight non-interacting, identical quantum particles of spin $-\frac{3}{2}$ in a one dimensional box of length L. The minimum excitation energy of the system, in units of $\frac{\pi^2\hbar^2}{2mL^2}$ is ________

Ans.: 5

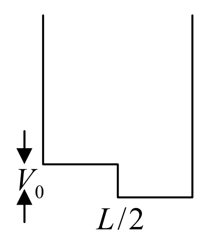
Solution: spin $\frac{3}{2}$ $\Rightarrow$ degeneracy $=(2S+1)=\left(2\times\frac{3}{2}+1\right)=4$

$$E_{ground}=4\times\frac{\pi^2\hbar^2}{2mL^2}+4\times\frac{4\pi^2\hbar^2}{2mL^2}=\frac{20\pi^2\hbar^2}{2mL^2}$$

$$E_{excited}^{I^{st}}=4\times\frac{\pi^2\hbar^2}{2mL^2}+3\times4\frac{\pi^2\hbar^2}{2mL^2}+1\times9\frac{\pi^2\hbar^2}{2mL^2}=25\frac{\pi^2\hbar^2}{2mL^2}$$

Now minimum excitation energy $\Delta E=E_{excited}^{I^{st}}-E_{ground}=25\frac{\pi^2\hbar^2}{2mL^2}-20\frac{\pi^2\hbar^2}{2mL^2}=5\frac{\pi^2\hbar^2}{2mL^2}$

Q39. A particle is confined in a box of length L as shown in the figure. If the potential V_0 is treated as a perturbation, including the first order correction, the ground state energy is



(a) $E=\frac{\hbar^2\pi^2}{2mL^2}+V_0$

(b) $E=\frac{\hbar^2\pi^2}{2mL^2}-\frac{V_0}{2}$

(c) $E=\frac{\hbar^2\pi^2}{2mL^2}+\frac{V_0}{4}$

(d) $E=\frac{\hbar^2\pi^2}{2mL^2}+\frac{V_0}{2}$

Ans.: (d)

Solution: $$E_0^1=\frac{2}{L}\left[\int_0^{\frac{L}{2}}V_0\sin^2\frac{\pi x}{L}dx+\int_{\frac{L}{2}}^{L}0\times\sin^2\frac{\pi x}{L}dx\right]$$

$$\Rightarrow E_0^1=\frac{2}{L}\frac{V_0}{2}\int_0^{\frac{L}{2}}\left(1-\cos\frac{2\pi x}{L}\right)dx=\frac{V_0}{L}\left[x-\sin\left(\frac{2\pi x}{L}\right)\frac{L}{2\pi}\right]_0^{\frac{L}{2}}$$

$$\Rightarrow E_0^1=\frac{V_0}{2}\Rightarrow E=\frac{\hbar^2\pi^2}{2mL^2}+\frac{V_0}{2}$$

Q40. Let the Hamiltonian for two spin-½ particles of equal masses m, momenta $\vec{p}_1$ and $\vec{p}_2$ and positions $\vec{r}_1$ and $\vec{r}_2$ be $H = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}m\omega^2\left(r_1^2 + r_2^2\right) + k\vec{\sigma}_1 \cdot \vec{\sigma}_2$, where $\vec{\sigma}_1$ and $\vec{\sigma}_2$ denote the corresponding Pauli matrices, $\hbar\omega = 0.1eV$ and $k = 0.2\ eV$. If the ground state has net spin zero, then the energy (in eV) is ___________

Ans.: -0.3

Solution: $H = \frac{1}{2m}p_1^2 + \frac{1}{2m}p_2^2 + \frac{1}{2}m\omega^2\left(r_1^2 + r_2^2\right) + k\vec{\sigma}_1.\vec{\sigma}_2$

$$\vec{\sigma} = \vec{\sigma}_1 + \vec{\sigma}_2 \quad \Rightarrow \vec{\sigma}^2 = \sigma_1^2 + \sigma_2^2 + 2\vec{\sigma}_1.\vec{\sigma}_2 \Rightarrow 2\vec{\sigma}_1.\vec{\sigma}_2 = \vec{\sigma}^2 - \sigma_1^2 - \sigma_2^2$$

$$\Rightarrow 2\vec{\sigma}_1.\vec{\sigma}_2 = 0 - 3I - 3I = -6I \Rightarrow \vec{\sigma}_1.\vec{\sigma}_2 = -3$$

Now energy $E = 2 \times \frac{3}{2}\hbar\omega + k(-3) = 3 \times (0.1) + (0.2)(-3) = -0.3\,eV$

Q41. Suppose a linear harmonic oscillator of frequency ω and mass m is in the state $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\psi_0\rangle + e^{i\frac{\pi}{2}}|\psi_1\rangle\right]$ at $t = 0$ where $|\psi_0\rangle$ and $|\psi_1\rangle$ are the ground and the first excited states, respectively. The value of $\langle\psi|x|\psi\rangle$ in the units of $\sqrt{\frac{\hbar}{m\omega}}$ at $t = 0$ is _____

Ans. : 0

Solution: $|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\psi_0\rangle + e^{\frac{i\pi}{2}}|\psi_1\rangle\right]$

$$\langle\psi|x|\psi\rangle = \langle\psi|\sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)|\psi\rangle \Rightarrow \langle\psi|x|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle\psi|a|\psi\rangle + \langle\psi|a^\dagger|\psi\rangle\right)$$

$$\because a|\psi\rangle = \frac{1}{\sqrt{2}}e^{\frac{i\pi}{2}}|\psi_0\rangle \quad \text{and} \quad a^\dagger|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1\rangle + \sqrt{2}e^{\frac{i\pi}{2}}|\psi_2\rangle\right)$$

$$\Rightarrow \langle\psi|x|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{\frac{i\pi}{2}}\langle\psi_0|\psi_0\rangle + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}e^{-\frac{i\pi}{2}}\langle\psi_1|\psi_1\rangle\right)$$

$$\Rightarrow \langle\psi|x|\psi\rangle = \sqrt{\frac{\hbar}{2m\omega}}(0) = 0$$

GATE-2016

Q42. Which of the following operators is Hermitian?

(a) $\frac{d}{dx}$ (b) $\frac{d^2}{dx^2}$ (c) $i\frac{d^2}{dx^2}$ (d) $\frac{d^3}{dx^3}$

Ans. : (b)

Q43. The scattering of particles by a potential can be analyzed by Born approximation. In particular, if the scattered wave is replaced by an appropriate plane wave, the corresponding Born approximation is known as the first Born approximation. Such an approximation is valid for

(a) large incident energies and weak scattering potentials.

(b) large incident energies and strong scattering potentials.

(c) small incident energies and weak scattering potentials.

(d) small incident energies and strong scattering potentials.

Ans.: (a)

Q44. Consider an elastic scattering of particles in $l = 0$ states. If the corresponding phase shift δ_0 is 90^0 and the magnitude of the incident wave vector is equal to $\sqrt{2\pi}\ fm^{-1}$ then the total scattering cross section in units of fm^2 is _______.

Ans.: 2

Solution: $\sigma = \frac{4\pi}{k^2}\sum_{l=0}(2l+1)\sin\delta_0$ for $l = 0$, it is given $\delta_0 = 90^0$ and $k = \sqrt{2\pi}\ fm^{-1}$

$$\sigma = \frac{4\pi}{2\pi}\sin 90 = 2$$

Q45. A hydrogen atom is in its ground state. In the presence of a uniform electric field $\vec{E} = E_0\hat{z}$, the leading order change in its energy is proportional to $(E_0)^n$. The value of the exponent n is _______.

Ans.: 2

Solution: First order energy correction is zero $\langle\psi_{1,0,0}|E_0 r\cos\theta|\psi_{1,0,0}\rangle = 0$

So one need to find correction of second $\sum_{n\neq 1}\frac{\left|\langle\psi_{n,l,m}|E_0 r\cos\theta|\psi_{1,0,0}\rangle\right|^2}{E_1^0 - E_m^0} \propto E_0^2$

So value of $n = 2$

Q46. If $\vec{s}_1$ and $\vec{s}_2$ are the spin operators of the two electrons of a He atom, the value of $\langle \vec{s}_1.\vec{s}_2 \rangle$ for the ground state is

(a) $-\frac{3}{2}\hbar^2$ (b) $-\frac{3}{4}\hbar^2$ (c) 0 (d) $\frac{1}{4}\hbar^2$

Ans.: (b)

Solution: $\vec{s} = \vec{s}_1 + \vec{s}_2$, $s_1 = \frac{1}{2}, s_1 = \frac{1}{2}, s = 0,1$, $\langle \vec{s}_1.\vec{s}_2 \rangle = \frac{s(s+1)\hbar^2 - s_1(s_1+1)\hbar^2 - s_2(s_2+1)\hbar^2}{2}$

For $s = 1, \langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{2\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2}{2} = \frac{3}{4}\hbar^2$

$s = 0, \langle \vec{s}_1 \cdot \vec{s}_2 \rangle = \frac{0\hbar^2 - \frac{3}{4}\hbar^2 - \frac{3}{4}\hbar^2}{2} = -\frac{3}{4}\hbar^2$

Q47. A two-dimensional square rigid box of side L contains six non-interacting electrons at $T = 0K$. The mass of the electron is m. The ground state energy of the system of electrons, in units of $\frac{\pi^2\hbar^2}{2mL^2}$ is ________.

Ans.: 24

Solution: $2 \times \frac{(1^2+1^2)\pi^2\hbar^2}{2mL^2} + 4 \times \frac{(2^2+1^2)\pi^2\hbar^2}{2mL^2} = \frac{24\pi^2\hbar^2}{2mL^2}$

Q48. σ_x, σ_y and σ_z are the Pauli matrices. The expression $2\sigma_x\sigma_y + \sigma_y\sigma_x$ is equal to

(a) $-3i\sigma_z$ (b) $-i\sigma_z$ (c) $i\sigma_z$ (d) $3i\sigma_z$

Ans.: (c)

Solution: $2\sigma_x\sigma_y + \sigma_y\sigma_x \Rightarrow \sigma_x\sigma_y + \sigma_x\sigma_y + \sigma_y\sigma_x \Rightarrow \sigma_x\sigma_y = i\sigma_z$

Q49. If x and p are the x components of the position and the momentum operators of a particle respectively, the commutator $[x^2, p^2]$ is

(a) $i\hbar(xp - px)$ (b) $2i\hbar(xp - px)$ (c) $i\hbar(xp + px)$ (d) $2i\hbar(xp + px)$

Ans.: (d)

Solution: $[x^2, p^2] = p[x^2, p] + [x^2 p]p = 2i\hbar px + 2i\hbar xp \Rightarrow 2i\hbar(xp + px)$

Q50. Let $|l,m\rangle$ be the simultaneous eigenstates of L^2 and L_z. Here $\vec{L}$ is the angular momentum operator with Cartesian components (L_x, L_y, L_z), l is the angular momentum quantum number and m is the azimuthal quantum number. The value of $\langle 1,0|(L_x + iL_y)|1,-1\rangle$ is

(a) 0 (b) $\hbar$ (c) $\sqrt{2}\hbar$ (d) $\sqrt{3}\hbar$

Ans.: (c)

Solution: $\langle 1,0|(L_x + iL_y|1,-1\rangle = \langle 1,0|L_+|1,-1\rangle = \sqrt{2}\hbar\langle 1,0|1,0\rangle = \sqrt{2}\hbar$

Q51. For the parity operator P, which of the following statements is **NOT** true?

(a) $P^\dagger = P$ (b) $P^2 = -P$ (c) $P^2 = I$ (d) $P^\dagger = P^{-1}$

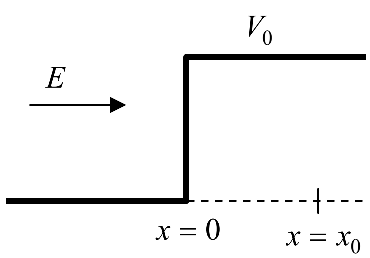
Ans.: (b)

Q52. The state of a system is given by $|\psi\rangle = |\phi_1\rangle + 2|\phi_2\rangle + 3|\phi_3\rangle$, where $|\phi_1\rangle, |\phi_2\rangle and |\phi_3\rangle$ form an orthonormal set. The probability of finding the system in the state $|\phi_2\rangle$ is ________. (Give your answer upto two decimal places)

Ans. : 0.28

Solution: Probability that ψ in state $|\phi_2\rangle = \frac{2^2}{1^2+2^2+3^2} = \frac{4}{1+4+9} = \frac{4}{14} = \frac{2}{7} = 0.28$

Q53. A particle of mass m and energy E, moving in the positive x direction, is incident on a step potential at $x = 0$, as indicated in the figure. The height of the potential is V_0, where $V_0 > E$. At $x = x_0$, where $x_0 > 0$, the probability of finding the electron is $\frac{1}{e}$ times the probability of finding it at $x = 0$. If $\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$, the value of x_0 is



(a) $\frac{2}{\alpha}$ (b) $\frac{1}{\alpha}$ (c) $\frac{1}{2\alpha}$ (d) $\frac{1}{4\alpha}$

Ans.: (c)

Solution: $\frac{1}{e} = e^{-2\alpha x_0} = e^{-1} = e^{-2\alpha x_0} \Rightarrow x_0 = \frac{1}{2\alpha}$

GATE- 2017

Q54. The Compton wavelength of a proton is…………….. fm. (up to two decimal places).

Ans. : 3×10^{8}

Solution: $\left(m_p=1.67\times10^{-27}\,kg, h=6.626\times10^{-34}\,Js, e=1.602\times10^{-19}\,C, c=3\times10^{8}\,ms^{-1}\right)$

Q55. A one dimensional simple harmonic oscillator with Hamiltonian $H_0=\frac{p^2}{2m}+\frac{1}{2}kx^2$ is subjected to a small perturbation, $H_1=\alpha x+\beta x^3+\gamma x^4$. The first order correction to the ground state energy is dependent on

(a) only β (b) α and γ (c) α and β (d) only γ

Ans. : (d)

Solution: $H_1=\alpha x+\beta x^3+\gamma x^4$, $E_g^1=\alpha\langle x\rangle+\beta\langle x^3\rangle+\gamma\langle x^4\rangle$, $\langle x\rangle=0,\langle x^3\rangle=0,\langle x^4\rangle\neq0$

Q56. For the Hamiltonian $H=a_0I+\vec{b}.\vec{\sigma}$ where $a_0\in R,\vec{b}$ is a real vector, I is the 2×2 identity matrix, and $\vec{\sigma}$ are the Pauli matrices, the ground state energy is

(a) $|b|$ (b) $2a_0-|b|$ (c) $a_0-|b|$ (d) a_0

Ans. : (c)

Solution: $$a_0I+\vec{b}.\vec{\sigma}=a_0\begin{pmatrix}1&0\\0&1\end{pmatrix}+b_x\begin{pmatrix}0&1\\1&0\end{pmatrix}+b_y\begin{pmatrix}0&-i\\i&0\end{pmatrix}+b_z\begin{pmatrix}1&0\\0&-1\end{pmatrix}=\begin{pmatrix}a_0+b_z&b_x-ib_y\\b_x+ib_y&a_0-b_z\end{pmatrix}$$

$$H=a_0I+\vec{b}.\vec{\sigma}=\begin{pmatrix}a_0+b_z&b_x-ib_y\\b_x+ib_y&a_0-b_z\end{pmatrix}$$

For eigen value $\begin{vmatrix}a_0+b_z-\lambda&b_x-ib_y\\b_x+ib_y&a_0-b_z-\lambda\end{vmatrix}=0$

$$(a_0+b_z-\lambda)(a_0-b_z-\lambda)-(b_x^2+b_y^2)=0$$

$$\lambda_1=a_0-|b|,\lambda_1=a_0+|b|$$

Q57. The degeneracy of the third energy level of a 3-dimensional isotropic quantum harmonic oscillator is

(a) 6 (b) 12 (c) 8 (d) 10

Ans. : (a)

Solution: First energy level is $n=0$

Second energy level is $n=1$

Third energy level is $n=2$

Degeneracy of third level $\frac{(n+1)(n+2)}{2}=\frac{3\times 4}{2}=6$

Q58. A free electron of energy $1eV$ is incident upon a one-dimensional finite potential step of height $0.75eV$. The probability of its reflection from the barrier is………… (up to two decimal places).

Ans. : 0.11

Solution: $R=\left(\frac{\sqrt{E}-\sqrt{E-V_0}}{\sqrt{E}+\sqrt{E-V_0}}\right)^2=\left(\frac{1-\sqrt{0.25}}{1+\sqrt{0.25}}\right)^2=\left(\frac{1-0.5}{1+0.5}\right)^2=0.11$

Q59. Consider a one-dimensional potential well of width $3nm$. Using the uncertainty principle $\left(\Delta x\cdot\Delta p\geq\frac{\hbar}{2}\right)$, an estimate of the minimum depth of the well such that it has at least one bound state for an electron is ($m_e=9.31\times10^{-31}kg$, $h=6.626\times10^{-34}Js$, $e=1.602\times10^{-19}C$)

(a) $1\mu eV$ (b) $1meV$ (c) $1eV$ (d) $1MeV$

Ans. : (b)

Solution: $E=\frac{p^2}{2m}$, $\Delta p=\frac{\hbar}{2\Delta x}\Rightarrow\Delta p=\frac{\hbar}{2a}$

So, $E=\frac{\hbar^2}{8ma^2}=\frac{h^2}{32\pi^2ma^2}=\frac{(6.6\times10^{-34})^2}{32\times10\times9.31\times10^{-31}\times9\times10^{-18}}=.001\times10^{-19}J\approx1meV$

Q60. The integral $\int_0^\infty x^2e^{-x^2}dx$ is equal to………. (up to two decimal places).

Ans. : 0.44

Solution: The given integral is $\int_0^\infty x^2\,e^{-x^2}\,dx$

Let $x^2=t$ then $2xdx=dt\Rightarrow dx=\frac{dt}{2\sqrt{t}}$

Thus, the given integral can be written as

$$\int_0^{\infty} t\, e^{-t} \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^{\infty} e^{-t}\, t^{1/2}\, dt = \frac{1}{2}\int_0^{\infty} e^{-t}\, t^{\frac{3}{2}-1}\, dt = \frac{1}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4}$$

Hence the value of the integral up to two decimal places is 0.44.

Q61. Which one of the following operators is Hermitian?

(a) $i\dfrac{\left(p_x x^2 - x^2 p_x\right)}{2}$ (b) $i\dfrac{\left(p_x x^2 + x^2 p_x\right)}{2}$

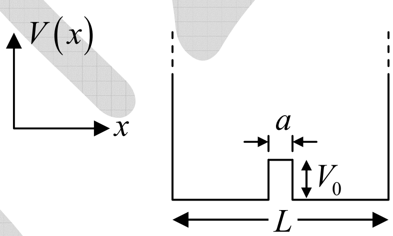
(c) $e^{i p_x a}$ (d) $e^{-i p_x a}$

Ans. : (a)

Solution: $A = i\dfrac{\left(p_x x^2 - x^2 p_x\right)}{2}, A^{\dagger} = -i\dfrac{\left(\left(p_x x^2\right)^{\dagger} - \left(x^2 p_x\right)^{\dagger}\right)}{2} = i\dfrac{\left(p_x x^2 - x^2 p_x\right)}{2}$

GATE-2018

Q62. The ground state energy of a particle of mass m in an infinite potential well is E_0. It changes to $E_0\left(1+\alpha\times 10^{-3}\right)$, when there is a small potential pump of height $V_0 = \dfrac{\pi^2\hbar^2}{50\, mL^2}$ and width $a = L/100$, as shown in the figure. The value of α is ________ (up to two decimal places).



Ans. : 0.81

Solution: $\alpha_1 = \left(\dfrac{L}{2} - \dfrac{a}{2}\right), \alpha_2 = \left(\dfrac{L}{2} + \dfrac{a}{2}\right), \quad a = \dfrac{L}{100}$

$$E_1 = V_0 \int_{\alpha_1}^{\alpha_2} \left(\sqrt{\frac{2}{L}}\right)^2 \sin^2\left(\frac{\pi x}{L}\right) dx$$

$$= \frac{V_0}{L}\int_{\alpha_1}^{\alpha_2}\left[1 - \cos\frac{2\pi x}{L}\right] dx = \frac{V_0}{L}\left[x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_{\alpha_1}^{\alpha_2}$$

$$=\frac{V_0}{L}\left[a-\frac{L}{2\pi}\left(\sin\frac{2\pi(L+a)}{2L}-\sin\frac{2\pi(L-a)}{2L}\right)\right]$$

$$=\frac{V_0}{L}\left[\frac{L}{100}-\frac{L}{2\pi}\left(\sin\left(\pi+\frac{\pi a}{L}\right)-\sin\left(\pi-\frac{\pi a}{L}\right)\right)\right]$$

$$=V_0\left[\frac{1}{100}+\frac{1}{2\pi}(0.0314+0.0314)\right]$$

$$=V_0\times10^{-3}(10+10)=E_0\times10^{-3}\times\left(\frac{20}{25}\right)\Rightarrow \alpha E_0\times10^{-3}=0.81\times E_0\times10^{-3}$$

Hence, $\alpha=0.81$

Q63. A two-state quantum system has energy eigenvalues $\pm\in$ corresponding to the normalized states $|\psi_\pm\rangle$. At time $t=0$, the system is in quantum state $\frac{1}{\sqrt{2}}\left[|\psi_+\rangle+|\psi_-\rangle\right]$. The probability that the system will be in the same state at $t=h/(6\in)$ is _________ (up to two decimal places).

Ans. : 0.25

Solution: $|\psi(0)\rangle=\frac{1}{\sqrt{2}}\left[|\psi_+\rangle+|\psi_-\rangle\right]$

And $|\psi(t)\rangle=\frac{1}{\sqrt{2}}\left[|\psi_+\rangle e^{-\frac{i\in t}{\hbar}}+|\psi_-\rangle e^{\frac{i\in t}{\hbar}}\right]$

At $t=\frac{\hbar}{6\in}$,

$$|\psi(t)\rangle=\frac{1}{\sqrt{2}}\left[|\psi_+\rangle e^{-\frac{i\in h\times2\pi}{6\in h}}+|\psi\rangle e^{\frac{i\in h\times2\pi}{6\in h}}\right]=\frac{1}{\sqrt{2}}\left[|\psi_+\rangle e^{\frac{-i\pi}{3}}+|\psi_-\rangle e^{\frac{i\pi}{3}}\right]$$

Now, probability in same state

$$P=\frac{\left|\langle\psi(t)|\psi(0)\rangle\right|^2}{\langle\psi|\psi\rangle}=\frac{1}{4}\left|e^{-i\pi/3}+e^{i\pi/3}\right|^2=\frac{1}{4}\left|2\cos\frac{\pi}{3}\right|^2=\frac{1}{4}\times\left|2\times\frac{1}{2}\right|^2=0.25$$

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Q64. An electric field $\vec{E} = E_0\hat{z}$ is applied to a Hydrogen atom in $n=2$ excited state. Ignoring spin the $n=2$ state is fourfold degenerate, which in the $|l,m\rangle$ basis are given by $|0,0\rangle, |1,1\rangle, |1,0\rangle$ and $|1,-1\rangle$. If H' is the interaction Hamiltonian corresponding to the applied electric field, which of the following matrix elements is nonzero?

(a) $\langle 0,0|H'|0,0\rangle$ (b) $\langle 0,0|H'|1,1\rangle$

(c) $\langle 0,0|H'|1,0\rangle$ (d) $\langle 0,0|H'|1,-1\rangle$

Ans. : (c)

Q65. For a spin $\frac{1}{2}$ particle, let $|\uparrow\rangle$ and $|\downarrow\rangle$ denote its spin up and spin down states respectively. If $|a\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$ and $|b\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$ are composite states of two such particles, which of the following statements is true for their total spin S?

(a) $S=1$ for $|a\rangle$ and $|b\rangle$ is not an eigenstate of the operator $\hat{S}^2$

(b) $|a\rangle$ is not an eigenstate of the operator $\hat{S}^2$ and $S=0$ for $|b\rangle$

(c) $S=0$ for $|a\rangle$, and $S=1$ for $|b\rangle$

(d) $S=1$ for $|a\rangle$, and $S=0$ for $|b\rangle$

Ans. : (d)

Solution: $S=1$ is triplet $|a\rangle$, and $S=0$ for singlet for $|b\rangle$

Q66. The Hamiltonian for a quantum harmonic oscillator of mass m in three dimensions is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 r^2$$

where ω is the angular frequency. The expectation value of r^2 in the first excited state of the oscillator in units of $\frac{\hbar}{m\omega}$ (rounded off to one decimal place) is___________

Ans. : 2.5

Solution: $\langle r^2\rangle = \langle x^2\rangle + \langle y^2\rangle + \langle z^2\rangle$

$$=\frac{\hbar}{2m\omega}\left[\left(2n_x+1\right)+\left(2n_y+1\right)+\left(2n_z+1\right)\right]$$

For first excited state $n_x=1, n_y=0, n_z=0$

Hence it is triply degenerate one can take

$n_x=0,\ n_y=1,\ n_z=0$ or $n_x=0, n_y=0,\ n_z=1$

putting any one combination, expectation value of $r^2=\frac{5}{2}\frac{\hbar}{m\omega}=2.5\frac{\hbar}{m\omega}$

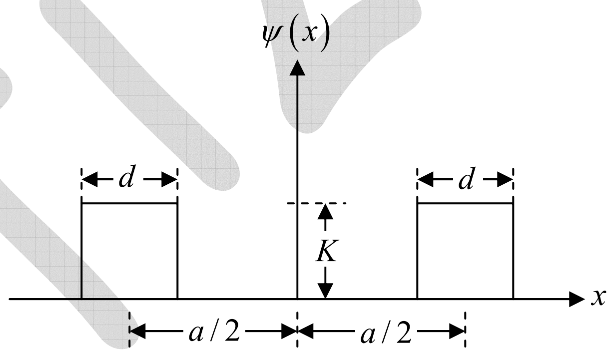
Q67. Let $|\psi_2\rangle=\begin{pmatrix}1\\0\end{pmatrix}$, $|\psi_2\rangle=\begin{pmatrix}0\\1\end{pmatrix}$ represent two possible states of a two-level quantum system. The state obtained by the incoherent superposition of $|\psi_1\rangle$ and $|\psi_2\rangle$ is given by a density matrix that is defined as $\rho\equiv c_1|\psi_1\rangle\langle\psi_1|+c_2|\psi_2\rangle\langle\psi_2|$. If $c_1=0.4$ and $c_2=0.6$, the matrix element ρ_{22} (rounded off to one decimal place) is __________

Ans. : 0.6

Solution: $\rho_{2,2}=\langle\psi_2|\rho|\psi_2\rangle=\rho\equiv c_1\langle\psi_2|\psi_1\rangle\langle\psi_1|\psi_2\rangle+c_2\langle\psi_2|\psi_2\rangle\langle\psi_2|\psi_2\rangle$

$\Rightarrow c_2=0.6$

Q68. The wave function $\psi(x)$ of a particle is as shown below



Here K is a constant, and $a>d$. The position uncertainty (Δx) of the particle is

(a) $\sqrt{\frac{a^2+3d^2}{12}}$ (b) $\sqrt{\frac{3a^2+d^2}{12}}$ (c) $\sqrt{\frac{d^2}{6}}$ (d) $\sqrt{\frac{d^2}{24}}$

Ans. : (b)

Solution: $\psi(x)=\begin{cases} k, & -\frac{a}{2}-\frac{d}{2}<x<-\frac{a}{2}+\frac{d}{2} \\ 0, & -\frac{a}{2}+\frac{d}{2}<x<\frac{a}{2}-\frac{d}{2} \\ k, & \frac{a}{2}-\frac{d}{2}<x<\frac{a}{2}+\frac{d}{2} \\ 0, & \frac{a}{2}+\frac{d}{2}>0 \end{cases}$

$$\langle\psi|\psi\rangle=1$$

$$k^2\int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}dx+k^2\int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}dx=1$$

$$k^2\left[\left(-\frac{a}{2}+\frac{d}{2}\right)-\left(-\frac{a}{2}-\frac{d}{2}\right)\right]+k^2\left[\left(\frac{a}{2}+\frac{d}{2}\right)-\left(\frac{a}{2}-\frac{d}{2}\right)\right]=1$$

$$k^2\left[\frac{d}{2}+\frac{d}{2}+\frac{d}{2}+\frac{d}{2}\right]=1\Rightarrow k=\frac{1}{\sqrt{2d}}$$

Hence wavefunction is symmetric about $x=0$, so $\langle x\rangle=0$

$$\langle x^2\rangle=k^2\int_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}x^2dx+k^2\int_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}x^2dx$$

$$=\frac{k^2}{3}\left[\left[x^3\right]_{-\frac{a}{2}-\frac{d}{2}}^{-\frac{a}{2}+\frac{d}{2}}+\left[x^3\right]_{\frac{a}{2}-\frac{d}{2}}^{\frac{a}{2}+\frac{d}{2}}\right]$$

$$=\frac{k^2}{3\times 8}\left[(-a+d)^3-(-a-d)^3+(a+d)^3-(a-d)^3\right]$$

$$=\frac{k^2}{24}\left\{\left(-a^3+d^3-3a^2d+3ad^2\right)+\left(a^3+d^3+3a^2d+3ad^2\right)+\left(a^3+d^3+3a^2d+3ad^2\right)\right.$$
$$\left.-\left\{a^3-d^3-3ad(a-d)\right\}\right]$$

$$\langle x^2\rangle=\frac{k^2}{24}\left[4a^3+12a^2d\right]=\frac{4d\left(d^2+3a^2\right)}{24\times 2d}=\frac{3a^2+d^2}{12}$$

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\frac{3a^2+d^2}{12}}$$

Q69. Consider the motion of a particle along the x - axis in a potential $V(x) = F|x|$. Its ground state energy E_0 is estimated using the uncertainty principle. Then E_0 is proportional to

(a) $F^{1/3}$ (b) $F^{1/2}$ (c) $F^{2/5}$ (d) $F^{2/3}$

Ans. : (d)

Solution: $E = \frac{p^2}{2m} + F|x|$ $E = \frac{p^2}{2m} + Fx$ for $x > 0$ $E = \frac{p^2}{2m} - Fx$ < 0 from uncertainty theory

$$\Delta x.\Delta p = \hbar \Rightarrow \Delta p = \frac{\hbar}{\Delta x}$$

$$E = \frac{(\Delta p)^2}{2m} + F(\Delta x) \Rightarrow E = \frac{\hbar^2}{2m(\Delta x)^2} + F\Delta x$$

For minimum energy,

$$\frac{dE}{d\Delta x} = -\frac{\hbar^2}{m(\Delta x)^3} + F = 0 \Rightarrow \Delta x = \left(\frac{\hbar^2}{mF}\right)^{1/3} \frac{\hbar^2}{2m}\left(\frac{mF}{\hbar^2}\right)^{2/3} + F\left(\frac{\hbar^2}{mF}\right)^{1/3} \Rightarrow E \propto F^{2/3}$$

Q70. The Hamiltonian operator for a two-level quantum system is $H = \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$. If the state of the system at $t = 0$ is given by $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $|\langle\psi(0)|\psi(t)\rangle|^2$ at a later time t is

(a) $\frac{1}{2}\left(1 + e^{-(E_1-E_2)t/\hbar}\right)$ (b) $\frac{1}{2}\left(1 - e^{-(E_1-E_2)t/\hbar}\right)$

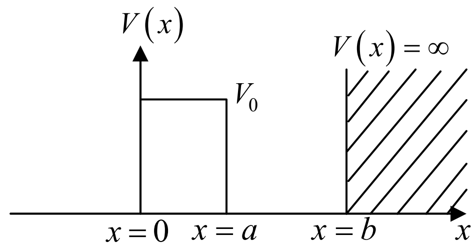
(c) $\frac{1}{2}\left(1 + \cos[E_1 - E_2]t/\hbar\right)$ (d) $\frac{1}{2}\left(1 - \cos[E_1 - E_2]t/\hbar\right)$

Ans. : (c)

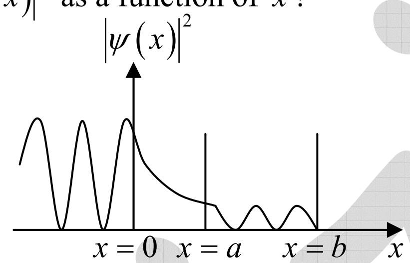
Solution: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $|\psi(t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} \exp -\frac{iE_1 t}{\hbar} \\ \exp -\frac{iE_2 t}{\hbar} \end{pmatrix}$

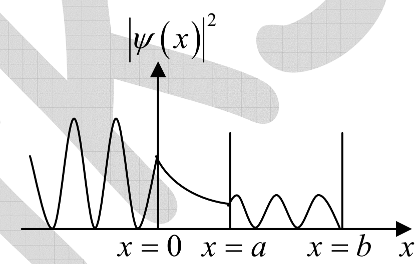
$$\left|\langle\psi(0)|\psi(t)\rangle\right|^2 = \frac{1}{4}\left|\exp-\frac{iE_1t}{\hbar}+\exp-\frac{iE_2t}{\hbar}\right|^2 = \frac{1}{2}\left(1+\cos\left[E_1-E_2\right]t/\hbar\right)$$

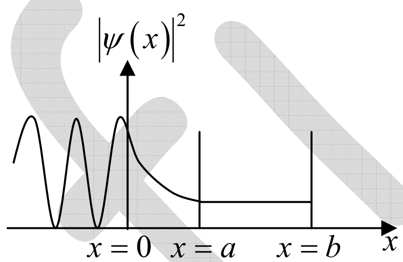
Q71. Consider a potential barrier $V(x)$ of the form:

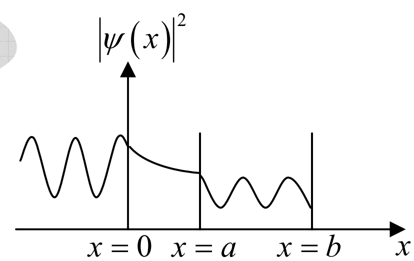


where V_0 is a constant. For particles of energy $E<V_0$ incident on this barrier from the left which of the following schematic diagrams best represents the probability density $|\psi(x)|^2$ as a function of x?

(a)


(b)


(c)


(d)


Ans. : (a)

Q72. The Hamiltonian of a system is $H=\begin{pmatrix}1 & \varepsilon\\ \varepsilon & -1\end{pmatrix}$ with $\varepsilon \ll 1$. The fourth order contribution to the ground state energy of H is $\gamma\varepsilon^4$. The value of γ (rounded off to three decimal places) is_________.

Ans. : 0.125

Solution: $H=\begin{pmatrix}1 & \varepsilon\\ \varepsilon & -1\end{pmatrix}$ the eigen value of the hamiltonion is $E_g=-\sqrt{1-\varepsilon^2}, E_f=+\sqrt{1-\varepsilon^2}$

The ground state is $E_g = -\sqrt{1-\varepsilon^2}$

Taylor expansion of $-\sqrt{1-\varepsilon^2} = -\left(1-\frac{\varepsilon^2}{2}-\frac{\varepsilon^4}{8}.....\right) = -1+\frac{\varepsilon^2}{2}+\frac{\varepsilon^4}{8}.....$

$\gamma = \frac{1}{8} = 0.125$

Q73. Electrons with spin in the z - direction $(\hat{z})$ are passed through a Stern-Gerlach (SG) set up with the magnetic field at $\theta = 60^0$ from $\hat{z}$. The fraction of electrons that will emerge with their spin parallel to the magnetic field in the SG set up (rounded off to two decimal places) is___________

$$\left[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

Ans. : 0.25

Solution: $|\psi\rangle = \begin{pmatrix} \cos 60^0 \\ \sin 60^0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ \sqrt{3}/2 \end{pmatrix}$ state related to up state is $\left|\frac{1}{2},\frac{1}{2}\right\rangle = |\chi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

The fraction of electrons that will emerge with their spin parallel to the magnetic field $|\langle\chi|\psi\rangle|^2 = \frac{1}{4} = 0.25$