Quantum Mechanics

JEST-2012

Q1. The ground state (apart from normalization) of a particle of unit mass moving in a one-dimensional potential V(x) is $\exp\left(-x^2/2\right)\cosh\left(\sqrt{2}x\right)$. The potential V(x), in suitable units so that h = 1, is (up to an addiative constant.)

(a) $\pi^2/2$ (b) $\pi^2/2-\sqrt{2}x\tanh\left(\sqrt{2}x\right)$

(c) $\pi^2/2-\sqrt{2}x\tan\left(\sqrt{2}x\right)$ (d) $\pi^2/2-\sqrt{2}x\coth\left(\sqrt{2}x\right)$

Ans. : (b)

Q2. Consider the Bohr model of the hydrogen atom. If α is the fine-structure constant, the velocity of the electron in its lowest orbit is

(a) $\dfrac{c}{1+\alpha}$ (b) $\dfrac{c}{1+\alpha^2}$ $or\,(1-\alpha)c$ (c) $\alpha^2 c$ (d) αc

Ans. : (d)

Solution: $mvr = n\hbar$

$$\frac{mv^2}{r}=\frac{1}{4\pi\in_0}\frac{ze^2}{r^2}\Rightarrow r=\frac{1}{4\pi\in_0}\frac{ze^2}{mr^2}$$

$$mv\cdot\frac{1}{4\pi\in_0}\frac{ze^2}{mv^2}=n\hbar$$

$v=\dfrac{ze^2}{4\pi\in_0 n\hbar}$ and fine structure constant $\alpha=\dfrac{e^2}{4\pi\in_0\hbar c}$

For lowest orbit, $v=\dfrac{ze^2}{4\pi\in_0\hbar}\Rightarrow v=\dfrac{ze^2}{4\pi\in_0\hbar c}c$

$$v=\alpha c$$

Q3. Define $\sigma_x=\left(f^\dagger+f\right)$, and $\sigma_y=-i\left(f^\dagger-f\right)$, where the σ' are Pauli spin matrices and $f, f^\dagger$ obey anti-commutation relations $\{f,f\}=0,\{f,f^\dagger\}=1$. Then σ_z is given by

(a) $f^\dagger f-1$ (b) $2f^\dagger f-1$ (c) $2f^\dagger f+1$ (d) $f^\dagger f$

Ans. : (c)

Solution: $\sigma_x\sigma_y=i\sigma_z$

$$i\sigma_z=\sigma_x\sigma_y$$

$$\sigma_z = \frac{1}{i}\sigma_x\sigma_y = \frac{-i}{i}\left(f^\dagger + f\right)\left(f^\dagger - f\right) = -\left[\left(f^\dagger\right)^2 - f^\dagger f + ff^\dagger - f^2\right]$$

$$= -\left[-f^\dagger f + \left(1 - f^\dagger .\ f\right)\right] = -\left[1 - 2f^\dagger f\right] = 2f^\dagger f - 1$$

Q4. Consider a system of two spin-$\frac{1}{2}$ particles with total spin $S = S_1 + S_2$, where S_1 and S_2 are in terms of Pauli matrices σ_i. The spin triplet projection operator is

(a) $\frac{1}{4} + S_1 \cdot S_2$ (b) $\frac{3}{4} - S_1 \cdot S_2$ (c) $\frac{3}{4} + S_1 \cdot S_2$ (d) $\frac{1}{4} - S_1 \cdot S_2$

Ans. : (c)

Solution: $\Rightarrow S = S_1 + S_2$ $\quad S^2 = S_1^2 + S_2^2 + 2S_1 \cdot S_2$

$$S^2 = \left(\frac{3}{4} + \frac{3}{4} + 2.S_1 \cdot S_2\right)\hbar^2 \qquad [\because S = 0,\ 1]$$

$$S^2 = 2\left[\frac{3}{4} + S_1 \cdot S_2\right]\hbar^2 \text{ for Triplet projection operator}$$

$$s(s+1)\hbar^2 = 2\left[\frac{3}{4} + S_1 \cdot S_2\right]\hbar^2 \qquad S = 1$$

$$1(1+1) = 2\left(\frac{3}{4} + S_1 \cdot S_2\right) \qquad \Rightarrow \frac{3}{4} + S_1 \cdot S_2 = I$$

Q5. Consider a spin-$\frac{1}{2}$ particle in the homogeneous magnetic field of magnitude B along z-axis which is prepared initially in a state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$ at time $t = 0$. At what time t will the particles be in the state $-|\psi\rangle$ (μ_B is Bohr magneton)?

(a) $t = \frac{\pi\hbar}{\mu_B B}$ (b) $t = \frac{2\pi\hbar}{\mu_B B}$ (c) $t = \frac{\pi\hbar}{2\mu_B B}$ (d) Never

Ans.: (a)

Solution: $\vec{E} = \mu_B \cdot B\hat{z}$ $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

$$|\psi(x,t)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}e^{-\frac{iEt}{b}} \Rightarrow |\psi(x,t)\rangle = -|\psi\rangle$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}e^{\frac{-i\mu_B Bt}{\hbar}}=\frac{-1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$e^{\frac{-i\mu_B Bt}{\hbar}}=-1$$

$$\cos\left(\frac{\mu_B Bt}{\hbar}\right)=\cos\pi$$

$$\frac{\mu_B Bt}{\hbar}=\pi\Rightarrow t=\frac{\hbar\pi}{\mu_B B}$$

Q6. The ground state energy of 5 identical spin-$\frac{1}{2}$ particles which are subject to a one-dimensional simple harmonic oscillator potential of frequency ω is

(a) $\frac{15}{2}\hbar\omega$ (b) $\frac{13}{2}\hbar\omega$ (c) $\frac{1}{2}\hbar\omega$ (d) $5\hbar\omega$

Ans. : (b)

Solution: Degeneracy $=2s+1=2\times\frac{1}{2}+1=2$

$$E_{ground}=2\times\frac{1}{2}\hbar\omega+2\times\frac{3}{2}\hbar\omega+1\times\frac{5}{2}\hbar\omega=\frac{13}{2}\hbar\omega$$

Q7. The spatial part of a two-electron state is symmetric under exchange. If $|\uparrow\rangle$ and $|\downarrow\rangle$ represent the spin-up and spin-down states respectively of each particle, the spin-part of the two-particle state is

(a) $|\uparrow\rangle|\downarrow\rangle$ (b) $|\downarrow\rangle|\uparrow\rangle$

(c) $\left(|\downarrow\rangle|\uparrow\rangle-|\uparrow\rangle|\downarrow\rangle\right)/\sqrt{2}$ (d) $\left(|\downarrow\rangle|\uparrow\rangle+|\uparrow\rangle|\downarrow\rangle\right)/\sqrt{2}$

Ans. : (c)

Solution: Since, electrons are Fermions and Fermions have anti-symmetric wave function

$\because$ spatial part is symmetric then its spin part is antisymmetric to maintain antisymmtric wave function

$$\psi(x)=\frac{1}{\sqrt{2}}\left(|\downarrow\rangle|\uparrow\rangle-|\uparrow\rangle|\downarrow\rangle\right)$$

Q8. The wave function of a free particle in one dimension is given by $\psi(x) = A\sin x + B\sin 3x$. Then $\psi(x)$ is an eigenstate of

(a) the position operator (b) the Hamiltonian

(c) the momentum operator (d) the parity operator

Ans. : (d) $\psi(-x) = \psi(x)$

$= -\psi(x)$ {parity (even and odd)

$\psi(-x) = A\sin(-x) + B\sin(-3x) = -[A\sin x + B\sin 3x]$

$\psi(-x) = -\psi(x) \Rightarrow$ negative parity i.e. parity operator

Q9. The quantum state $\sin x|\uparrow\rangle + \exp(i\phi)\cos x|\downarrow\rangle$, where $\langle\uparrow|\downarrow\rangle = 0$ and x, ϕ are, real, is orthogonal to:

(a) $\sin x|\uparrow\rangle$ (b) $\cos x|\uparrow\rangle + \exp(i\phi)\sin x|\downarrow\rangle$

(c) $-\cos x|\uparrow\rangle - \exp(i\phi)\sin x|\downarrow\rangle$ (d) $-\exp(-i\phi)\cos x|\uparrow\rangle + \sin x|\downarrow\rangle$

Ans.: (d)

Solution: $\langle\uparrow\|\downarrow\rangle = 0$, $|\psi\rangle = \sin x|\uparrow\rangle + \exp(i\phi)\cos x|\downarrow\rangle$

$$\langle\psi'|\psi\rangle = -\exp(i\phi)\cos x\sin x\langle\uparrow|\uparrow\rangle - \exp(i\phi)\exp(i\phi)\cos x\langle\downarrow|\uparrow\rangle$$
$$+\sin^2 x\langle\downarrow|\uparrow\rangle + \exp(i\phi)\cos x\sin x\langle\downarrow|\downarrow\rangle$$

$$= -\exp(i\phi)\cos x\sin x + \exp(i\phi)\cos x\sin x = 0$$

JEST-2013

Q10. A particle of mass m is contained in a one-dimensional infinite well extending from $x=-\frac{L}{2}$ to $x=\frac{L}{2}$. The particle is in its ground state given by $\varphi_0(x)=\sqrt{2/L}\cos(\pi x/L)$. The walls of the box are moved suddenly to form a box extending from $x=-L$ to $x=L$. what is the probability that the particle will be in the ground state after this sudden expansion?

(a) $(8/3\pi)^2$ (b) 0 (c) $(16/3\pi)^2$ (d) $(4/3\pi)^2$

Ans.: (a)

Solution: Probability $|\langle\phi_0|\phi_1\rangle|^2, \phi_0=\sqrt{\frac{2}{L}}\cos\frac{\pi x}{L}, \phi_1\sqrt{\frac{2}{2L}}\cos\frac{\pi x}{2L}$

Since the wall of box are moved suddenly then

$$\text{Probability} =\left|\int_{-L/2}^{L/2}\sqrt{\frac{2}{L}}\cdot\sqrt{\frac{1}{L}}\frac{\cos\pi x}{L}\cdot\frac{\cos\pi x}{2L}dx\right|^2=\left|\frac{\sqrt{2}}{L}\frac{1}{2}\int_{-L/2}^{L/2}\frac{2\cos\pi x}{L}\cdot\frac{\cos\pi x}{2L}dx\right|^2$$

$$\Rightarrow\left|\frac{\sqrt{2}}{L}\cdot\frac{1}{2}\int_{-L/2}^{L/2}\left[\cos\left(\frac{3\pi x}{2L}\right)+\cos\left(\frac{\pi x}{2L}\right)\right]dx\right|^2\Rightarrow\left|\frac{\sqrt{2}}{L}\cdot\frac{1}{2}\left[\frac{2L}{3\pi}\sin\frac{3\pi x}{2L}+\frac{2L}{\pi}\sin\frac{\pi x}{2L}\right]_{-L/2}^{L/2}\right|^2$$

$$\Rightarrow\left|\frac{\sqrt{2}}{L}\cdot\frac{1}{2}\left[\frac{2L}{3\pi}\left(\sin\frac{3\pi}{4}+\sin\frac{3\pi}{4}\right)+\frac{2L}{\pi}\left(\sin\frac{\pi}{4}+\sin\frac{\pi}{4}\right)\right]\right|^2\Rightarrow\left|\frac{2}{3\pi}+\frac{2}{\pi}\right|^2=\left|\frac{8}{3\pi}\right|^2$$

Q11. A quantum mechanical particle in a harmonic oscillator potential has the initial wave function $\psi_0(x)+\psi_1(x)$, where ψ_0 and ψ_1 are the real wavefunctions in the ground and first excited state of the harmonic oscillator Hamiltonian. For convenience we take $m=\hbar=\omega=1$ for the oscillator. What is the probability density of finding the particle at x at time $t=\pi$?

(a) $(\psi_1(x)-\psi_0(x))^2$ (b) $(\psi_1(x))^2-(\psi_0(x))^2$

(c) $(\psi_1(x)+\psi_0(x))^2$ (d) $(\psi_1(x))^2+(\psi_0(x))^2$

Ans.: (a)

Solution: $\psi(x)=\psi_0(x)+\psi_1(x)$

$$\psi(x,t)=\psi_0(x)e^{-i}\frac{E_0t}{\hbar}+\psi_1(x)e^{-i}\frac{E_1t}{\hbar}$$

Now probability density at time t

$$|\psi(x,t)|^2=\psi^*(x,t)\psi(x,t)=|\psi_0(x)|^2+|\psi_1(x)|^2+2\operatorname{Re}\psi_0^*(x)\psi_1(x)\cos(E_1-E_0)\frac{t}{\hbar}$$

putting $t=\pi$

$$|\psi(x,t)|^2=|\psi_0(x)|^2+|\psi_1(x)|^2+2\operatorname{Re}\psi_0^*(x)\psi_1(x)\cos\pi \qquad [\because E_1-E_0=\hbar\omega=1]$$

$$|\psi(x,t)|^2=|\psi_0(x)|^2+|\psi_1(x)|^2-2\operatorname{Re}\psi_0^*(x)\psi_1(x)=[\psi_1(x)-\psi_0(x)]^2$$

Q12. If J_x, J_y and J_z are angular momentum operators, the eigenvalues of the operator $\frac{(J_x+J_y)}{\hbar}$ are:

(a) real and discrete with rational spacing

(b) real and discrete with irrational spacing

(c) real and continuous

(d) not all real

Ans.: (b)

Solution: $J_x=\frac{1}{2}(J_++J_-)$, $J_y=\frac{i}{2}(J_--J_+)\Rightarrow J_+=\hbar\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $J_-=\hbar\begin{bmatrix}0&0\\1&0\end{bmatrix}$

$$J_x=\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad J_y=\frac{i\hbar}{2}\begin{bmatrix}0&-1\\1&0\end{bmatrix}\Rightarrow\frac{J_x+J_y}{\hbar}=\frac{1}{2}\begin{bmatrix}0&1-i\\1+i&0\end{bmatrix}$$

eigen value $\frac{1}{2}\begin{pmatrix}-\lambda&1-i\\1+i&-\lambda\end{pmatrix}\Rightarrow\lambda^2-2=0\Rightarrow\lambda=\pm\sqrt{2}$

Q13. A simple model of a helium-like atom with electron-electron interaction is replaced by Hooke's law force is described by Hamiltonian

$$\frac{-\hbar^2}{2m}(\nabla_1^2+\nabla_2^2)+\frac{1}{2}m\omega^2(r_1^2+r_2^2)-\frac{\lambda}{4}\ m\omega^2|\vec{r}_1-\vec{r}_2|^2.$$

What is the exact ground state energy?

(a) $E=\frac{3}{2}\hbar\omega(1+\sqrt{1+\lambda})$

(b) $E=\frac{3}{2}\hbar\omega(1+\sqrt{\lambda})$

(c) $E=\frac{3}{2}\hbar\omega\sqrt{1-\lambda}$

(d) $E=\frac{3}{2}\hbar\omega(1+\sqrt{1-\lambda})$

Ans.: (b)

Q14. Consider the state $\begin{pmatrix} 1/2 \\ 1/2 \\ 1/\sqrt{2} \end{pmatrix}$ corresponding to the angular momentum $l=1$ in the L_z basis of states with $m=+1,\ 0,\ -1$. If L_z^2 is measured in this state yielding a result 1, what is the state after the measurement?

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 1/\sqrt{3} \\ 0 \\ \sqrt{2/3} \end{pmatrix}$ (c) $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$

Ans.: (d)

Solution: $L_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $L_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, eigenvector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Corresponding eigenvalue 1, 0, 1

Now state after measurement yielding $1 \Rightarrow |\phi_1\rangle + |\phi_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Q15. What are the eigenvalues of the operator $H = \vec{\sigma}\cdot\vec{a}$, where $\vec{\sigma}$ are the three Pauli matrices and $\vec{a}$ is a vector?

(a) $a_x + a_y$ and a_z (b) $a_x + a_z \pm i a_y$ (c) $\pm\left(a_x + a_y + a_z\right)$ (d) $\pm|\vec{a}|$

Ans.: (d)

Solution: $H = \vec{\sigma}\cdot\vec{a} = \left(\sigma_x.a_x + \sigma_y.a_y + \sigma_z.a_z\right)$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} a_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} a_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} a_z = \begin{pmatrix} a_z & \left(a_x - i a_y\right) \\ \left(a_x + i a_y\right) & -a_z \end{pmatrix}$$

For eigen value,

$$\begin{pmatrix} \left(a_z - \lambda\right) & \left(a_x - i a_y\right) \\ \left(a_x + i a_y\right) & -\left(a_z + \lambda\right) \end{pmatrix} = 0 \Rightarrow -\left(a_z - \lambda\right)\left(a_z + \lambda\right) - \left(a_x - i a_y\right)\left(a_x + i a_y\right) = 0$$

$$\Rightarrow -a_z^2 + \lambda^2 - a_x^2 - a_y^2 = 0 \Rightarrow \lambda^2 = a_x^2 + a_y^2 + a_z^2 \Rightarrow \lambda = \pm|\,\vec{a}\,|$$

Q16. The hermitian conjugate of the operator $\left(\frac{-\partial}{\partial x}\right)$ is

(a) $\frac{\partial}{\partial x}$ (b) $-\frac{\partial}{\partial x}$ (c) $i\frac{\partial}{\partial x}$ (d) $-i\frac{\partial}{\partial x}$

Ans.: (a)

Solution: $\Rightarrow\left(\psi^*(x)-\frac{\partial}{\partial x}\psi(x)\right)^\dagger=\left(\frac{-\partial\psi^*(x)}{\partial x}\psi(x)\right)$

$$\Rightarrow\int_{-\infty}^{\infty}\psi^*(x)\left[-\frac{\partial}{\partial x}\psi(x)\right]dx=\psi^*(x)\psi(x)\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}-\frac{\partial\psi^*(x)}{\partial x}\psi(x)dx$$

$$=\int_{-\infty}^{\infty}\frac{\partial\psi^*(x)}{\partial x}\psi(x)dx$$

Q17. If the expectation value of the momentum is $\langle p\rangle$ for the wavefunction $\psi(x)$, then the expectation value of momentum for the wavefunction $e^{ikx/\hbar}\psi(x)$ is

(a) k (b) $\langle p\rangle-k$ (c) $\langle p\rangle+k$ (d) $\langle p\rangle$

Ans.: (c)

Solution: $\int_{-\infty}^{\infty}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)\psi(x)dx=\langle p\rangle$

Now

$$\int_{-\infty}^{\infty}e^{-\frac{ikx}{\hbar}}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\right)e^{\frac{ikx}{\hbar}}\psi(x)dx\Rightarrow\int_{-\infty}^{\infty}e^{\frac{-ikx}{\hbar}}\psi^*(x)(-i\hbar)\left[e^{\frac{ikx}{\hbar}}\frac{\partial}{\partial x}\psi(x)+\frac{ik}{\hbar}e^{\frac{ikx}{\hbar}}\psi(x)\right]$$

$$\Rightarrow\int_{-\infty}^{\infty}e^{-\frac{ikx}{\hbar}}\psi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)e^{\frac{ikx}{\hbar}}+\int_{-\infty}^{\infty}-i\hbar.\frac{ik}{\hbar}e^{\frac{-ikx}{\hbar}}\psi^*(x)\psi(x)dx$$

$$\Rightarrow\int_{-\infty}^{\infty}\psi^*(x)\left[-i\hbar\frac{\partial}{\partial x}\psi(x)\right]+k\int_{-\infty}^{\infty}\psi^*(x)\psi(x)\Rightarrow\langle p\rangle+K$$

Q18. Two electrons are confined in a one dimensional box of length L. The one-electron states are given by $\psi_n(x)=\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)$. What would be the ground state wave function $\psi(x_1,x_2)$ if both electrons are arranged to have the same spin state?

(a) $\psi(x_1,x_2)=\frac{1}{\sqrt{2}}\left[\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)+\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]$

(b) $\psi(x_1,x_2)=\frac{1}{\sqrt{2}}\left[\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)-\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)\right]$

(c) $\psi(x_1,x_2)=\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right)\sin\left(\frac{2\pi x_2}{L}\right)$

(d) $\psi(x_1,x_2)=\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right)\sin\left(\frac{\pi x_2}{L}\right)$

Ans.: (b)

Solution: Electrons are Fermions of spin $\frac{1}{2}$ and its wave functions are anti-symmetric

Since, spin part is symmetric, therefore, space part will be anti-symmetric (since as total wave function is anti-symmetric)

Then,

$$\psi(x_1,x_2)=\frac{1}{\sqrt{2}}\left[\frac{2}{L}\sin\left(\frac{\pi x_1}{L}\right).\sin\left(\frac{2\pi x_2}{L}\right)-\frac{2}{L}\sin\left(\frac{2\pi x_1}{L}\right).\sin\left(\frac{\pi x_2}{L}\right)\right]$$

Q19. The operator $\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)$ is equivalent to

(a) $\frac{d^2}{dx^2}-x^2$

(b) $\frac{d^2}{dx^2}-x^2+1$

(c) $\frac{d^2}{dx^2}-x\frac{d}{dx}x^2+1$

(d) $\frac{d^2}{dx^2}-2x\frac{d}{dx}-x^2$

Ans.: (b)

Solution: $\Rightarrow\left(\frac{d}{dx}-x\right)\left(\frac{d}{dx}+x\right)f(x)\Rightarrow\left(\frac{d}{dx}-x\right)\left[\frac{d}{dx}f(x)+xf(x)\right]$

$$\Rightarrow \frac{d}{dx}\left[\frac{d}{dx} f(x)+xf(x)\right]-x\frac{d}{dx} f(x)-x^2 f(x)$$

$$\Rightarrow \frac{d^2}{dx^2} f(x)+f(x)+x\frac{df(x)}{dx}-x\frac{d}{dx} f(x)-x^2 f(x)$$

$$\Rightarrow \frac{d^2}{dx^2} f(x)-x^2 f(x)+f(x)=\left(\frac{d^2}{dx^2}-x^2+1\right) f(x)$$

JEST-2014

Q20. Suppose a spin $1/2$ particle is in the state

$$|\psi\rangle=\frac{1}{\sqrt{6}}\begin{bmatrix}1+i\\2\end{bmatrix}$$

If S_x (x component of the spin angular momentum operator) is measured what is the probability of getting $+\hbar/2$?

(a) $1/3$ (b) $2/3$ (c) $5/6$ (d) $1/6$

Ans.: (c)

Solution: $S_x=\frac{\hbar}{2}\begin{bmatrix}0&1\\1&0\end{bmatrix}$ with eigenvalues $\pm\frac{\hbar}{2}$ and eigenvector corresponding to $\frac{\hbar}{2}$ is $\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

Now probability getting $+\frac{\hbar}{2}$

$$p\left(\frac{\hbar}{2}\right)=\frac{|\langle\phi|\psi\rangle|}{\langle\psi|\psi\rangle}=\frac{\left|\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{6}}\begin{bmatrix}1&1\end{bmatrix}\begin{bmatrix}1+i\\2\end{bmatrix}\right|^2}{\frac{1}{6}\begin{bmatrix}1-i&2\end{bmatrix}\begin{bmatrix}1+i\\2\end{bmatrix}}=\frac{\frac{1}{12}|1+i+2|^2}{6\times\frac{1}{6}}=\frac{5}{6}$$

Q21. The Hamiltonian operator for a two-state system is given by

$$H=\alpha\left(|1\rangle\langle 1|-|2\rangle\langle 2|+|1\rangle\langle 2|+|2\rangle\langle 1|\right),$$

where α is a positive number with the dimension of energy. The energy eigenstates corresponding to the larger and smaller eigenvalues respectively are:

(a) $|1\rangle-\left(\sqrt{2}+1\right)|2\rangle,\ |1\rangle+\left(\sqrt{2}-1\right)|2\rangle$

(b) $|1\rangle+\left(\sqrt{2}-1\right)|2\rangle,\ |1\rangle-\left(\sqrt{2}+1\right)|2\rangle$

(c) $|1\rangle+\left(\sqrt{2}-1\right)|2\rangle,\ \left(\sqrt{2}+1\right)|1\rangle-|2\rangle$

(d) $|1\rangle-\left(\sqrt{2}+1\right)|2\rangle,\ \left(\sqrt{2}-1\right)|1\rangle+|2\rangle$

Ans.: (b)

Solution: $H=\alpha\left(|1\rangle\langle 1|-|2\rangle\langle 2|+|1\rangle\langle 2|+|2\rangle\langle 1|\right)\Rightarrow H|1\rangle=\alpha\left(|1\rangle+|2\rangle\right),\ \ H|2\rangle=\alpha\left(|1\rangle-|2\rangle\right)$

Lets check for option (b): $|1\rangle+\left(\sqrt{2}-1\right)|2\rangle,\ |1\rangle-\left(\sqrt{2}+1\right)|2\rangle$

Now $H|\psi\rangle=\alpha|\psi\rangle\Rightarrow H\left[|1\rangle+\left(\sqrt{2}-1\right)|2\rangle\right]=H|1\rangle+H\left(\sqrt{2}+1\right)|2\rangle$

$$H\left[|1\rangle+\left(\sqrt{2}-1\right)|2\rangle\right]\Rightarrow H\left(|1\rangle\right)+\left(\sqrt{2}-1\right)H|2\rangle\Rightarrow\alpha\left(|1\rangle+|2\rangle\right)+\left(\sqrt{2}-1\right)\alpha\left(|1\rangle-|2\rangle\right)$$

$$\Rightarrow\alpha\left[1+\sqrt{2}-1\right]|1\rangle+\alpha\left[1-\left(\sqrt{2}-1\right)\right]|2\rangle\Rightarrow\alpha\sqrt{2}|1\rangle+\alpha\left(2-\sqrt{2}\right)|2\rangle$$

$$\Rightarrow\alpha\sqrt{2}\left[|1\rangle+\left(\sqrt{2}-1\right)|2\rangle\right]$$

Now $H\left(|1\rangle-\sqrt{2}+1\right)|2\rangle\Rightarrow H\left[|1\rangle-\left(\sqrt{2}+1\right)|2\rangle\right]\Rightarrow H|1\rangle-H\left(\sqrt{2}+1\right)|2\rangle$

$$\Rightarrow\alpha\left(|1\rangle+|2\rangle\right)-\alpha\left[\left(\sqrt{2}+1\right)\left(|1\rangle-|2\rangle\right)\right]\Rightarrow\alpha\left(1-\sqrt{2}-1\right)|1\rangle+\alpha\left(1+\sqrt{2}+1\right)|2\rangle$$

$$\Rightarrow-\sqrt{2}\alpha|1\rangle+\left(2+\sqrt{2}\right)\alpha|2\rangle\Rightarrow-\alpha\sqrt{2}\left[|1\rangle-\left(1+\sqrt{2}\right)|2\rangle\right]$$

Q22. Consider an eigenstate of $\vec{L}^2$ and L_z operator denoted by $|l,\,m\rangle$. Let $A=\hat{n}\cdot\vec{L}$ denote an operator, where $\hat{n}$ is a unit vector parametrized in terms of two angles as $\left(n_x,n_y,n_z\right)=\left(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta\right)$. The width ΔA in $|l,\,m\rangle$ state is:

(a) $\sqrt{\dfrac{l(l+1)-m^2}{2}}\hbar\cos\theta$

(b) $\sqrt{\dfrac{l(l+1)-m^2}{2}}\hbar\sin\theta$

(c) $\sqrt{l(l+1)-m^2}\,\hbar\sin\theta$

(d) $\sqrt{l(l+1)-m^2}\,\hbar\cos\theta$

Ans.: (c)

Solution: $A=\hat{n}\cdot\vec{L}\Rightarrow A=L_x\cdot\dfrac{x}{r}+L_y\cdot\dfrac{y}{r}+L_z\cdot\dfrac{z}{r}$

$$\Rightarrow A=L_x\cdot\frac{r\sin\theta\cos\phi}{r}+L_y\cdot\frac{r\sin\theta\ \sin\phi}{r}+L_z\cdot\frac{r\cos\theta}{r}$$

$$\Rightarrow A=L_x\sin\theta\cos\phi+L_y\sin\theta\cdot\sin\phi+L_z\cos\theta$$

Now $\Delta A=\sqrt{\langle A^2\rangle-\langle A\rangle^2}$

$$\langle A\rangle = \langle L_x\rangle \sin\theta\cos\phi + \langle L_y\rangle \sin\theta\sin\phi + \langle L_z\rangle\cos\theta$$

$$\langle A\rangle = (m\hbar)\cos\theta \qquad \because \langle L_x\rangle = 0, \langle L_y\rangle = 0$$

$$\langle A^2\rangle = \langle L_x^2\rangle \sin^2\theta\cos^2\phi + \langle L_y^2\rangle \sin^2\theta\sin^2\phi + \langle L_z^2\rangle\cos^2\theta$$

$$= \left(\langle L_x^2\rangle + \langle L_y^2\rangle\right)\sin^2\theta + \langle L_z^2\rangle\cos^2\theta$$

$$= \left(\langle L^2\rangle - \langle L_z^2\rangle\right)\sin^2\theta - \langle L_z^2\rangle\cos^2\theta$$

$$\Rightarrow \langle A^2\rangle = \left[l(l+1) - m^2\right]\hbar^2\sin^2\theta + m^2\hbar^2\cos^2\theta$$

$$\Rightarrow \langle A^2\rangle = \left[l(l+1) - m^2\right]\hbar^2\sin^2\theta + m^2\hbar^2\cos^2\theta$$

$$\Delta A = \sqrt{\langle A^2\rangle - \langle A\rangle^2} = \sqrt{\left(l(l+1) - m^2\right)\hbar^2\sin^2\theta + m^2\hbar^2\cos^2\theta - m^2\hbar^2\cos^2\theta}$$

$$\Delta A = \sqrt{\left[l(l+1) - m^2\right]}\hbar\sin\theta$$

Q23. Consider a three-state system with energies E, E and $E-3g$ (where g is a constant) and respective eigenstates $|\psi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $|\psi_2\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\1\\-2\end{pmatrix}$ and $|\psi_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$

If the system is initially (at $t=0$), in state $|\psi_i\rangle = \begin{pmatrix}1\\0\\0\end{pmatrix}$

what is the probability that at a later time t system will be in state $|\psi_f\rangle = \begin{pmatrix}0\\0\\1\end{pmatrix}$

(a) 0

(b) $\frac{4}{9}\sin^2\left(\frac{3gt}{2\hbar}\right)$

(c) $\frac{4}{9}\cos^2\left(\frac{3gt}{2\hbar}\right)$

(d) $\frac{4}{9}\sin^2\left(\frac{E-3gt}{2\hbar}\right)$

Ans.: (b)

Q24. The lowest quantum mechanical energy of a particle confined in a one-dimensional box of size L is $2eV$. The energy of the quantum mechanical ground state for a system of three non-interacting spin $\frac{1}{2}$ particles is

(a) $6eV$ (b) $10eV$ (c) $12eV$ (d) $16eV$

Ans.: (c)

Solution: $E_1 = \frac{\pi^2\hbar^2}{2ml^2} = 2eV$, $E_2 = 4E_1 = 8\ eV$

Spin, spin is $\frac{1}{2}$, therefore, degeneracy $g_i = 2S+1 = 2\times\frac{1}{2}+1 = 2$

$\Rightarrow$ ground state energy $= 2\times 2\ eV + 1\times 8eV = 12\ eV$

Q25. A ball bounces off earth. You are asked to solve this quantum mechanically assuming the earth is an infinitely hard sphere. Consider surface of earth as the origin implying $V(0) = \infty$ and a linear potential elsewhere (i.e. $V(x) = -mgx$ for $x > 0$). Which of the following wave functions is physically admissible for this problem (with $k > 0$):

(a) $\psi = e^{-kx}/x$ (b) $\psi = xe^{-kx^2}$ (c) $\psi = -Axe^{kx}$ (d) $\psi = Ae^{-kx^2}$

Ans.: (b)

Solution: $\psi = xe^{-kx^2}$

For given potential, at $x = 0$ and $x = \infty$ wave function must vanish.

Q26. The operator A and B share all the eigenstates. Then the least possible value of the product of uncertainties $\Delta A \Delta B$ is

(a) $\hbar$ (b) 0 (c) $\hbar/2$ (d) Determinant (AB)

Ans.: (b)

Solution: $\Delta A\cdot\Delta B \geq \left|\frac{[AB]}{2}\right|$

$\Delta A\cdot\Delta B \geq 0$ [$\because A$ and B have share their eigen values, so $[AB] = 0$]

Q27. Consider a square well of depth $-V_0$ and width a with V_0 as fixed. Let $V_0 \to \infty$ and $a \to 0$. This potential well has

(a) No bound states
(b) 1 bound state
(c) 2 bound states
(d) Infinitely many bound states

Ans.: (b)

Solution: It forms delta potential, so it has only one bound state.

JEST-2015

Q28. Consider a harmonic oscillator in the state $|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^+}|0\rangle$, where $|0\rangle$ is the ground state, a^+ is the raising operator and α is a complex number. What is the probability that the harmonic oscillator is in the nth eigenstate $|n\rangle$?

(a) $e^{-|\alpha^2|}\dfrac{|\alpha|^{2n}}{n!}$
(b) $e^{-\frac{|\alpha|^2}{2}\frac{|\alpha|^n}{\sqrt{n!}}}$
(c) $e^{-|\alpha|^2}\dfrac{|\alpha|^n}{n!}$
(d) $e^{-\frac{|\alpha|^2}{2}}\dfrac{|\alpha|^{2n}}{n!}$

Ans.: (a)

Solution: $|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^+}|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{(\alpha a^+)^n}{\lfloor n}|0\rangle$ and $|n\rangle = \frac{(a^+)^n}{\sqrt{\lfloor n}}|0\rangle \Rightarrow (a^+)^n|0\rangle = \sqrt{\lfloor n}\,|n\rangle$

$$|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{(\alpha)^n\sqrt{\lfloor n}}{\lfloor n}|n\rangle \Rightarrow \langle\psi|\psi\rangle = e^{-|\alpha|^2}\sum \frac{(\alpha^*\alpha)^n\lfloor n}{(\lfloor n)^2}\langle n|n\rangle = e^{-|\alpha|^2}\sum_n \frac{|\alpha|^n}{\lfloor n} = e^{-|\alpha|^2}e^{|\alpha|^2} = 1$$

Probability that $|\psi\rangle$ is in $|n\rangle$ state is, $\dfrac{|\langle n|\psi\rangle|^2}{\langle\psi|\psi\rangle} = |\langle n|\psi\rangle|^2$

$$|\psi\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_n \frac{(\alpha)^n\sqrt{\lfloor n}}{\lfloor n}|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \alpha^n\frac{1}{\sqrt{\lfloor n}}|n\rangle$$

$$\Rightarrow \langle n|\psi\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \alpha^n \frac{1}{\sqrt{\lfloor n}}\langle n|n\rangle = \frac{e^{-\frac{|\alpha|^2}{2}}}{\sqrt{\lfloor n}}\alpha^n \Rightarrow |\langle n|\psi\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{\lfloor n}$$

Q29. A particle of mass m moves in $1-$dimensional potential $V(x)$, which vanishes at infinity. The exact ground state eigenfunction is $\psi(x)=A\sec h(\lambda x)$, where A and λ are constants. The ground state energy eigenvalue of this system is,

(a) $E=\frac{\hbar^2\lambda^2}{m}$ (b) $E=-\frac{\hbar^2\lambda^2}{m}$ (c) $E=-\frac{\hbar^2\lambda^2}{2m}$ (d) $E=\frac{\hbar^2\lambda^2}{2m}$

Ans.: (d)

Solution: $\because \psi(x)=A\sec h(\lambda x)\Rightarrow \frac{d\psi}{dx}=-A\lambda\sec h(\lambda x)\tanh(\lambda x)$

$$\Rightarrow \frac{d^2\psi}{dx^2}=-A\lambda\left[-\sec h(\lambda x)\tan^2 h(\lambda x)\lambda+\lambda\sec h(\lambda x)\sec^2 h(\lambda x)\right]$$

$$=-A\lambda^2\left[\sec h(\lambda x)\left[-\tan^2 h(\lambda x)+\sec^2 h(\lambda x)\right]\right]$$

$$=-A\lambda^2\left[\sec h(\lambda x)\left[\sec^2 h(\lambda x)-\tan^2 h(\lambda x)\right]\right]$$

$$=-A\lambda^2\left[\sec h(\lambda x)\left[\sec^2 h(\lambda x)-\left[1-\sec^2 h(\lambda x)\right]\right]\right]$$

$\because \tan^2 h(\lambda x)=1-\sec^2 h(\lambda x)$

$$=-A\lambda^2\left[\sec h(\lambda x)\left[\sec^2 h(\lambda x)-1+\sec^2 h(\lambda x)\right]\right]$$

$$\Rightarrow \frac{d^2\psi}{dx^2}=-A\lambda^2\left[2\sec^3 h(\lambda x)-\sec h(\lambda x)\right]$$

Now put the value $\frac{d^2\psi}{dx^2}$ in equation $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+V(x)\psi(x)=E\psi(x)$

$$-\frac{\hbar^2}{2m}\lambda^2 A\left[2\sec^3 h(\lambda x)-\sec h(\lambda x)\right]+V(x)A\sec h(\lambda x)=EA\sec h(\lambda x)$$

$\because V(x)\to 0 \quad as \quad x\to\infty$

$$\Rightarrow +\frac{\hbar^2}{2m}\lambda^2 A\sec h(\lambda x)-\frac{\hbar^2\lambda^2}{2m}2A\sec^3 h(\lambda x)=EA\sec h(\lambda x)$$

Now we have to do approximation i.e. $\sec^3 h(\lambda x)$ dacays very fastly as $x\to\infty$ so second term

$\frac{\hbar^2\lambda^2}{2m}2A\sec^3 h(\lambda x)=0$. Thus $\frac{\hbar^2\lambda^2}{2m}A\sec h(\lambda x)=EA\sec h(\lambda x)\Rightarrow E=\frac{\hbar^2\lambda^2}{2m}$

Q30. Consider a spin $-\frac{1}{2}$ particle characterized by the Hamiltonian $H = \omega S_z$. Under a perturbation $H' = gS_x$, the second order correction to the ground state energy is given by,

(a) $-\frac{g^2}{4\omega}$ (b) $\frac{g^2}{4\omega}$ (c) $-\frac{g^2}{2\omega}$ (d) $\frac{g^2}{2\omega}$

Ans.: (a)

Solution: $\because H = \omega s_z$ *and* $s_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\Rightarrow H = \frac{\omega\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $H' = gs_x = \frac{g\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Ground state energy is $-\frac{\omega\hbar}{2}$ with eigenvector $|\phi_1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

and first excited state energy is $\frac{\omega\hbar}{2}$ with eigenvector $|\phi_2\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Second order correction in ground state $E_1^2 = \sum_{m\neq 1} \frac{|\langle \phi_m | H' | \phi_1 \rangle|^2}{E_1^0 - E_m^0} = \frac{|\langle \phi_m | H' | \phi_1 \rangle|^2}{-\frac{\omega\hbar}{2} - \frac{\omega\hbar}{2}}$

$$\Rightarrow E_1^2 = \frac{g^2\hbar^2}{4} \frac{\left| (1 \quad 0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2}{-\frac{2\omega\hbar}{2}} = -\frac{g^2\hbar^2}{4\omega\hbar} = -\frac{g^2}{4\omega}\hbar$$

Q31. Given that ψ_1 and ψ_2 are eigenstates of a Hamiltonian with eigenvalues E_1 and E_2 respectively, what is the energy uncertainty in the state $(\psi_1 + \psi_2)$?

(a) $-\sqrt{E_1 E_2}$ (b) $\frac{1}{2}|E_1 - E_2|$

(c) $\frac{1}{2}(E_1 + E_2)$ (d) $\frac{1}{\sqrt{2}}|E_2 - E_1|$

Ans.: (b)

Solution: $\langle E^2 \rangle = \frac{1}{2}E_1^2 + \frac{1}{2}E_2^2 = \frac{(E_1^2 + E_2^2)}{2}$ and $\langle E \rangle = \frac{1}{2}E_1 + \frac{1}{2}E_2$

$$\because \Delta E=\sqrt{\langle E^2\rangle-\langle E\rangle^2}=\sqrt{\frac{\left(E_1^2+E_2^2\right)}{2}-\frac{1}{4}\left(E_1+E_2\right)^2}=\sqrt{\frac{2E_1^2+2E_2^2-E_1^2-E_2^2-2E_1E_2}{4}}$$

$$\Rightarrow \Delta E=\sqrt{\frac{E_1^2+E_2^2-2E_1E_2}{4}}=\frac{1}{2}\left|E_1-E_2\right|$$

Q32. A particle moving under the influence of a potential $V(r)=\frac{kr^2}{2}$ has a wavefunction $\psi(r,t)$. If the wavefunction changes to $\psi(\alpha r,t)$, the ratio of the average final kinetic energy to the initial kinetic energy will be,

(a) $\frac{1}{\alpha^2}$ (b) α (c) $\frac{1}{\alpha}$ (d) α^2

Ans.: (c)

Solution: For $\psi(r,t)$ the average kinetic energy $\langle T\rangle=\int_0^\infty \psi^*(r,t)\left(-\frac{\hbar^2}{2m}\right)\left(\nabla^2\psi\right)r^2dr$, ∇^2 is written in spherical polar coordinate, which is dimension of $(\text{length})^{-2}$

For wave function $\psi(\alpha r,t)$

$$\langle T_\alpha\rangle=\int_0^\infty \psi^*(\alpha r,\, t)\left(-\frac{\hbar^2}{2m}\right)\left(\nabla^2\psi(\alpha r,\, t)\right)r^2dr$$

Put $\alpha r=r'$ or $r=\frac{r'}{\alpha}\Rightarrow dr=\frac{dr'}{\alpha}$ and $\nabla_r^2=\alpha^2\nabla_{r'}^2$

$$\langle T_\alpha\rangle=\frac{\alpha^2}{\alpha^3}\int_0^\infty \psi^*(r',t)\left(-\frac{\hbar^2}{2m}\right)\nabla^2\psi(r',t)r'^2dr'=\frac{1}{\alpha}\int_0^\infty \psi^*(r',t)\left(-\frac{\hbar^2}{2m}\right)\nabla^2\psi(r',t)r'^2dr'$$

$$\Rightarrow \langle T_\alpha\rangle=\frac{\langle T\rangle}{\alpha}\Rightarrow \frac{\langle T_\alpha\rangle}{\langle T\rangle}=\frac{1}{\alpha}$$

Q33. If a Hamiltonian H is given as $H=|0\rangle\langle 0|-|1\rangle\langle 1|+i\left(|0\rangle\langle 1|-|1\rangle\langle 0|\right)$, where $|0\rangle$ and $|1\rangle$ are orthonormal states, the eigenvalues of H are

(a) ± 1 (b) $\pm i$ (c) $\pm\sqrt{2}$ (d) $\pm i\sqrt{2}$

Ans: (c)

Solution: $H=|0\rangle\langle 0|-|1\rangle\langle 1|+i\left(|0\rangle\langle 1|-|1\rangle\langle 0|\right)$

$H|0\rangle = |0\rangle - i|1\rangle$ and $H|1\rangle = -|1\rangle + i|0\rangle$

The matrix representation of H is $\begin{vmatrix} \langle 0|H|0\rangle & \langle 0|H|1\rangle \\ \langle 1|H|0\rangle & \langle 1|H|1\rangle \end{vmatrix} = \begin{pmatrix} 1 & i \\ -i & -1 \end{pmatrix}$

Eigenvalue of H $\begin{pmatrix} 1-\lambda & i \\ -i & -1-\lambda \end{pmatrix} = 0 \Rightarrow -\left(1-\lambda^2\right) - 1 = -0 \Rightarrow \lambda = \pm\sqrt{2}$

Q34. A particle of mass m is confined in a potential well given by $V(x) = 0$ for $\frac{-L}{2} < x < \frac{L}{2}$ $\frac{L}{2}$ and $V(x) = \infty$ elsewhere. A perturbing potential $H'(x) = ax$ has been applied to the system. Let the first and second order corrections to the ground state be $E_0^{(1)}$ and $E_0^{(2)}$, respectively. Which one of the following statements is correct?

(a) $E_0^{(1)} < 0$ and $E_0^{(2)} > 0$

(b) $E_0^{(1)} = 0$ and $E_0^{(2)} > 0$

(c) $E_0^{(1)} > 0$ and $E_0^{(2)} < 0$

(d) $E_0^{(1)} = 0$ and $E_0^{(2)} < 0$

Ans.: (d)

Solution: $V(x) = \begin{cases} 0 & -L/2 < x < +L/2 \\ \infty & elsewhere \end{cases}$ and $H'(x) = \alpha x$

For ground state $|\phi_0\rangle = \sqrt{\frac{2}{L}} \cos\frac{\pi x}{L}$

$$E_0^{(1)} = \frac{\langle \phi_0 | H' | \phi_0 \rangle}{\langle \phi_0 | \phi_0 \rangle} = \frac{2}{L}\alpha \int_{-L/2}^{L/2} x \cos^2 \frac{\pi x}{L} = 0$$

$$E_0^{(2)} = \sum_{m \neq 0} \frac{\left|\langle \phi_m | H' | \phi_0 \rangle\right|^2}{E_0^0 - E_m^0} \Rightarrow E_0^{(2)} < 0 \quad \because E_0^0 < E_m^0$$

JEST-2016

Q35. The wavefunction of a hydrogen atom is given by the following superposition of energy eigen functions $\psi_{nlm}(\vec{r})$ (n,l,m are the usual quantum numbers):

$$\psi(\vec{r}) = \frac{\sqrt{2}}{\sqrt{7}}\psi_{100}(\vec{r}) - \frac{3}{\sqrt{14}}\psi_{210}(\vec{r}) + \frac{1}{\sqrt{14}}\psi_{322}(\vec{r})$$

The ratio of expectation value of the energy to the ground state energy and the expectation value of L^2 are, respectively:

(a) $\frac{229}{504}$ and $\frac{12\hbar^2}{7}$ (b) $\frac{101}{504}$ and $\frac{12\hbar^2}{7}$

(c) $\frac{101}{504}$ and $\hbar^2$ (d) $\frac{229}{504}$ and $\hbar^2$

Ans.: (a)

Solution: $\langle E\rangle = \frac{2}{7}\times\frac{E_0}{1} + \frac{9}{14}\times\frac{E_0}{4} + \frac{1}{14}\times\frac{E_0}{9} = \frac{229}{504}E_0$

$$\langle L^2\rangle = \frac{2}{7}\times 0\hbar^2 + \frac{9}{14}\times 2\hbar^2 + \frac{1}{14}\times 6\hbar^2 = \frac{24}{14}\hbar^2 = \frac{12}{7}\hbar^2$$

Q36. A spin-$\frac{1}{2}$ particle in a uniform external magnetic field has energy eigenstates $|1\rangle$ and $|2\rangle$. The system is prepared in ket-state $\frac{(|1\rangle+|2\rangle)}{\sqrt{2}}$ at time $t=0$. It evolves to the state described by the ket $\frac{(|1\rangle-|2\rangle)}{\sqrt{2}}$ in time T. The minimum energy difference between two levels is:

(a) $\frac{h}{6T}$ (b) $\frac{h}{4T}$ (c) $\frac{h}{2T}$ (d) $\frac{h}{T}$

Ans.: (c)

Solution: $|\psi(t=0)\rangle = \frac{(|1\rangle+|2\rangle)}{\sqrt{2}} \Rightarrow |\psi(t=t)\rangle = \frac{\left(|1\rangle\left(-i\frac{E_1 t}{\hbar}\right)+|2\rangle\exp\left(-i\frac{E_2 t}{\hbar}\right)\right)}{\sqrt{2}}$

$$|\psi(t=t)\rangle = \left(-i\frac{E_1 t}{\hbar}\right)\frac{\left(|1\rangle+|2\rangle\exp\left(-i\frac{(E_2-E_1)t}{\hbar}\right)\right)}{\sqrt{2}}$$

$$\exp\left(-i\frac{(E_2-E_1)t}{\hbar}\right)=-1$$

$$\frac{(E_2-E_1)T}{\hbar}=\pi\Rightarrow(E_2-E_1)=\frac{\pi\hbar}{T}=\frac{h}{2T}$$

Q37. The energy of a particle is given by $E=|p|+|q|$ where p and q are the generalized momentum and coordinate, respectively. All the states with $E\le E_0$ are equally probable and states with $E>E_0$ are inaccessible. The probability density of finding the particle at coordinate q, with $q>0$ is:

(a) $\frac{(E_0+q)}{E_0^2}$ (b) $\frac{q}{E_0^2}$ (c) $\frac{(E_0-q)}{E_0^2}$ (d) $\frac{1}{E_0}$

Ans.: (c)

Solution: For condition, $E=|p|+|q|$ total number of accessible state upto energy E_0 for $q>0$ is area under the curve $=\frac{1}{2}\times 2\times E_0^2=E_0^2$

The probability density of finding the particle at coordinate q, with $q>0$

$$\frac{dpdq}{E_0^2}=\frac{pdq}{E_0^2}\Rightarrow\frac{(E_0-q)dq}{E_0^2}$$

For probability at point q, dq is insignificant so $p(q)=\frac{(E_0-q)}{E_0^2}$

Q38. Consider a quantum particle of mass m in one dimension in an infinite potential well, i.e., $V(x)=0$ for $\frac{-a}{2}<x<\frac{a}{2}$ and $V(x)=\infty$ for $|x|\ge\frac{a}{2}$. A small perturbation, $V'(x)=\frac{2\in|x|}{a}$ is added. The change in the ground state energy to $O(\in)$ is:

(a) $\frac{\in}{2\pi^2}(\pi^2-4)$ (b) $\frac{\in}{2\pi^2}(\pi^2+4)$

(c) $\frac{\in\pi^2}{2}(\pi^2+4)$ (d) $\frac{\in\pi^2}{2}(\pi^2-4)$

Ans.: (a)

Solution: $E_1^1 = \int_{-\frac{a}{2}}^{\frac{a}{2}} \phi_1^* V'(x)\phi_1 dx \Rightarrow \frac{2\in}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} |x| \frac{2}{a}\cos^2\frac{\pi x}{a}dx$

$$= \frac{2\in}{a}.2.\int_0^{\frac{a}{2}} x\,\frac{2}{a}\cos^2\frac{\pi x}{a}dx \Rightarrow \frac{4\in}{a^2}\int_0^{\frac{a}{2}} x\frac{2}{2}\left(\cos\frac{2\pi x}{a}+1\right)dx \Rightarrow \frac{4\in}{a^2}\int_0^{\frac{a}{2}} x\left(\cos\frac{2\pi x}{a}+1\right)dx$$

$$\Rightarrow \frac{4\in}{a^2}\int_0^{\frac{a}{2}} x\left(\cos\frac{2\pi x}{a}+1\right)dx = \frac{\in}{2\pi^2}\left(\pi^2-4\right)$$

Q39. If $Y_{xy} = \frac{1}{\sqrt{2}}\left(Y_{2,2} - Y_{2,-2}\right)$ where $Y_{l,m}$ are spherical harmonics then which of the following is true?

(a) Y_{xy} is an eigenfunction of both L^2 and L_z

(b) Y_{xy} is an eigenfunction of L^2 but not L_z

(c) Y_{xy} is an eigenfunction both of L_z but not L^2

(d) Y_{xy} is not an eigenfunction of either L^2 and L_z

Ans.: (b)

Solution: The $L^2 Y_{xy} = l(l+1)\hbar^2 Y_{xy}$, where $l = 2$ and $L_z Y_{xy} \neq m Y_{xy}$

So, Y_{xy} is an eigenfunction of L^2 but not L_z

Q40. A spin-1 particle is in a state $|\psi\rangle$ described by the column matrix $\frac{1}{\sqrt{10}}\begin{pmatrix}2\\ \sqrt{2}\\ 2i\end{pmatrix}$ in the S_z basis. What is the probability that a measurement of operator S_z will yield the result $\hbar$ for the state $S_x|\psi\rangle$?

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

Ans.: (c)

Solution: $S_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $|\psi\rangle = \frac{1}{\sqrt{10}}\begin{pmatrix} 2 \\ \sqrt{2} \\ 2i \end{pmatrix}$

$$S_x|\psi\rangle = \frac{\sqrt{2}}{\sqrt{10}}\hbar\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix}$$

$$S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The eigen state corresponding to eigen value $\hbar$ of S_z is $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\therefore \quad P(\hbar) = \frac{\left|(1\ 0\ 0)\frac{\sqrt{2}}{\sqrt{10}}\hbar\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix}\right|^2}{\frac{2}{10}\hbar^2(1\ 1-i\ 1)\begin{pmatrix} 1 \\ 1+i \\ 1 \end{pmatrix}} = \frac{1}{4}$$

Q41. The Hamiltonian of a quantum particle of mass m confined to a ring of unit radius is:

$$H = \frac{\hbar^2}{2m}\left(-i\frac{\partial}{\partial\theta} - \alpha\right)^2$$

where θ is the angular coordinate, α is a constant. The energy eigenvalues and eigenfunctions of the particle are (n is an integer):

(a) $\psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n-\alpha)^2$ (b) $\psi_n(\theta) = \frac{\sin(n\theta)}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n-\alpha)^2$

(c) $\psi_n(\theta) = \frac{\cos(n\theta)}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n-\alpha)^2$ (d) $\psi_n(\theta) = \frac{e^{in\theta}}{\sqrt{2\pi}}$ and $E_n = \frac{\hbar^2}{2m}(n+\alpha)^2$

Ans.: (a)

Solution: $H=\frac{\hbar^2}{2m}\left(-i\frac{\partial}{\partial\theta}-\alpha\right)^2 \Rightarrow \frac{\hbar^2}{2m}\left[-\frac{\partial^2\psi}{\partial\theta^2}+2i\alpha\frac{\partial\psi}{\partial\theta}+\alpha^2\psi\right]=E\psi$

By inspection, $|\psi_n(\theta)\rangle=\frac{e^{in\theta}}{\sqrt{2\pi}}$, which will also satisfy boundary condition $|\psi_n(\theta+2\pi)\rangle=|\psi_n(\theta)\rangle$ and satisfies the eigen value equation with eigen value $E=\frac{\hbar^2(n-\alpha)^2}{2m}$

Q42. The adjoint of a differential operator $\frac{d}{dx}$ acting on a wavefunction $\psi(x)$ for a quantum mechanical system is:

(a) $\frac{d}{dx}$ (b) $-i\hbar\frac{d}{dx}$ (c) $-\frac{d}{dx}$ (d) $i\hbar\frac{d}{dx}$

Ans.: (c)

Q43. In the ground state of hydrogen atom, the most probable distance of the electron from the nucleus, in units of Bohr radius a_0 is:

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) $\frac{3}{2}$

Ans.: (d)

Solution: $\psi_{100}=\frac{1}{\sqrt{\pi a_0^3}}e^{\frac{-r}{a_0}}$

$$P=\psi^*\psi=\frac{1}{\pi a_0^3}e^{\frac{-r}{a_0}}\Rightarrow r_p=\frac{dP}{dr}=0\Rightarrow r_p=a_0$$

Q44. For operators P and Q, the commutator $\left[P,Q^{-1}\right]$ is

(a) $Q^{-1}[P,Q]Q^{-1}$ (b) $-Q^{-1}[P,Q]Q^{-1}$ (c) $Q^{-1}[P,Q]Q$ (d) $-Q[P,Q]Q^{-1}$

Ans.: (b)

Solution: $\left[P,Q^{-1}\right]=PQ^{-1}-Q^{-1}P$

$$-Q^{-1}[P,Q]Q^{-1}=-Q^{-1}[PQ-QP]Q^{-1}=-Q^{-1}\left[PQQ^{-1}-QPQ^{-1}\right]=-Q^{-1}P+PQ^{-1}=\left[P,Q^{-1}\right]$$

Q45. A spin $\frac{1}{2}$ particle is in a state $\frac{(|\uparrow\rangle+|\downarrow\rangle)}{\sqrt{2}}$ where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenstates of S_z operator. The expectation value of the spin angular momentum measured along x direction is:

(a) $\hbar$ (b) $-\hbar$ (c) 0 (d) $\frac{\hbar}{2}$

Ans.: (d)

Solution: $|\psi\rangle = \frac{(|\uparrow\rangle+|\downarrow\rangle)}{\sqrt{2}} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\langle S_x \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{\hbar}{2}$$

JEST 2017

Q46. What is the dimension of $\frac{\hbar \partial \psi}{i \partial x}$, where ψ is a wavefunction in two dimensions?

(a) $kg\, m^{-1} s^{-2}$ (b) $kg\, s^{-2}$ (c) $kg\, m^2 s^{-2}$ (d) $kg\, s^{-1}$

Ans. : (d)

Solution: Dimension of $\frac{\hbar \partial \psi}{i \partial x} = \frac{\text{dim of } \hbar}{\text{dim of } x} = \frac{kg \cdot m \cdot \sec^{-2} \cdot \sec}{m} = kg \sec^{-1}$

Q47. Suppose the spin degrees of freedom of a 2 - particle system can be described by a 21 - dimensional Hilbert subspace. Which among the following could be the spin of one of the particles?

(a) $\frac{1}{2}$ (b) 3 (c) $\frac{3}{2}$ (d) 2

Ans. : (b)

Solution: Dimension of Hilbert space $= (2s_1+1) \otimes (2s_2+1) = 7 \times 3 = 21$

So, $s_1 = 3, s_2 = 1$

Q48. If the ground state wavefunction of a particle moving in a one dimensional potential is proportional to $\exp\left(-x^2/2\right)\cosh\left(\sqrt{2}x\right)$, then the potential in suitable units such that $\hbar=1$, is proportional to

(a) x^2

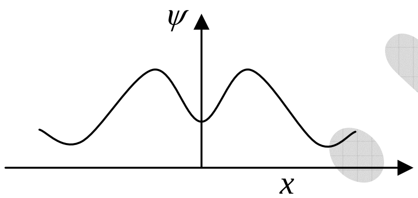
(b) $x^2-2\sqrt{2}x\tanh\left(\sqrt{2}x\right)$

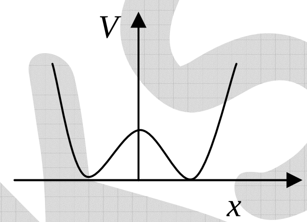
(c) $x^2-2\sqrt{2}x\tan\left(\sqrt{2}x\right)$

(d) $x^2-2\sqrt{2}x\coth\left(\sqrt{2}x\right)$

Ans. : (b)

Solution: From figure, we can conclude that option (b) is the correct answer.





Q49. A particle is described by the following Hamiltonian

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2+\lambda\hat{x}^4$$

where the quartic term can be treated perturbatively. If ΔE_0 and ΔE_1 denote the energy correction of $O(\lambda)$ to the ground state and the first excited state respectively, what is the fraction $\Delta E_1/\Delta E_0$?

Ans. : 5

Solution: $\hat{H}=\frac{\hat{P}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2+\lambda\hat{x}^4$

Now, energy correction of $O(\lambda)$ to ground state is

$$\Delta E_0=\langle 0|\hat{x}^4|0\rangle=\left(\frac{\hbar}{2m\omega}\right)^2\langle 0|6n^2+6n+3|0\rangle=\left(\frac{\hbar}{2m\omega}\right)^2\times 3$$

And energy correction of $O(\lambda)$ to first excited state is

$$\Delta E_1=\langle 1|\hat{x}^4|1\rangle=\left(\frac{\hbar}{2m\omega}\right)^2\langle 1|6n^2+6n+3|1\rangle$$

$$=\left(\frac{\hbar}{2m\omega}\right)^2\times\left[6+6+3\right]=15\left(\frac{\hbar}{2m\omega}\right)^2. \text{ Hence, } \frac{\Delta E_1}{\Delta E_0}=\frac{15}{3}=5$$

Q50. If $\hat{x}(t)$ be the position operator at a time t in the Heisenberg picture for a particle described by the Hamiltonian, $\hat{H}=\frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2$ what is $e^{i\omega t}\langle 0|\hat{x}(t)\hat{x}(0)|0\rangle$ in units of $\frac{\hbar}{2m\omega}$ where $|0\rangle$ is the ground state?

Solution: Operator $\hat{X}(t)$ in Hisenburg picture is written as

$$\hat{X}(t)=e^{iHt/\hbar}\hat{X}(0)e^{iHt/\hbar}$$

Thus, $\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle=\langle 0|e^{iHt/\hbar}X(0)e^{-iHt/\hbar}X(0)|0\rangle$

Here, $\hat{X}(0)|0\rangle=\sqrt{\frac{\hbar}{2m\omega}}|1\rangle$

So, above equation reduces as,

$$\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle=\sqrt{\frac{\hbar}{2m\omega}}\langle 0|e^{iHt/\hbar}\hat{X}(0)e^{-iHt/\hbar}|1\rangle$$

In integral form,

$$\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle=\sqrt{\frac{\hbar}{2m\omega}}\int\phi_0^*(t)\hat{X}(0)\phi_1(t)dx$$

$$=\sqrt{\frac{\hbar}{2m\omega}}\int\phi_0^*e^{\frac{i\hbar\omega t}{2\hbar}}\hat{X}(0)\phi_1e^{\frac{-i3\hbar\omega t}{2\hbar}}dx=\sqrt{\frac{\hbar}{2m\omega}}e^{-i\omega t}\int\phi_0^*x\phi_1dx$$

Therefore, $e^{i\omega t}\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle=\left(\sqrt{\frac{\hbar}{2m\omega}}\right)^2\langle 0|a+a^\dagger|1\rangle$

$$e^{i\omega t}\langle 0|\hat{X}(t)\hat{X}(0)|0\rangle=\frac{\hbar}{2m\omega}$$

Q51. Consider a particle confined by a potential $V(x)=k|x|$, where k is a positive constant. The spectrum E_n of the system, within the WKB approximation is proportional to

(a) $\left(n+\frac{1}{2}\right)^{3/2}$ (b) $\left(n+\frac{1}{2}\right)^{2/3}$ (c) $\left(n+\frac{1}{2}\right)^{1/2}$ (d) $\left(n+\frac{1}{2}\right)^{4/3}$

Ans. : (b)

Solution: $V(x)=\begin{cases}kx & x>0\\ -kx & x<0\end{cases}$

$$\therefore \sqrt{2m}\int_0^b \sqrt{E-V(x)}dx = \left(n+\frac{1}{2}\right)\hbar\pi = 2\sqrt{2m}\int_0^{E/k}\sqrt{E-kx}dx = 2\sqrt{2m}\int_0^{E/k}\sqrt{E}\cdot\sqrt{1-\frac{k}{E}x}\,dx$$

$$= 2\sqrt{2mE}\int_0^1 \sqrt{1-t}\,\frac{E}{k}dt = \frac{2E}{k}\sqrt{2mE}\int_0^1\sqrt{1-t}\,dt = 2E^{3/2}\frac{\sqrt{2m}}{k}\times\frac{2}{3}$$

$$= \left(n+\frac{1}{2}\right)\hbar\pi \Rightarrow E_n^{3/2} = \frac{3\hbar\pi k}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)$$

$$E_n = \left[\frac{3\hbar\pi k}{4\sqrt{2m}}\left(n+\frac{1}{2}\right)\right]^{2/3}$$

Q52. Consider the Hamiltonian

$$H(t) = \alpha\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix} + \beta t\begin{pmatrix}0&0&1\\0&0&0\\1&0&-2\end{pmatrix}$$

The time dependent function $\beta(t)=\alpha$ for $t\le 0$ and zero for $t>0$. Find $\left|\langle\Psi(t<0)|\Psi(t>0)\rangle\right|^2$, where $|\Psi(t<0)\rangle$ is the normalised ground state of the system at a time $t<0$ and $|\Psi(t>0)\rangle$ is the state of the system at $t>0$.

(a) $\frac{1}{2}(1+\cos(2\alpha t))$ (b) $\frac{1}{2}(1+\cos(\alpha t))$

(c) $\frac{1}{2}(1+\sin(2\alpha t))$ (d) $\frac{1}{2}(1+\sin(\alpha t))$

Ans. : (a)

Solution: $H(t) = \alpha\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix} + \beta(t)\begin{pmatrix}0&0&1\\0&0&0\\1&0&-2\end{pmatrix}$

Time dependent function $\beta(t) = \begin{cases}\alpha & , t\le 0\\ 0 & , t>0\end{cases}$

When $t\le 0$

$$H(t) = \alpha\begin{pmatrix}1&0&1\\0&2&0\\1&0&1\end{pmatrix}$$

Eigen value are 0 , 2α , 2α .

For Eigen value zero, the ground state wave function is $\left|\psi(t\le 0)\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$.

And $\left|\psi(t\ge 0)\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\end{pmatrix}e^{\frac{-i\alpha t}{\hbar}} - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\end{pmatrix}e^{\frac{-i3\alpha t}{\hbar}}$

Now, $\left|\left\langle\psi(t<0)\middle|\psi(t>0)\right\rangle\right|^2 = \frac{1}{4}\left|e^{\frac{-i\alpha t}{\hbar}} + e^{\frac{-i3\alpha t}{\hbar}}\right|^2$

$$= \frac{1}{4}\left[\left(\cos\frac{\alpha t}{\hbar} + \cos\frac{3\alpha t}{\hbar}\right)^2 + \left(-\sin\frac{\alpha t}{\hbar} - \sin\frac{3\alpha t}{\hbar}\right)^2\right]$$

$$= \frac{1}{4}\left[1+1+2\left(\cos\frac{\alpha t}{\hbar}.\cos\frac{3\alpha t}{\hbar} + \sin\frac{\alpha t}{\hbar}\cos\frac{3\alpha}{\hbar}\right)\right] = \frac{1}{4}\left[2+2.\cos\frac{2\alpha t}{\hbar}\right] = \frac{1}{2}\left[1+\cos\frac{2\alpha t}{\hbar}\right]$$

JEST-2018

Q53. If $\psi(x)$ is an infinitely differentiable function, then $\hat{D}\psi(x)$, where the operator $\hat{D} = \exp\left(ax\frac{d}{dx}\right)$, is

(a) $\psi(x+a)$ (b) $\psi(ae^a + x)$ (c) $\psi(e^a x)$ (d) $e^a\psi(x)$

Ans. : (c)

Q54. A one dimensional harmonic oscillator (mass m and frequency ω) is in a state $|\psi\rangle$ such that the only possible outcomes of an energy measurement are E_0, E_1 or E_2, where E_n is the energy of the n-th excited state. If H is the Hamiltonian of the oscillator, $\langle\psi|H|\psi\rangle = \frac{3\hbar\omega}{2}$ and $\langle\psi|H^2|\psi\rangle = \frac{11\hbar^2\omega^2}{4}$, then the probability that the energy measurement yields E_0 is

(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) 0

Ans. : (b)

Solution: $|\psi\rangle = a|\phi_0\rangle + b|\phi_1\rangle + c|\phi_2\rangle$ let us assume a,b,c is real

$$\langle H\rangle = \frac{a^2\times\frac{\hbar\omega}{2}+b^2\times\frac{3\hbar\omega}{2}+c^2\times\frac{5\hbar\omega}{2}}{a^2+b^2+c^2}=\frac{3}{4}\hbar\omega \Rightarrow \frac{a^2}{2}+\frac{3b^2}{2}+\frac{5c^2}{2}=\frac{3}{4}\hbar\omega \quad \text{....(i)}$$

$$\langle H\rangle = \frac{a^2\times\left(\frac{\hbar\omega}{2}\right)^2+b^2\times\left(\frac{3\hbar\omega}{2}\right)^2+c^2\times\left(\frac{5\hbar\omega}{2}\right)^2}{a^2+b^2+c^2}=\frac{11\hbar^2\omega^2}{4}$$

$$\Rightarrow \frac{a^2}{4}+\frac{9b^2}{4}+\frac{25c^2}{4}=\frac{11\hbar^2\omega^2}{4} \quad \text{.....(ii)}$$

$$a^2+b^2+c^2=1 \quad \text{....(iii)}$$

Solving $a^2=\frac{1}{4}, b^2=\frac{1}{2}, c^2=\frac{1}{4}$

$$P\left(\frac{\hbar\omega}{2}\right)=\frac{a^2}{a^2+b^2+c^2}=a^2=\frac{1}{4}$$

Q55. A quantum particle of mass m is moving on a horizontal circular path of radius a. The particle is prepared in a quantum state described by the wavefunction

$$\psi=\sqrt{\frac{4}{3\pi}}\cos^2\phi,$$

ϕ being the azimuthal angle. If a measurement of the z-component of orbital angular momentum of die particle is carried out, the possible outcomes and the corresponding probabilities are

(a) $L_z=0,\pm\hbar,\pm2\hbar$ with 0 $P(0)=\frac{1}{5}, P(\pm\hbar)=\frac{1}{5}$ and $P(\pm2\hbar)=\frac{1}{5}$

(b) $L_z=0$ with $P(0)=1$

(c) $L_z=0,\pm\hbar$ with $P(0)=\frac{1}{3}$ and $P(\pm\hbar)=\frac{1}{3}$

(d) $L_z=0,\pm2\hbar$ with $P(0)=\frac{2}{3}$ and $P(\pm2\hbar)=\frac{1}{6}$

Ans. : (d)

Solution: $\psi=\sqrt{\frac{4}{3\pi}}\cos^2\phi=\sqrt{\frac{4}{3\pi}}\left(\frac{1+\cos2\phi}{2}\right)\Rightarrow\psi=\sqrt{\frac{4}{3\pi}}.\frac{1}{2}\left[\frac{\sqrt{2\pi}}{\sqrt{2\pi}}+\frac{\sqrt{2\pi}\left(\exp 2i\phi+\exp-2i\phi\right)}{\sqrt{2\pi}\quad 2}\right]$

$$\psi = \sqrt{\frac{2}{3}}|0\rangle + \frac{1}{\sqrt{6}}|2\rangle + \frac{1}{\sqrt{6}}|-2\rangle$$

$L_z = 0, \pm 2\hbar$ with $P(0) = \frac{2}{3}$ and $P(\pm 2\hbar) = \frac{1}{6}$

Q56. Consider two canonically conjugate operators $\hat{X}$ and $\hat{Y}$ such that $[\hat{X}, \hat{Y}] = i\hbar I$, where I is identity operator. If $\hat{X} = \alpha_{11}\hat{Q}_1 + \alpha_{12}\hat{Q}_2, \hat{Y} = \alpha_{21}\hat{Q}_1 + \alpha_{22}\hat{Q}_2$, where α_{ij} are complex numbers and $[\hat{Q}_1, \hat{Q}_2] = zI$, the value of $\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}$ is

(a) $i\hbar z$ (b) $\frac{i\hbar}{z}$ (c) $i\hbar$ (d) z

Ans. : (b)

Solution: $[\hat{X}, \hat{Y}] = i\hbar I$, $[\alpha_{11}\hat{Q}_1 + \alpha_{12}\hat{Q}_2, \alpha_{21}\hat{Q}_1 + \alpha_{22}\hat{Q}_2] = i\hbar I$

$$\Rightarrow [\alpha_{11}\hat{Q}_1, \alpha_{22}\hat{Q}_2] + [\alpha_{12}\hat{Q}_2, \alpha_{21}\hat{Q}_1] = \alpha_{11}\alpha_{22}[\hat{Q}_1, \hat{Q}_2] + \alpha_{12}\alpha_{21}[\hat{Q}_2, \hat{Q}_1]$$

$$[\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}]zI = i\hbar I \Rightarrow [\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}] = \frac{i\hbar}{z}$$

Q57. Suppose the spin degree of freedom of two particles (nonzero rest mass and nonzero spin) is described completely by a Hilbert space of dimension twenty one. Which of the following could be the spin of one of the particles?

(a) 2 (b) $\frac{3}{2}$ (c) 1 (d) $\frac{1}{2}$

Ans. : (c)

Solution: $(2s_1 + 1) \otimes (2s_2 + 1) = 21 = 7 \times 3 \Rightarrow s_1 = 3, s_2 = 1$

Q58. The normalized eigenfunctions and eigenvalues of the Hamiltonian of a Particle confined to move between $0 \le x \le a$ in one dimension are

$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\frac{n\pi x}{a} \text{ and } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

respectively. Here $1, 2, 3 \ldots$. Suppose the state of the particle is

$$\psi(x) = A\sin\left(\frac{\pi x}{a}\right)\left[1 + \cos\left(\frac{\pi x}{a}\right)\right]$$

where A is the normalization constant. If the energy of the particle is measured, the probability to get the result as $\frac{\pi^2\hbar^2}{2ma^2}$ is $\frac{x}{100}$. What is the value of x?

Ans. : 80

Solution: $\psi(x) = A\sin\left(\frac{\pi x}{a}\right)\left[1+\cos\left(\frac{\pi x}{a}\right)\right] = \sqrt{\frac{a}{2}}A\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)+\frac{2}{2}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\right]$

$$\psi(x) = \sqrt{\frac{a}{2}}A\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)+\frac{2}{2}\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)\right]$$

$$|\psi\rangle = \sqrt{\frac{a}{2}}A\left[|\phi_1\rangle+\frac{1}{2}|\phi_2\rangle\right]$$

$$\langle\psi|\psi\rangle = 1 \Rightarrow \frac{a}{2}A^2\left[1+\frac{1}{4}\right] \Rightarrow A^2\frac{a}{2}\times\frac{5}{4} = 1 \Rightarrow A = \sqrt{\frac{8}{5a}}$$

$$|\psi\rangle = \sqrt{\frac{a}{2}}.\sqrt{\frac{8}{5a}}\left[|\phi_1\rangle+\frac{1}{2}|\phi_2\rangle\right] \Rightarrow \sqrt{\frac{4}{5}}|\phi_1\rangle+\sqrt{\frac{1}{5}}|\phi_2\rangle$$

$$P\left(\frac{\pi^2\hbar^2}{2ma^2}\right) = \frac{4}{5} = \frac{x}{100} \Rightarrow x = \frac{4}{5}\times 100 = 80$$

Q59. A harmonic oscillator has the following Hamiltonian

$$H_0 = \frac{\hat{p}^2}{2m}+\frac{1}{2}m\omega^2\hat{x}^2$$

It is perturbed with a potential $V = \lambda\hat{x}^4$. Some of the matrix elements of $\hat{x}^2$ in terms of its expectation value in the ground state are given as follows:

$$\langle 0|\hat{x}^2|0\rangle = C$$

$$\langle 0|\hat{x}^2|2\rangle = \sqrt{2}C$$

$$\langle 1|\hat{x}^2|1\rangle = 3C$$

$$\langle 1|\hat{x}^2|3\rangle = \sqrt{6}C$$

where $|n\rangle$ is the normalized eigenstate of H_0 corresponding to the eigenvalue $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$. Suppose ΔE_0 and ΔE_1 denote the energy correction of $O(\lambda)$ to thee ground state and the first excited state, respectively. What is the fraction $\frac{\Delta E_1}{\Delta E_0}$?

Ans. : 5

Solution: For n^{th} state $\Delta E_n = \langle n | X^4 | n \rangle = \frac{\hbar^2}{4m^2\omega^2}\left(6n^2 + 6n + 3\right)$

$$\Delta E_0 = \langle 0 | X^4 | 0 \rangle = \frac{3\hbar^2}{4m^2\omega^2} \quad \langle 1 | X^4 | 1 \rangle = \frac{\hbar^2}{4m^2\omega^2}\left(6.1^2 + 6.1 + 3\right) = \frac{15\hbar^2}{4m^2\omega^2}$$

$$\frac{\Delta E_1}{\Delta E_0} = 5$$

Q60. Consider a wavepacket defined by

$$\psi(x) = \int_{-\infty}^{\infty} dk f(k) \exp\left[i(kx)\right]$$

Further, $f(k) = 0$ for $|k| > \frac{K}{2}$ and $f(k) = a$ for $|k| \le \frac{K}{2}$. Then, the form of normalized $\psi(x)$ is

(a) $\frac{\sqrt{8\pi K}}{x}\sin\frac{Kx}{2}$

(b) $\sqrt{\frac{2}{\pi K}}\frac{\sin\frac{Kx}{2}}{x}$

(c) $\frac{\sqrt{8\pi K}}{x}\cos\frac{Kx}{2}$

(d) $\sqrt{\frac{2}{\pi K}}\frac{\sin\frac{Kx}{2}}{x}$

Ans. : (b)

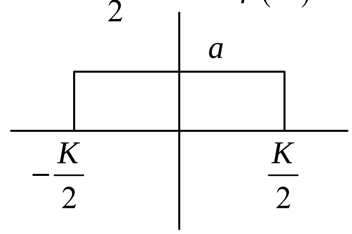
Solution: Given $\psi(x) = \int_{-\infty}^{\infty} dk f(k) e^{ikx}$

$$\psi(x) = \int_{-K/2}^{K/2} dK\, a\, e^{iKx}$$

$$|K| > \frac{K}{2}$$

$$= \frac{q}{ix} e^{ikx}\Big|_{-K/2}^{K/2} = \frac{q}{ix} e^{i\frac{K}{2}x} - e^{-i\frac{K}{2}x}$$

$$K > \frac{K}{2} \qquad f(K) = 0$$



$$\psi(x) = \frac{2}{x}\sin\frac{kx}{2}$$

$$A^2\int_{-\infty}^{\infty}\frac{2^2}{x^2}\frac{Kx}{2}dx = 1$$

$$4A^2\int_{-\infty}^{\infty}\frac{h^2 Kx/2}{x^2} = 1$$

$$4A^2\frac{\pi x}{2}=1$$

$$A^2=\frac{1}{2\pi K}\Rightarrow A=\sqrt{\frac{1}{2\pi K}}$$

$$\psi(x)=\frac{2}{x}\sqrt{\frac{1}{2\pi K}}\cdot\sin\frac{Kx}{2}$$

$$\therefore\ \psi(x)=\sqrt{\frac{2}{\pi K}}\frac{\sin\frac{Kx}{2}}{x}$$

JEST-2019

Q61. What is the binding energy of an electron in the ground state of a He^+ ion?

(a) $6.8\,eV$ (b) $13.6\,eV$ (c) $27.2\,eV$ (d) $54.4\,eV$

Ans. : (d)

Solution: $E=-\frac{13.6}{n^2}z^2(eV)$

$He^+ : z=2$

$$\therefore E=\frac{-13.6\times 4}{n^2}(eV)$$

The binding energy of an electron in ground state is

$$E=\frac{-13.6\times 4}{(1)^2}(eV)=54.4\,eV$$

Q62. The wave function $\psi(x)=A\exp\left(-\frac{b^2x^2}{2}\right)$ (for real constants A and b) is a normalized eigen-function of the Schrodinger equation for a particle of mass m and energy E in a one dimensional potential $V(x)$ such that $V(x)=0$ at $x=0$. Which of the following is correct?

(a) $V=\frac{\hbar^2b^4x^2}{m}$ (b) $V=\frac{\hbar^2b^4x^2}{2m}$ (c) $E=\frac{\hbar^2b^2}{4m}$ (d) $E=\frac{\hbar^2b^2}{m}$

Ans. : (b)

Solution: Comparing with harmonic oscillator $\psi(x)=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)$ the potential is

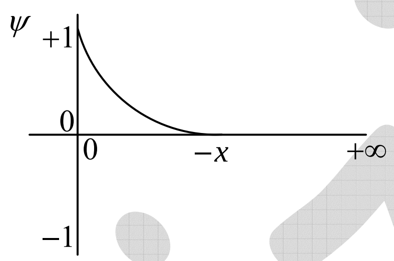
$V(x)=\frac{1}{2}m\omega^2x^2$ and energy is $E=\frac{\hbar\omega}{2}$

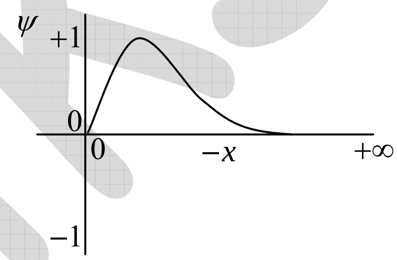
$\psi(x)=A\exp\left(-\frac{b^2x^2}{2}\right)$ $\quad\omega=\frac{b^2\hbar}{m}$ so $V(x)=\frac{b^4\hbar^2x^2}{2m}$ and energy $E=\frac{\hbar\omega}{2}\Rightarrow\frac{b^2\hbar^2}{2m}$

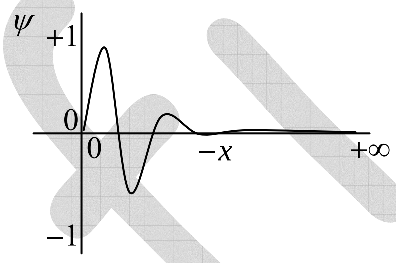
Q63. A quantum particle of mass m is in a one dimensional potential of the form

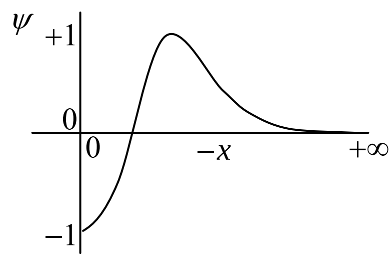
$$V(x)=\begin{cases}\frac{1}{2}m\omega^2x^2, & \text{if } x>0\\ \infty & \text{if } x\le 0\end{cases}$$

where ω is a constant. Which one of the following represents the possible ground state wave function of the particle?

(a)


(b)


(c)


(d)


Ans. : (b)

Q64. For a spin $\frac{1}{2}$ particle placed in a magnetic field B, the Hamiltonian is $H=-\gamma BS_y=-\omega S_y$, where S_y is the y-component of the spin operator. The state of the system at time $t=0$ is $|\psi(t=0)\rangle=|+\rangle$, where $S_z|\pm\rangle=\pm\frac{\hbar}{2}|\pm\rangle$. At a later time t, if S_z measured then what is the probability to get a value $-\frac{\hbar}{2}$?

(a) $\cos^2(\omega t)$ (b) $\sin^2(\omega t)$ (c) 0 (d) $\sin^2\left(\frac{\omega t}{2}\right)$

Ans. : (d)

Solution: $H = -\gamma B S_y = -\omega S_y$ Eigen value is $\frac{-\omega\hbar}{2}, \frac{\omega\hbar}{2}$ with eigen vector $|\phi_1\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle + |-\rangle\right]$

and $|\phi_2\rangle = \frac{1}{\sqrt{2}}\left[|+\rangle - |-\rangle\right]$ respectively.

$$|\psi(t=0)\rangle = |+\rangle \Rightarrow I|+\rangle \Rightarrow |\phi_1\rangle\langle\phi_1|+\rangle + |\phi_2\rangle\langle\phi_2|+\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{2}}|\phi_2\rangle$$

$$|\psi(t=t)\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle \exp\left(\frac{i\omega t}{2}\right) + \frac{1}{\sqrt{2}}|\phi_2\rangle \exp\left(-\frac{i\omega t}{2}\right)$$

If S_z is measured on $|\psi(t)\rangle$ then probability to find $-\frac{\hbar}{2}$ is

$$P\left(-\frac{\hbar}{2}\right) = \frac{|\langle -|\psi(t)\rangle|^2}{\langle\psi(t)|\psi(t)\rangle} = \frac{1}{4}\left|\left(\exp\left(\frac{i\omega t}{2}\right) - \exp\left(-\frac{i\omega t}{2}\right)\right)\right|^2 = \sin^2\frac{\omega t}{2}$$

Q65. Consider a quantum particle in a one-dimensional box of length L. The coordinates of the leftmost wall of the box is at $x = 0$ and that of the rightmost wall is at $x = L$. The particle is in the ground state at $t = 0$. At $t = 0$, we suddenly change the length of the box to $3L$ by moving the right wall. What is the probability that the particle is in the ground state of the new system immediately after the change?

(a) 0.36 (b) $\frac{9}{8\pi}$ (c) $\frac{81}{64\pi^2}$ (d) $\frac{0.5}{\pi}L$

Ans. : (c)

Solution: $|\phi_1\rangle = \begin{cases} \sqrt{\frac{2}{3a}}\sin\frac{\pi x}{3a}. & 0 < x < 3a \\ 0, & otherwise \end{cases}$ $\quad |\psi\rangle = \begin{cases} \sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}. & 0 < x < a \\ 0, & otherwise \end{cases}$

$$P\left(\frac{\pi^2\hbar^2}{2m(3a)^2}\right) = \frac{|\langle\phi_1|\psi\rangle|^2}{\langle\psi|\psi\rangle} = \int_0^a \sqrt{\frac{2}{3a}}\sin\frac{\pi x}{3a}\sqrt{\frac{2}{a}}\sin\frac{\pi x}{a}\,dx = \frac{81}{64\pi^2}$$

Q66. Consider a quantum particle of mass m and a charge e moving in a two dimensional potential given as:

$$V(x,y)=\frac{k}{2}(x-y)^2+k(x+y)^2$$

The particle is also subject to an external electric field $\vec{E}=\lambda(\hat{i}-\hat{j})$, where λ is a constant $\hat{i}$ and $\hat{j}$ corresponds to unit vectors along x and y directions, respectively. Let E_1 and E_0 be the energies of the first excited state and ground state, respectively. What is the value of E_1-E_0?

(a) $\hbar\sqrt{\frac{2k}{m}}$ (b) $\hbar\sqrt{\frac{2k}{m}}+e\lambda^2$ (c) $3\hbar\sqrt{\frac{2k}{m}}$ (d) $3\hbar\sqrt{\frac{2k}{m}}+e\lambda^2$

Ans. : (a)

Solution: For constant electric field we know there is not any change in frequency and energy of each level is changed by constant value.

The total potential is

$$V(x,y)=\frac{k}{2}(x-y)^2+k(x+y)^2-\lambda x+\lambda y\Rightarrow V(x,y)=\frac{3}{2}kx^2+\frac{3}{2}ky^2+kxy-\lambda x+\lambda y$$

$$T=\begin{pmatrix} m & 0\\ 0 & m\end{pmatrix}\quad \text{and}\quad V=\begin{pmatrix} 3k & k\\ k & 3k\end{pmatrix}$$

Secular equation is given by

$$\left|V-\omega^2 m\right|=0\Rightarrow\left(3k-\omega^2 m\right)^2-k^2=0\Rightarrow\omega_x=\sqrt{\frac{4k}{m}},\omega_y=\sqrt{\frac{2k}{m}}$$

The equivalent quantum mechanical energy is $E_{n_x,n_y}=\left(n_x+\frac{1}{2}\right)\hbar\omega_x+\left(n_y+\frac{1}{2}\right)\hbar\omega_y+V_0$

Where $n_x=0,1,2,3...$ and $n_y=0,1,2,3...$

The ground state energy $E_0=E_{0,0}=\frac{\hbar}{2}\sqrt{\frac{4k}{m}}+\frac{\hbar}{2}\sqrt{\frac{2k}{m}}$

The first excited state energy $E_1=E_{0,1}=\frac{\hbar}{2}\sqrt{\frac{4k}{m}}+\frac{3\hbar}{2}\sqrt{\frac{2k}{m}}$

$$E_1-E_0=\hbar\sqrt{\frac{2k}{m}}$$

Q67. A one-dimensional harmonic oscillator is in the state

$$|\psi\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} |n\rangle$$

where $|n\rangle$ is the normalized energy eigenstate with eigenvalue $\left(n+\frac{1}{2}\right)\hbar\omega$. Let the expectation value of the Hamiltonian in the state $|\psi\rangle$ be expressed as $\frac{1}{2}\alpha\hbar\omega$. What is the value of α?

Ans. : 3

Solution: $\langle H\rangle = \sum_{n=0}^{\infty} \frac{\left(n+\frac{1}{2}\right)\hbar\omega}{\lfloor n} = \frac{1}{2}\hbar\omega + \hbar\omega\sum_{n=1}^{\infty}\frac{n}{\lfloor n} = \left[\frac{1}{2}+e\right]\hbar\omega = 3.2\hbar\omega$

Q68. Consider a system of 15 non-interacting spin-polarized electrons. They are trapped in a two dimensional isotropic harmonic oscillator potential $V(x,y) = \frac{1}{2}m\omega^2\left(x^2+y^2\right)$. The angular frequency ω is such that $\hbar\omega = 1$ in some chosen unit. What is the ground state energy of the system in the same units?

Ans. : 55

Solution: Non-interacting spin-polarized electrons means direction of spin is fixed

$$1\times\hbar\omega + 2\times 2\hbar\omega + 3\times 3\hbar\omega + 4\times 4\hbar\omega + 5\times 5\hbar\omega = 55\hbar\omega$$