Thermodynamics & Statistical Mechanics

JEST-2012

Q1. A monatomic ideal gas at 170^0C is adiabatically compressed to $1/8$ of its original volume. The temperature after compression is

(a) 2.1^0C (b) 17^0C (c) -200.5^0C (d) 887^0C

Ans. : (d)

Solution: $PV^\gamma = \text{costant}, PV = RT$

$$\frac{TV^\gamma}{V} = \text{costant}$$

$$\Rightarrow TV^{\gamma-1} = \text{costant}$$

$$\Rightarrow T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \Rightarrow T_2 = T_1\left(\frac{V_1}{V_2}\right)^x \Rightarrow 443(8)^{\frac{5}{3}-1} = 443\times(8)^{2/3} = 443\times 4$$

Temperature in ${}^0C = 1772 - 273 = 1499$

$\therefore$ Most appropriate answer is option (d)

Q2. Consider a system of particles in three dimensions with momentum $\vec{p}$ and energy $E = c|\vec{p}|, c$ being a constant. The system is maintained at inverse temperature β, volume V and chemical potential μ. What is the grand partition function of the system?

(a) $\exp\left[e^{\beta\mu}8\pi V/(\beta ch)^3\right]$ (b) $e^{\beta\mu}6\pi V/(\beta ch)^2$

(c) $\exp\left[e^{\beta\mu}6\pi V/(\beta ch)^3\right]$ (d) $e^{\beta\mu}8\pi V/(\beta ch)^2$

Ans. : (a)

Solution: Canonical partition function,

$$z_N = \frac{1}{h^3}\int e^{-\beta H}dp_xdp_ydp_zdxdydz, \qquad E = pc$$

$$z_N = \frac{V}{h^3}\int_0^\infty 4\pi\, p^2 e^{-E\beta}dp = \frac{V}{h^3}\int_0^\infty 4\pi \quad p^2 e^{-\beta pc}dp = \frac{4\pi \quad V}{h^3}\cdot\frac{\lceil 3}{(\beta c)^3} = \frac{8\pi \quad V}{(\beta hc)^3}$$

Grand canonical partition function, $z_u = \exp\left[e^{\frac{\mu}{kT}}z_N\right] = \exp\left[e^{\frac{\mu}{kT}}\cdot\frac{8\pi V}{(\beta hc)^3}\right]$

$$\Rightarrow \exp\left[e^{\beta\mu}\cdot\frac{8\pi V}{(\beta hc)^3}\right]$$

Q3. Consider a system maintained at temperature T, with two available energy states E_1 and E_2 each with degeneracies g_1 and g_2. If p_1 and p_2 are probabilities of occupancy of the two energy states, what is the entropy of the system?

(a) $S = -k_B[p_1 \ln(p_1/g_1) + p_2 \ln(p_2/g_2)]$

(b) $S = -k_B[p_1 \ln(p_1 g_1) + p_2 \ln(p_2 g_2)]$

(c) $S = -k_B\left[p_1 \ln\left(p_1^{g_1}\right) + p_2 \ln\left(p_2^{g_2}\right)\right]$

(d) $S = -k_B[(1/p_1)\ln(p_1/g_1) + (1/p_2)\ln(p_2/g_2)]$

Ans. : (a)

Solution: $p_i = \dfrac{\Sigma g_i e^{-\beta E_i}}{z}$, where z is partition function

$$\Rightarrow \ln p_i = \ln g_i - \beta E_i - \ln z$$

$$\Rightarrow \ln \frac{p_i}{g_i} = -\beta E_i + \frac{F}{kT} \qquad [\because F = -kT \ln z]$$

$$\Rightarrow \left\langle \ln \frac{p_i}{g_i} \right\rangle = -\beta \langle E_i \rangle + \beta \langle F \rangle$$

$$\Rightarrow \left\langle \ln \frac{p_i}{g_i} \right\rangle = \beta [F - U] \qquad [\because F = U - TS]$$

$$\left\langle \ln \frac{p_i}{g_i} \right\rangle = -\beta \times TS, \qquad \left[\because \beta = \frac{1}{kT}\right]$$

$$S = -k \left\langle \ln \frac{p_i}{g_i} \right\rangle = -k \left(\sum p_i \ln \frac{p_i}{g_i} \right) = -k \left[p_1 \ln \frac{p_1}{g_1} + p_2 \ln \frac{p_2}{g_2} \right]$$

Q4. Efficiency of a perfectly reversible (Carnot) heat engine operating between absolute temperature T and zero is equal to

(a) 0 (b) 0.5 (c) 0.75 (d) 1

Ans. : (d)

Solution: $\eta = 1 - \dfrac{T_2}{T_1} = 1 - \dfrac{0}{T} = 1$

Q5. Consider an ideal gas of mass m at temperature T_1 which is mixed isobarically (i.e. at constant pressure) with an equal mass of same gas at temperature T_2 in a thermally insulated container. What is the change of entropy of the universe?

(a) $2mC_p \ln\left(\frac{T_1+T_2}{2\sqrt{T_1T_2}}\right)$ (b) $2mC_p \ln\left(\frac{T_1-T_2}{2\sqrt{T_1T_2}}\right)$

(c) $2mC_p \ln\left(\frac{T_1+T_2}{2T_1T_2}\right)$ (d) $2mC_p \ln\left(\frac{T_1-T_2}{2\sqrt{T_1T_2}}\right)$

Ans. : (a)

Solution: Let us consider final temperature will be T

$$mC(T_1-T)=mC(T-T_2)\Rightarrow T=\frac{T_1+T_2}{2}$$

$$\Delta S_1 = mC_p\frac{\Delta T}{T}$$

Now, $\Delta S=\Delta S_1+\Delta S_2 \Rightarrow \Delta S = mC_p\int_{T_1}^{T}\frac{dT}{T}+mC_p\int_{T_2}^{T}\frac{dT}{T}$

$$\Rightarrow \Delta S = mC_p\ln\left(\frac{T}{T_1}\right)+mC_p\ln\left(\frac{T}{T_2}\right)$$

$$\Rightarrow \Delta S = 2mC_p\ln\frac{T}{\sqrt{T_1T_2}} = mC_p\ln\left(\frac{T_1+T_2}{2\sqrt{T_1T_2}}\right)^2 \Rightarrow \Delta S = 2mC_p\ln\left(\frac{T_1+T_2}{2\sqrt{T_1T_2}}\right)$$

Q6. A collection of N two-level systems with energies 0 and $E>0$ is in thermal equilibrium at temperature T. For $T\to\infty$, the specific heat approaches to,

(a) 0 (b) Nk_B (c) $\frac{3Nk_B}{2}$ (d) ∞

Ans.: (a)

Solution: $Z=\sum e^{-\beta E_i}=e^{-\beta\times 0}+e^{-\beta E_i}\Rightarrow Z=1+e^{-\beta E}\Rightarrow \ln z=\ln\left(1+e^{-\beta E}\right)$

$$U=\langle E\rangle = -\frac{\partial}{\partial\beta}\ln z = -\frac{\partial}{\partial\beta}\ln\left(1+e^{-\beta E}\right) = -\frac{1}{1+e^{-\beta E}}\times e^{-\beta E}(-E)=\frac{Ee^{-\beta E}}{1+e^{-\beta E}}$$

Now, $\left(\frac{\partial U}{\partial T}\right)_V = C_V = \frac{\partial}{\partial T}\left(\frac{Ee^{-\frac{E}{kT}}}{1+e^{-\frac{E}{kT}}}\right)$

$$\Rightarrow C_V = \frac{\left(\frac{E^2}{kT^2}e^{\frac{-E}{kT}} + \frac{E^2}{kT^2}e^{\frac{-2E}{kT}} - \frac{E^2}{kT^2}e^{\frac{-2E}{kT}}\right)}{\left(1+e^{\frac{-E}{kT}}\right)^2} \Rightarrow C_V = \frac{\frac{E^2}{kT^2}e^{\frac{-E}{kT}}}{\left(1+e^{\frac{-E}{kT}}\right)^2} \Rightarrow C_V\big|_{T\to\infty} = 0$$

Q7. A thermally insulated ideal gas of volume V_1 and temperature T expands to another enclosure of volume V_2 through a porous plug. What is the change in the temperature of the gas?

(a) 0 (b) $T\ \ln\left(\frac{V_1}{V_2}\right)$ (c) $T\ \ln\left(\frac{V_2}{V_1}\right)$ (d) $T\ln\left(\frac{V_2-V_1}{V_2}\right)$

Ans. : (c)

Solution: $dH = TdS + VdP$, for porous plug Joul Thomshon $dH = 0$ and $TdS = 0$ since it is thermally insulated ideal gas

$VdP = 0$

$$\because VdP = 0 \Rightarrow nRdT = pdV \Rightarrow nRdT = \frac{nRTdV}{V}$$

$$dT = T\frac{dV}{V} \Rightarrow dT = T\int_{V_1}^{V_2}\frac{dV}{V} \Rightarrow dT = T\ln\frac{V_2}{V_1}$$

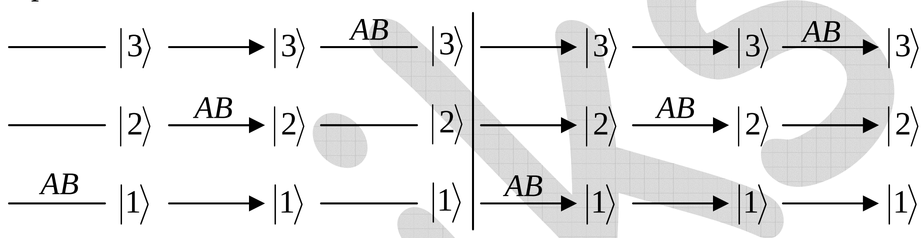
JEST-2013

Q8. Consider a system of two particles A and B. Each particle can occupy one of three possible quantum states $|1\rangle, |2\rangle$ and $|3\rangle$. The ratio of the probability that the two particles are in the same state to the probability that the two particles are in different states is calculated for bosons and classical (Maxwell-Boltzmann) particles. They are respectively

(a) 1, 0 (b) $\frac{1}{2}$, 1 (c) 1, $\frac{1}{2}$ (d) 0, $\frac{1}{2}$

Ans.: (c)

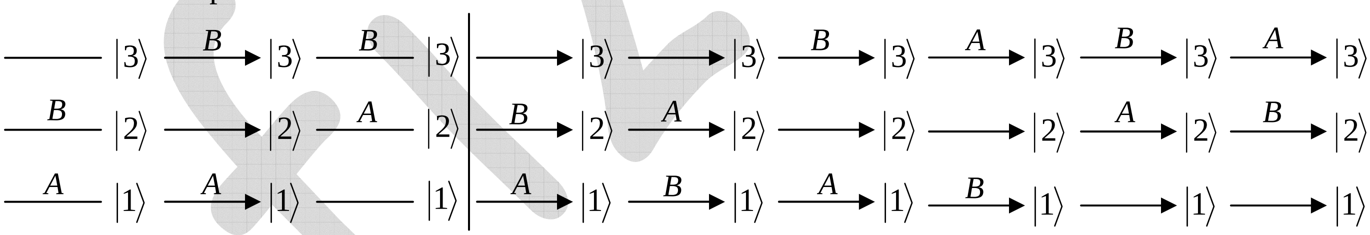
Solution: For two particle in same state:



Boson Classical (Maxwell - Boltzman)

Probability ratio: $\frac{1/3}{1/3} = 1$

For two particle in different states



Boson Classical (Maxwell-Boltzmann)

Probability ratio: $\frac{1/3}{2/3} = \frac{1}{2}$

Q9. For a diatomic ideal gas near room temperature, what fraction of the heat supplied is available for external work if the gas is expanded at constant pressure?

(a) $\frac{1}{7}$ (b) $\frac{5}{7}$ (c) $\frac{3}{4}$ (d) $\frac{2}{7}$

Ans.: (d)

Solution: It is isobaric process (constant pressure). Then $\delta Q = nC_p\Delta T \Rightarrow \Delta W = nR\Delta T$

In this process δQ is heat exchange during process.

Function of heat supplied

$$=\frac{\delta W}{\Delta Q}=\frac{nR\Delta T}{nC_p\Delta T}=\frac{R}{R\frac{\gamma}{\gamma-1}}=\frac{\gamma-1}{\gamma}=1-\frac{1}{\gamma}$$

$$=1-\frac{1}{\left(1+\frac{2}{f}\right)} \qquad \left[\because \quad \gamma=\frac{C_p}{C_V}\Rightarrow C_p=\frac{\gamma R}{\gamma-1}\right]$$

$$=1-\frac{f}{f+2}$$ [f = degree of freedom, for diatomic molecule $f=5$]

$$\Rightarrow 1-\frac{5}{5+2}=\frac{2}{7}$$

Q10. A metal bullet comes to rest after hitting its target with a velocity of $80\,m/s$. If 50% of the heat generated remains in the bullet, what is the increase in its temperature? (The specific heat of the bullet $=160$ $Joule/kg/{}^0C$)

(a) $14^0\,C$ (b) $12.5^0\,C$ (c) $10^0\,C$ (d) $8.2^0\,C$

Ans.: (c)

Solution: Conservation of momentum 50% of $\frac{1}{2}mv^2=mc\Delta T\Rightarrow\frac{1}{2}\frac{80\times80}{2}=160\quad\Delta T$

$$\Rightarrow\Delta T=\frac{80\times80}{4}\times\frac{1}{160}=10^0C$$

Q11. Consider a particle with three possible spin states: $s=0$ and ±1. There is a magnetic field h present and the energy for a spin state s is $-hs$. The system is at a temperature T. Which of the following statements is true about the entropy $S(T)$?

(a) $S(T)=\ln 3$ at $T=0$, and 3 at high T (b) $S(T)=\ln 3$ at $T=0$, and 0 at high T

(c) $S(T)=0$ at $T=0$, and 3 at high T (d) $S(T)=0$ at $T=0$, and $\ln 3$ at high T

Ans.: (d)

Solution: $S=k\ln\omega$, where $\omega=$ number of microstates

$S=k\ln 3$ $\quad\omega=3$, at height T and at $T=0$, it is perfect ordered i.e. $S=0$

Q12. Consider three situations of 4 particles in one dimensional box of width L with hard walls. In case (i), the particles are fermions, in case (ii) they are bosons, and in case (iii) they are classical. If the total ground state energy of the four particles in these three cases are E_F, E_B and E_{cl} respectively, which of the following is true?

(a) $E_F = E_B = E_{cl}$

(b) $E_F > E_B = E_{cl}$

(c) $E_F < E_B < E_{cl}$

(d) $E_F > E_B > E_{cl}$

Ans.: (b)

Solution: For fermions, in 1-D box of width L, the ground state energy for single particle is written as, $\frac{\pi^2\hbar^2}{2ml^2} = \epsilon_0$

$\Rightarrow 1\times\epsilon_0 + 1\times 4\epsilon_0 + 1\times 9\epsilon_0 + 1\times 16\epsilon_0 = 30\epsilon_0$

For Boson $= 4\times\epsilon_0$, For Maxwell $= 4\times\epsilon_0$

$E_F > E_B = E_{cl}$

JEST-2014

Q13. A monoatomic gas consists of atoms with two internal energy levels, ground state $E_0 = 0$ and an excited state $E_1 = E$. The specific heat of the gas is given by

(a) $\frac{3}{2}k$

(b) $\frac{E^2 e^{E/kT}}{kT^2\left(1+e^{E/kT}\right)^2}$

(c) $\frac{3}{2}k + \frac{E^2 e^{E/kT}}{kT^2\left(1+e^{E/kT}\right)^2}$

(d) $\frac{3}{2}k - \frac{E^2 e^{E/kT}}{kT^2\left(1+e^{E/kT}\right)^2}$

Ans.: (c)

Solution: $E_0 = 0, \quad E_1 = E$

Then partition function is

$$z = \sum e^{-\beta E_i} \Rightarrow z = e^{-\beta\times 0} + e^{-\beta E} \Rightarrow \ln z = \ln\left(1+e^{-\beta E_1}\right)$$

$$U = \langle E\rangle = \frac{-\partial}{\partial\beta}\ln z = -\frac{\partial}{\partial\beta}\ln\left(1+e^{-\beta E}\right) = -\frac{1}{\left(1+e^{-\beta E}\right)}(-E)e^{-\beta E} = \frac{Ee^{-\beta E}}{1+e^{-\beta E}} \qquad [\because \beta = k_B T]$$

$$\left(\frac{\partial U}{\partial T}\right)_v = C_V = \frac{\left(1+e^{-\frac{E}{k_BT}}\right)E.e^{-\frac{E}{k_BT}}.\left(\frac{E}{k_BT^2}\right) - Ee^{-\frac{E}{k_BT}}\cdot e^{-\frac{E}{k_BT}}\left(\frac{E}{k_BT^2}\right)}{\left(1+e^{-\frac{E}{k_BT}}\right)^2}$$

$$C_V = \frac{\frac{E^2}{k_BT^2}e^{-\frac{E}{k_BT}} + \frac{E^2}{k_BT^2}e^{-\frac{2E}{k_BT}} - \frac{E^2}{k_BT^2}e^{-\frac{2E}{k_BT}}}{\left(1+e^{-\frac{E}{k_BT}}\right)^2} = \frac{E^2e^{-\frac{E}{k_BT}}}{k_BT^2\left(1+e^{-\frac{E}{k_BT}}\right)^2} = \frac{E^2e^{\frac{E}{k_BT}}}{k_BT^2\left(1+e^{\frac{E}{k_BT}}\right)^2}$$

If gas will classically allowed, then $C_V = \frac{3}{2}k_B$

and quantum mechanically, $C_V = \frac{E^2e^{\frac{E}{k_BT}}}{k_BT^2\left(1+e^{\frac{E}{k_BT}}\right)^2}$

$$\therefore \quad C_V = \frac{3}{2}k_B + \frac{E^2e^{E/kT}}{kT^2\left(1+e^{E/kT}\right)^2}$$

Q14. The temperature of a thin bulb filament (assuming that the resistance of the filament is nearly constant) of radius r and length L is proportional to

(a) $r^{1/4}L^{-1/2}$ (b) L^2r (c) $L^{1/4}r^{-1}$ (d) r^2L^{-1}

Ans.: (a)

Q15. Ice of density ρ_1 melts at pressure P and absolute temperature T to form water of density ρ_2. The latent heat of melting of 1 gram of ice is L. What is the change in the internal energy ΔU resulting from the melting of 1 gram of ice?

(a) $L+P\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)$ (b) $L-P\left(\frac{1}{\rho_2}-\frac{1}{\rho_1}\right)$

(c) $L-P\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)$ (d) $L+P\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)$

Ans.: (d)

Solution: $dU = dQ - \delta W = dQ - PdV$

$$dU = mL - PdV \Rightarrow dU = L - P\int_{\rho_1}^{\rho_2}\left(-\frac{1}{\rho^2}\right)d\rho = L + P\left[\frac{1}{\rho_1} - \frac{1}{\rho_2}\right]$$

$$\because \quad V = \frac{1}{\rho} \Rightarrow dV = -\frac{1}{\rho^2}d\rho$$

Q16. What is the contribution of the conduction electrons in the molar entropy of a metal with electronic coefficient of specific heat?

(a) γT (b) γT^2 (c) γT^3 (d) γT^4

Ans.: (a)

Solution: $C_V = BT^3 + AT$

Q17. Consider a system of $2N$ non-interacting spin $1/2$ particles each fixed in position and carrying a magnetic moment μ. The system is immersed in a uniform magnetic field B. The number of spin up particles for which the entropy of the system will be maximum is

(a) 0 (b) N (c) $2N$ (d) $N/2$

Ans.: (b)

Solution: Let us consider n number of spin out of $2N$ particle have spin up remaining $2N-n$ is down.

$$\text{Number of ways, } \omega = \begin{cases} {}^{2N}C_n & \text{for spin } \frac{1}{2} \text{(up)} \\ {}^{2N}C_{2N-n} & \text{for spin } \frac{1}{2} \text{(down)} \end{cases},$$

$$\text{Entropy, } S = k\ln\omega \Rightarrow S = k\ln {}^{2N}C_{2N-n} + k\ln {}^{2N}C_n$$

$$S = k\left\{\left[\ln\frac{2N!}{(n!)(2N-n)!}\right] + \left[\ln\frac{2N!}{(n!)(2N-n)!}\right]\right\}$$

$$S = 2k\left[(\ln 2N! - \ln n! - \ln(2N-n)!)\right]$$

$$S = 2k\left[2N\ln 2N - 2N - n\ln n + n - \{(2N-n)\ln(2N-n) - (2N-n)\}\right]$$

$$[\because \ln N! = N\ln N - N!]$$

$$S = 2k\left[2N\ln 2N - 2N - n\ln n + n - 2N\ln(2N-n) + n\ln(2N-n) + (2N-n)\right]$$

$S = 2k\left[2N\ln 2N - n\ln n - 2N\ln(2N-n) + n\ln(2N-n)\right]$

Now for maximum entropy at equilibrium for spin $\frac{1}{2}$ up particle,

$$\frac{dS}{dn} = 0$$

$$\frac{dS}{dn} = 2k\left[-\frac{n}{n}\cdot 1 - \ln n - \frac{2N}{2N-n}(-1) + \frac{n}{2N-n}(-1) + \ln(2N-n)\right]$$

$$= 2k\left[-1 - \ln n + \frac{2N}{2N-n} - \frac{n}{2N-n} + \ln(2N-n)\right]$$

$$= 2k\left[-1 + \frac{2N-n}{2N-n} + \ln(2N-n) - \ln n\right] \Rightarrow 2k\left[-1+1+\ln\frac{(2N-n)}{n}\right] = 0$$

$\because \quad 2k \neq 0$

$$\therefore \ln\left(\frac{2N-n}{n}\right) = 0 \Rightarrow \frac{2N-n}{n} = 1 \Rightarrow 2N = 2n \Rightarrow n = N$$

Q18. For which gas the ratio of specific heats (C_p / C_v) will be the largest?

(a) mono-atomic (b) di-atomic (c) tri-atomic (d) hexa-atomic

Ans.: (a)

Solution: $\frac{C_P}{C_V} = \gamma = \left(1 + \frac{2}{f}\right)$, where f is degree of freedom.

For monoatomic: $f = 3$, For diatomic: $f = 6$, For Triatomic: $f = 9$

For hexaatomic: $f = 18$

JEST-2015

Q19. For a system in thermal equilibrium with a heat bath at temperature T, which one of the following equalities is correct? $\left(\beta = \frac{1}{k_B T}\right)$

(a) $\frac{\partial}{\partial \beta}\langle E\rangle = \langle E\rangle^2 - \langle E^2\rangle$ (b) $\frac{\partial}{\partial \beta}\langle E\rangle = \langle E^2\rangle - \langle E\rangle^2$

(c) $\frac{\partial}{\partial \beta}\langle E\rangle = \langle E^2\rangle + \langle E\rangle^2$ (d) $\frac{\partial}{\partial \beta}\langle E\rangle = -\left(\langle E^2\rangle + \langle E\rangle^2\right)$

Ans.: (a)

Solution: $\because \langle E\rangle = \frac{\sum_i E_i e^{-\beta E_i}}{\sum_i e^{-\beta E_i}}$

$$\frac{\partial\langle E\rangle}{\partial \beta} = -\frac{\sum_i E_i^2 e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} + \frac{\sum_i E_i^2 e^{-\beta E_i}\cdot e^{-\beta E_i}}{\left(\sum_i e^{-\beta E_i}\right)^2} = -\frac{\sum_i E_i^2 e^{-\beta E_i}}{\sum_i e^{-\beta E_i}} + \frac{\sum_i E_i^2 e^{-2\beta E_i}}{\left(\sum_i e^{-\beta E_i}\right)^2}$$

$$\Rightarrow \frac{\partial\langle E\rangle}{\partial \beta} = \langle E\rangle^2 - \langle E^2\rangle$$

Q20. An ideal gas is compressed adiabatically from an initial volume V to a final volume αV and a work W is done on the system in doing so. The final pressure of the gas will be $\left(\gamma = \frac{C_P}{C_V}\right)$

(a) $\frac{W}{V^\gamma}\frac{1-\gamma}{\alpha-\alpha^\gamma}$ (b) $\frac{W}{V^\gamma}\frac{\gamma-1}{\alpha-\alpha^\gamma}$ (c) $\frac{W}{V}\frac{1-\gamma}{\alpha-\alpha^\gamma}$ (d) $\frac{W}{V}\frac{\gamma-1}{\alpha-\alpha^\gamma}$

Ans.: (c)

Solution: Work done in adiabatic process,

$$W = \frac{P_2V_2 - P_1V_1}{1-\gamma}$$

$$\because \quad P_2V_2^\gamma = P_1V_1^\gamma \Rightarrow P_1 = P_2\left(\frac{V_2}{V_1}\right)^\gamma \Rightarrow P_1 = P_2(\alpha)^\gamma$$

$$W = \frac{P_2\alpha V - P_2\alpha^\gamma V}{(1-\gamma)} \Rightarrow P_2 = \frac{W}{V}\frac{(1-\gamma)}{(\alpha-\alpha^\gamma)}$$

Q21. A particle in thermal equilibrium has only 3 possible states with energies $-\in, 0, \in$. If the system is maintained at a temperature, $T >> \frac{\in}{k_B}$, then the average energy of the particle can be approximated to,

(a) $\frac{2\in^2}{3k_BT}$ (b) $\frac{-2\in^2}{3k_BT}$ (c) $\frac{-\in^2}{k_BT}$ (d) 0

Ans.: (b)

Solution: $\langle E\rangle = \frac{-\in e^{+\frac{\in}{kT}}+0+\in e^{-\frac{\in}{kT}}}{e^{\frac{\in}{kT}}+1+e^{-\frac{\in}{kT}}} = \in\left(\frac{e^{-\frac{\in}{kT}}-e^{\frac{\in}{kT}}}{1+e^{-\frac{\in}{kT}}+e^{\frac{\in}{kT}}}\right)$

$$\Rightarrow \langle E\rangle = \frac{\left[\left(1-\frac{\in}{kT}\right)-\left(1+\frac{\in}{kT}\right)\right]}{1+\left(1-\frac{\in}{kT}\right)+\left(1+\frac{\in}{kT}\right)} = \frac{-2\in^2}{3kT}$$

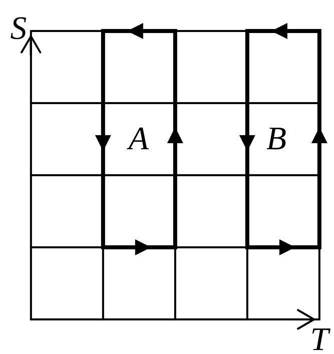
Q22. The blackbody at a temperature of $6000K$ emits a radiation whose intensity spectrum peaks at $600\,nm$. If the temperature is reduced to $300K$, the spectrum will peak at,

(a) $120\,\mu m$ (b) $12\,\mu m$ (c) $12\,mm$ (d) $120\,mm$

Ans.: (b)

Solution: $\lambda_1 T_1 = \lambda_2 T_2 \Rightarrow \lambda_2 = \frac{\lambda_1 T_1}{T_2} = \frac{600\times 6000}{300} = 12000\ nm = 12\ \mu m$

Q23. The entropy-temperature diagram of two Carnot engines, A and B, are shown in the figure 4. The efficiencies of the engines are η_A and η_B respectively. Which one of the following equalities is correct?



(a) $\eta_A = \frac{\eta_B}{2}$

(b) $\eta_A = \eta_B$

(c) $\eta_A = 3\eta_B$

(d) $\eta_A = 2\eta_B$

Ans. : (d)

Solution: $\eta = \frac{\Delta W}{Q_1}$, where ΔW = area under the curve , Q_1 = area under high temperature

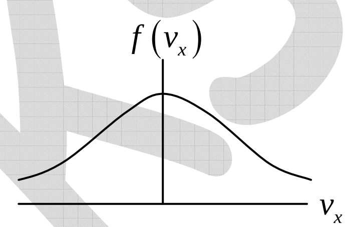
$$\eta_A = \frac{(2T-T)(3S-0)}{2T(3S-0)} = \frac{T}{2T} = \frac{1}{2} \quad \text{and} \quad \eta_B = \frac{(4T-3T)}{4T}\frac{(3S-0)}{(3S-0)} = \frac{T}{4T} = \frac{1}{4}$$

$$\Rightarrow \frac{\eta_A}{\eta_B} = \frac{1/2}{1/4} = 2 \Rightarrow \eta_A = 2\eta_B$$

Q24. Electrons of mass m in a thin, long wire at a temperature T follow a one-dimensional Maxwellian velocity distribution. The most probable speed of these electrons is,

(a) $\sqrt{\left(\frac{kT}{2\pi m}\right)}$ (b) $\sqrt{\left(\frac{2kT}{m}\right)}$ (c) 0 (d) $\sqrt{\left(\frac{8kT}{\pi m}\right)}$.

Ans.: (c)



Solution: $f(v_x) = \left(\frac{m}{2\pi kT}\right)^{1/2} e^{-\frac{mv_x^2}{2kT}} dv_x; \quad -\infty < v_x < \infty$

Most probable speed $v_x = 0$

JEST-2016

Q25. An ideal gas with adiabatic exponent γ undergoes a process in which its pressure P is related to its volume V by the relation $P = P_0 - \alpha V$, where P_0 and α are positive constants. The volume starts from being very close to zero and increases monotonically to $\frac{P_0}{\alpha}$. At what value of the volume during the process does the gas have maximum entropy?

(a) $\frac{P_0}{\alpha(1+\gamma)}$ (b) $\frac{\gamma P_0}{\alpha(1-\gamma)}$ (c) $\frac{\gamma P_0}{\alpha(1+\gamma)}$ (d) $\frac{P_0}{\alpha(1-\gamma)}$

Ans.: (c)

Solution: $TdS = nC_V dT + PdV \Rightarrow TdS = \frac{nRdT}{(\gamma-1)} + PdV$

For maximum entropy, $dS = 0$

For Ideal gas, $PV = nRT \Rightarrow PdV + VdP = nRdT$

$$\Rightarrow TdS = \frac{PdV + VdP}{(\gamma - 1)} + PdV \Rightarrow \frac{PV}{nR} dS = \frac{\gamma}{(\gamma - 1)} PdV + \frac{VdP}{(\gamma - 1)}$$

Since, $P = P_0 - \alpha V \Rightarrow dP = -\alpha dV$

$$\frac{PV}{nR} dS = \frac{\gamma}{(\gamma - 1)} PdV - \frac{\alpha VdV}{(\gamma - 1)} \Rightarrow \frac{dS}{dV} = \frac{\gamma nRP}{(\gamma - 1) PV} - \frac{nR}{(\gamma - 1) PV} \alpha V$$

For maximum entropy, $\frac{dS}{dV} = 0 \Rightarrow \gamma P - \alpha V = 0 \Rightarrow \gamma (P_0 - \alpha V) = \alpha V$

$$\Rightarrow V = \frac{\gamma P_0}{\alpha (1 + \gamma)}$$

Q26. An ideal gas has a specific heat ratio $\frac{C_P}{C_V} = 2$. Starting at a temperature T_1 the gas under goes an isothermal compression to increase its density by a factor of two. After this an adiabatic compression increases its pressure by a factor of two. The temperature of the gas at the end of the second process would be:

(a) $\frac{T_1}{2}$ (b) $\sqrt{2} T_1$ (c) $2T_1$ (d) $\frac{T_1}{\sqrt{2}}$

Ans.: (b)

Solution: During the isothermal process, $T = T_1$ is constant

Let us assume, the adiabatic process started at point $A(P_1, T_1)$ and at point B the coordinate is (P_2, T_2), it is given $P_1^{1-\gamma} T_1^{\gamma} = P_2^{1-\gamma} T_2^{\gamma} \Rightarrow T_2 = \left(\frac{P_1}{P_2}\right)^{\frac{1-\gamma}{\gamma}} T_1 \Rightarrow T_2 = \left(\frac{P_1}{2P_1}\right)^{\frac{1-2}{2}} T_1$

$$\Rightarrow T_2 = \sqrt{2} T_1$$

Q27. A two dimensional box in a uniform magnetic field B contains $\frac{N}{2}$ localised spin-$\frac{1}{2}$ particles with magnetic moment μ, and $\frac{N}{2}$ free spinless particles which do not interact with each other. The average energy of the system at a temperature T is:

(a) $3NkT - \frac{1}{2}N\mu B\sinh\left(\frac{\mu B}{k_B T}\right)$

(b) $NkT - \frac{1}{2}N\mu B\tanh\left(\frac{\mu B}{k_B T}\right)$

(c) $\frac{1}{2}NkT - \frac{1}{2}N\mu B\tanh\left(\frac{\mu B}{k_B T}\right)$

(d) $\frac{3}{2}NkT + \frac{1}{2}N\mu B\cosh\left(\frac{\mu B}{k_B T}\right)$

Ans.: (c)

Solution: For $\frac{N}{2}$ free particles in two dimension, average energy is $\frac{N}{2}kT$ and for $\frac{N}{2}$ localized spin-$\frac{1}{2}$ particle, the average energy is $-\frac{1}{2}N\mu B\tanh\left(\frac{\mu B}{k_B T}\right)$

Then average energy of system at temperature T is

$$\langle E\rangle = \frac{NkT}{2} - \frac{1}{2}N\mu B\tanh\left(\frac{\mu B}{k_B T}\right).$$

Q28. A gas of N molecules of mass m is confined in a cube of volume $V = L^3$ at temperature T. The box is in a uniform gravitational field $-g\hat{z}$. Assume that the potential energy of a molecule is $U = mgz$ where $z \in [0, L]$ is the vertical coordinate inside the box. The pressure $P(z)$ at height z is:

(a) $P(z) = \frac{N}{V}\frac{mgL}{2}\frac{\exp\left(-\frac{mg\left(z-\frac{L}{2}\right)}{k_B T}\right)}{\sinh\left(\frac{mgL}{2k_B T}\right)}$

(b) $P(z) = \frac{N}{V}\frac{mgL}{2}\frac{\exp\left(-\frac{mg\left(z-\frac{L}{2}\right)}{k_B T}\right)}{\cosh\left(\frac{mgL}{2k_B T}\right)}$

(c) $P(z) = \frac{k_B T N}{V}$

(d) $P(z) = \frac{N}{V}mgz$

Ans.: (c)

Solution: The partition function of a system is given by,

$$Z_N = \left(\frac{2\pi mk_BT}{h^2}\right)^{\frac{3N}{2}} \left(\frac{k_BTV}{mgL}\right)^N \left(1-\exp\left(-\frac{mgl}{k_BT}\right)\right)^N$$

Helmohtz free energy is given by, $F = -k_BT\ln Z_N$

Pressure is given by $P = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = \frac{k_BTN}{V}$

Q29. For a quantum mechanical harmonic oscillator with energies, $E_n = \left(n+\frac{1}{2}\right)\hbar\omega$, where $n = 0,1,2...$, the partition function is:

(a) $\dfrac{e^{\frac{\hbar\omega}{k_BT}}}{e^{\frac{\hbar\omega}{2k_BT}}-1}$ (b) $e^{\frac{\hbar\omega}{2k_BT}}-1$ (c) $e^{\frac{\hbar\omega}{2k_BT}}+1$ (d) $\dfrac{e^{\frac{\hbar\omega}{2k_BT}}}{e^{\frac{\hbar\omega}{k_BT}}-1}$

Ans.: (d)

Solution: $z = \exp\left(-\frac{\hbar\omega}{2kT}\right)+\exp\left(-\frac{3\hbar\omega}{2kT}\right)+\exp\left(-\frac{5\hbar\omega}{2kT}\right)+\exp\left(-\frac{7\hbar\omega}{2kT}\right)......$

$$z = \exp\left(-\frac{\hbar\omega}{2kT}\right)\left(1+\exp\left(-\frac{\hbar\omega}{kT}\right)+\exp\left(-\frac{2\hbar\omega}{kT}\right).....\right)$$

$$z = \frac{\exp\left(-\frac{\hbar\omega}{2kT}\right)}{1-\exp\left(\frac{\hbar\omega}{kT}\right)} \Rightarrow \frac{1}{\exp\left(\frac{\hbar\omega}{2kT}\right)-\exp\left(-\frac{\hbar\omega}{2kT}\right)} \Rightarrow \frac{\exp\left(\frac{\hbar\omega}{2kT}\right)}{\exp\left(\frac{\hbar\omega}{kT}\right)-1}$$

JEST 2017

Part-A: 1-Mark Questions

Q30. After the detonation of an atom bomb, the spherical ball of gas was found to be of 15 meter radius at a temperature of $3\times10^5 K$. Given the adiabatic expansion coefficient $\gamma = 5/3$, what will be the radius of the ball when its temperature reduces to $3\times10^3 K$?

(a) $156m$ (b) $50m$ (c) $150m$ (d) $100m$

Ans. : (c)

Solution: $T_1V_1^{\gamma-1} = T_2V_2^{\gamma-1} \Rightarrow V_2 = \left(\frac{T_1}{T_2}\right)^{\frac{1}{(\gamma-1)}} V_1 \Rightarrow V_2 = \left(\frac{T_1}{T_2}\right)^{3/2} V_1$

$$\Rightarrow R = \left(\frac{T_1}{T_2}\right)^{\frac{1}{2}} V_1 \Rightarrow R = \left(\frac{3\times10^5}{3\times10^3}\right)^{\frac{1}{2}} 15 = 150$$

Q31. If the mean square fluctuations in energy of a system in equilibrium at temperature T is proportional to T^{α}, then the energy of the system is proportional to

(a) $T^{\alpha-2}$ (b) $T^{\frac{\alpha}{2}}$ (c) $T^{\alpha-1}$ (d) T^{α}

Ans. : (c)

Solution: $(\Delta E)^2 = kT^2C_V \Rightarrow T^{\alpha-2} \propto C_V \Rightarrow T^{\alpha-2} \propto \left(\frac{\partial U}{\partial T}\right)_V \Rightarrow U \propto T^{\alpha-1}$

Q32. Suppose that the number of microstates available to a system of N particles depends on N and the combined variable UV^2, where U is the internal energy and V is the volume of the system. The system initially has volume $2m^3$ and energy $200\,J$. It undergoes an isentropic expansion to volume $4m^3$. What is the final pressure of the system in SI units?

Ans. : 25

Solution: Here, $\Omega = \left(UV^2\right)N \Rightarrow S = Nk\ln\left(UV^2\right)$

From law of thermodynamics,

$TdS = dU + PdV$

$$\Rightarrow \left.\frac{\partial S}{\partial U}\right|_V = \frac{1}{T} \Rightarrow U = NkT \qquad \text{.......(1)}$$

and $$\left.\frac{\partial S}{\partial V}\right|_U = \frac{P}{T} \Rightarrow PV = 2NkT \qquad \text{.......(2)}$$

From equation (1) and (2),

$$PV = 2U \qquad \text{.......(3)}$$

Now, from equation (3),

$$P_iV_i = 2U_i \Rightarrow P_i = \frac{2U_i}{V_i} = \frac{2\times200}{2} = 200\,atm \qquad \text{.......(4)}$$

As the given process is isoentropic,

$$\therefore\ TdS = 0 = dU + PdV \Rightarrow \left.\frac{\partial U}{\partial V}\right|_S = -P$$

and from equation (3),

$$\left.\frac{\partial U}{\partial V}\right|_S = \frac{P}{2} + \frac{V}{2}\left.\frac{\partial P}{\partial V}\right|_S = -P \Rightarrow \frac{V}{2}\frac{\partial P}{\partial V} = -\frac{3P}{2} \Rightarrow \frac{\partial P}{P} = -3\frac{\partial V}{V}$$

On solving above equation, we have

$$PV^3 = K \quad \text{(constant)}$$

$$\Rightarrow P_f V_f^{\ 3} = P_i V_i^3 \Rightarrow P_f = \frac{P_i V_i^3}{V_f^{\ 3}} = \frac{200\times 2^3}{4^3} = 25\,atm$$

Q33. A cylinder at temperature $T = 0$ is separated into two compartments A and B by a free sliding piston. Compartments A and B are filled by Fermi gases made of spin $1/2$ and $3/2$ particles respectively. If particles in both the compartments have same mass, the ratio of equilibrium density of the gas in compartment A to that of gas in compartment B is

(a) 1 (b) $\frac{1}{3^{2/5}}$ (c) $\frac{1}{2^{2/5}}$ (d) $\frac{1}{2^{2/3}}$

Ans. : (c)

Solution: **Follow Pathria** Page 198 equation 20 for $\in_f$

And equation (38) at pages 200

From equation (38) at $T = 0$

$$p = \frac{2}{5}n\in_f$$

$$= \frac{2}{5}n\left(\frac{6\pi^2 n}{g}\right)^{2/3}\frac{\hbar^2}{2m} \qquad \text{(using equation (24))}$$

for equilibrium $\rho_A = \rho_B$

$$\Rightarrow n_A\left(\frac{n_A}{g_A}\right)^{2/3} = n_B\left(\frac{n_B}{g_B}\right)^{2/3}$$

$$\left(\frac{n_A}{n_B}\right)^{5/3} = \left(\frac{g_A}{g_B}\right)^{2/3}$$

$g_A = 25+1 = 2\times\frac{1}{2}+1 = 2$, $g_B = 25+1 = 2\times\frac{5}{2}+1 = 4$

$$\left(\frac{g_A}{g_B}\right) = \left(\frac{1}{2}\right)^{2/3} \Rightarrow \frac{n_A}{n_B} = \left(\frac{1}{2}\right)^{2/3\times 2/3}$$

$$\frac{n_A}{n_B} = \frac{1}{2^{2/5}}$$

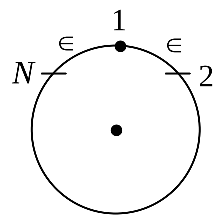
Q34. Two classical particles are distributed among $N(>2)$ sites on a ring. Each site can accommodate only one particle. If two particles occupy two nearest neighbour sites, then the energy of the system is increased by $\in$. The average energy of the system at temperature T is

(a) $\frac{2\in e^{-\beta\in}}{(N-3)+2e^{-\beta\in}}$ (b) $\frac{2N\in e^{-\beta\in}}{(N-3)+2e^{-\beta\in}}$

(c) $\frac{\in}{N}$ (d) $\frac{2\in e^{-\beta\in}}{(N-2)+2e^{-\beta\in}}$

Ans. : (a)

Solution: Since two particle two nearest neighbour sites, which energy of system increased by $\in$, and remaining $(N-3)$ particle has zero energy, then particle function is given



$$z = 2e^{-\beta\in} + (N-3)e^{-\beta.0} = (N-3)+2e^{-\beta\in}$$

then $\langle E\rangle = KT^2\frac{\partial}{\partial T}(\ln z)$

$$= \frac{KT^2}{z}.\left[0+2e^{-\beta\in}.\frac{\partial}{\partial T}\left(-\frac{\in}{KT}\right)\right] = \frac{KT^2}{z}.2e^{-\beta\in}\left(\frac{\in}{KT^2}\right)$$

$$\langle E\rangle = \frac{2\in e^{-\beta\in}}{\left[(N-3)+2e^{-\beta\in}\right]}$$

JEST-2018

Q35. When a collection of two-level systems is in equilibrium at temperature T_0, the ratio of the population in the lower and upper levels is $2:1$. When the temperature is changed to T, the ratio is $8:1$. Then

(a) $T = 2T_0$ (b) $T_0 = 2T$ (c) $T_0 = 3T$ (d) $T_0 = 4T$

Ans. : (c)

Solution: $N = N_0 \exp\left(-\frac{E}{k_B T}\right) \Rightarrow \frac{N_1}{N_2} = \exp\left(\frac{E_2 - E_1}{k_B T_0}\right) \Rightarrow 2 = \exp\left(\frac{E_2 - E_1}{kT_0}\right)$, and

$8 = \exp\left(\frac{E_2 - E_1}{kT}\right) \; \frac{\ln 2}{\ln 8} = \frac{T}{T_0} \Rightarrow T_0 = 3T$

Q36. A collection of N interacting magnetic moments, each of magnitude μ, is subjected to a magnetic field H along the z direction. Each magnetic moment has a doubly degenerate level of energy zero and two non-degenerate levels of energies $-\mu H$ and μH respectively. The collection is in thermal equilibrium at temperature T. The total energy $E(T, H)$ of the collection is

(a) $-\frac{\mu HN \sinh\left(\frac{\mu H}{k_B T}\right)}{1+\cosh\left(\frac{\mu H}{k_b T}\right)}$ (b) $-\frac{\mu HN}{2\left(1+\cosh\left(\frac{\mu H}{k_b T}\right)\right)}$

(c) $-\frac{\mu HN \cosh\left(\frac{\mu H}{k_B T}\right)}{1+\cosh\left(\frac{\mu H}{k_b T}\right)}$ (d) $-\mu HN \frac{\sinh\left(\frac{\mu H}{k_B T}\right)}{\cosh\left(\frac{\mu H}{k_b T}\right)}$

Ans. : (a)

Solution: $Z_1 = \left(2 \times \exp\left(-\frac{0}{k_B T}\right) + \exp\left(-\frac{-\mu H}{k_B T}\right) + \exp\left(-\frac{\mu H}{k_B T}\right)\right) \Rightarrow Z_1 = \left(2 + 2\cosh\frac{\mu H}{k_B T}\right)$

$Z_N = \left(2 + 2\cosh\frac{\mu H}{k_B T}\right)^N \; U = k_B T^2 \left(\frac{\partial \ln Z_N}{\partial T}\right)_{N,V} = -\frac{N\mu H \sinh\left(\frac{\mu H}{k_B T}\right)}{1+\cos\frac{\mu H}{k_B T}}$

Q37. In a thermodynamic process the volume of one mole of an ideal is varied as where $V = aT^{-1}$ a is a constant. The adiabatic exponent of the gas is γ. What is the amount of heat received by the gas if the temperature of the gas increases by ΔT in the process?

(a) $R\Delta T$ (b) $\frac{R\Delta T}{1-\gamma}$ (c) $\frac{R\Delta T}{2-\gamma}$ (d) $R\Delta T\frac{2-\gamma}{\gamma-1}$

Ans. : (d)

Solution: $V = \frac{a}{T} \Rightarrow dU = -\frac{a}{T_2}dT$

$PV = RT$

$W = \int PdV = \int \frac{RT}{V}dV \Rightarrow W = \int \frac{RT^2}{a} \times \left(-\frac{a}{T^2}\right)dT \Rightarrow W = -\int RdT = -R\Delta T$

$\Delta U = C_V \Delta T = \frac{R}{\gamma - 1}\Delta T$

$Q = W + \Delta U = \frac{R\Delta T}{\gamma - 1} - R\Delta T = R\Delta T\left(\frac{1}{\gamma-1} - 1\right) = R\Delta T\left(\frac{2-\gamma}{\gamma-1}\right)$

Q38. For a classical system of non-interacting particles in the presence of a spherically symmetric potential $V(r) = \gamma r^3$, what is the mean energy per particle? γ is a constant.

(a) $\frac{3}{2}k_BT$ (b) $\frac{5}{2}k_BT$ (c) $\frac{3}{2}\gamma k_BT$ (d) $\frac{3}{2}\gamma k_BT$

Ans. : (b)

Solution: $\langle V \rangle = \int e^{-\frac{\gamma r^3}{k_BT}} 4\pi r^2 dr$

$$= \frac{\int_0^\infty v r^3 e^{-\frac{\gamma r^3}{k_BT}} 4\pi r^2 dr}{\int_0^\infty e^{-\frac{\gamma r^3}{k_BT}} 4\pi r^2 dr} = \frac{\gamma \int_0^\infty r^5 e^{-\frac{\gamma r^3}{k_BT}} dr}{\int_0^\infty r^2 e^{-\frac{\gamma r^3}{k_BT}} dr}$$ put $u = r^3$ and solve the integral

$$= \frac{\gamma \frac{1}{3a^2}}{\frac{1}{3a}} = \frac{\gamma}{a} = \frac{\gamma}{\gamma} \cdot k_BT = k_BT$$

Put $u = r^3$ or $r = u^{1/3}$

$\int_0^\infty u^{5/3} e^{-au} \frac{1}{3}u^{-2/3} du dr = \frac{1}{3}x^{\frac{1}{3}-1} \Rightarrow \frac{1}{3}\int_0^\infty u e^{-au} du$

$dr = \frac{1}{3}u^{-2/3}du = \frac{3}{2}k_BT + k_BT = \frac{5}{2}k_BT$

Q39. An ideal fluid is subjected to a thermodynamic process described by $\rho = CV^{-\alpha}$ and $P = n\rho^{\Gamma}$ where ρ is energy density and P is pressure. For what values of n and Γ the process is adiabatic if the volume is changed slowly?

(a) $\Gamma = \alpha - 1, n = 1$ (b) $\Gamma = 1 - \alpha, n = \alpha$

(c) $\Gamma = 1, n = \alpha - 1$ (d) $\Gamma = \alpha, n = 1 - \alpha$

Ans. : (c)

Solution: As $\rho = \dfrac{U}{V} \Rightarrow U = \rho V = CV^{1-\alpha}$

$\rho = n\rho^{\Gamma} \Rightarrow \rho = n\left(CV^{-\alpha}\right)^{\Gamma} = ne^{\Gamma}\left(V^{-\alpha}\right)^{\Gamma}$

$TdS = dU + PdV$

$TdS = 0$, hence $dU + PdV = 0$

$dU = C(1-\alpha)V^{-\alpha}dV$

$PdV = ne^{\Gamma}V^{-\alpha\Gamma}dV \Rightarrow C(1-\alpha)V^{-\alpha}dV + ne^{\Gamma}V^{-\alpha\Gamma}dV = 0$

$\Rightarrow CV^{-\alpha}\left(1-\alpha+V(1-\Gamma)ne^{\Gamma-1}\right)dV = 0$

This is true only if $\Gamma = 1$ and for $\Gamma = 1$, $1-\alpha+n=0$

$\Rightarrow n = \alpha - 1$. Therefore, correct option is (c).

Q40. A frictionless heat conducting piston of negligible mass and heat capacity divides a vertical, insulated cylinder of height $2H$ and cross sectional area A into two halves. Each half contains one mole of an ideal gas at temperature T_0 and pressure P_0 corresponding to STP. The heat capacity ratio $\gamma = C_p / C_v$ is given. A load of weight W is tied to the piston and suddenly released. After the system comes to equilibrium, the piston is at rest and the temperatures of the gases in the two compartments are equal. What is the final displacement y of the piston from its initial position, assuming $yW >> T_0 C_v$?

(a) $\dfrac{2H}{\sqrt{\gamma}}$ (b) $H\gamma$ (c) $\dfrac{H}{\sqrt{\gamma}}$ (d) $\dfrac{2H}{\gamma}$

Ans. : (c)

Solution: $\frac{P_0V_0}{T_0}=\frac{P_2V_2}{T_2}$

$$\frac{P_0A\times H}{T_0}=\frac{P_2\left(A(H-y)\right)}{T_2}$$

$$P_x=\frac{T_2\times P_0H}{T_0\left(H-y\right)} \qquad \ldots.(i)$$

$$\Rightarrow \frac{P_0A\times H}{T_0}=P_1\frac{A(H+y)}{T_2}P_1=\frac{T_2}{T_0}\times\frac{P_0H}{(H+y)} \qquad \ldots.(ii)$$

Total change in internal energy of the system = Net energy input = wy

$$2xC_V\left(T_2-T_0\right)=wy$$

As $wy>>C_VT_0$ and $n=1$ mole

$$T_2=\frac{wy}{2C_V} \qquad (A)$$

$$C_V=\frac{R}{\gamma-1}$$

Also as equilibrium, $P_2-P_1=\frac{w}{A}$

Put the value of T_2 in (i) and (A) and substitute (ii) from (i)

$$\frac{wy}{2C_V}\frac{P_0H}{T_0\left(H-y\right)}-\frac{wy}{2C_V}\frac{P_0H}{T_0\left(H+y\right)}=\frac{w}{A}$$

$$\frac{HP_0y}{2C_VT_0}\left(\frac{1}{H-y}-\frac{1}{H+y}\right)=\frac{1}{A}$$

$$\frac{A\times H\times P_0y}{2\times\frac{R}{\gamma-1}T_0}\left[\frac{Hy-H+y}{H^2-y^2}\right]=1AH=V_0$$

$$\frac{P_0V_0}{T_0}\times\frac{y}{2R}\times\frac{2y\times\gamma^{-1}}{\left[H^2-y^2\right]}=1\Rightarrow\frac{R\times y^2\times\gamma^{-1}}{R\left(H^2-y^2\right)}=1$$

$$y^2\gamma-y^2=H^2-y^2\Rightarrow y=\sqrt{\frac{H}{\gamma}}$$

Q41. A theoretical model for a real (non-ideal) gas gives the following expressions for the internal energy (U) and the pressure (P),

$$U(T,V) = aV^{-2/3} + bV^{2/3}T^2 \text{ and } P(T,V) = \frac{2}{3}aV^{-5/3} + \frac{2}{3}bV^{-1/3}T^2$$

where a and b are constants. Let V_0 and T_0 be the initial volume and initial temperature respectively. If the gas expands adiabatically, the volume of the gas is proportional to

(a) T (b) $T^{3/2}$ (c) $T^{-3/2}$ (d) T^{-2}

Ans. : (c)

Solution: $U(T,V) = aV^{-2/3} + bV^{2/3}T^2$ and $P(T,V) = \frac{2}{3}aV^{-5/3} + \frac{2}{3}bV^{-1/3}T^2$

$$TdS = dU + PdV$$

$$dU = -PdV \qquad (ds = 0)$$

$$dU = \frac{\partial U}{\partial T}dT + \frac{\partial U}{\partial V}dV = -\left[\frac{2}{3}aV^{-5/3} + \frac{2}{3}bV^{-1/3}T^2\right]dV$$

$$= 2bV^{2/3}TdT\,\frac{2}{3}aV^{-\frac{2}{3}-1}dV + \frac{2}{3}bV^{-1/3}T^2dV = -\frac{2}{3}aV^{-5/3}dV - \frac{2}{3}bV^{-1/3}T^2dV$$

$$2bV^{2/3}\frac{T}{T^2}dT + \frac{2}{3}bV^{-1/3}T^2dV = -\frac{2}{3}bV^{-1/3}T^2dV$$

$$\frac{dT}{T} = -\frac{4}{3}\frac{bV^{-1/3}}{2bV^{2/3}}dV$$

$$\frac{dT}{T} = -\frac{2}{3}V^{-1}dV$$

$$\ln T = \ln V^{-2/3}$$

$$T \propto V^{-2/3} \qquad \therefore \quad V \propto T^{-3/2}$$

Q42. In an experiment, certain quantity of an ideal gas at temperature T_0 pressure P_0 and volume V_0 is heated by a current flowing through a Wire for a duration of t seconds. The volume is kept constant and the pressure changes to P_1. If the experiment is performed at constant pressure starting with the same initial conditions, the volume changes from V_0 to V_1. The ratio of the specific heats at constant pressure and constant volume is

(a) $\frac{P_1-P_0}{V_1-V_0}\frac{V_0}{P_0}$ (b) $\frac{P_1-P_0}{V_1-V_0}\frac{V_1}{P_1}$ (c) $\frac{P_1V_1}{P_0V_0}$ (d) $\frac{P_0V_0}{P_1V_1}$

Ans. : (a)

Solution: (I) Constant volume heating

$$\frac{P_0}{T_0}=\frac{P_1}{T_1}\Rightarrow T_1=\frac{P_1}{P_0}T_0$$

$$Q=C_V\left(T_1-T_0\right)=C_V\left(\frac{P_1}{P_0}-1\right)T_0$$

(II) Constant pressure heating

$$\frac{V_0}{T_0}=\frac{V_1}{T_1}\Rightarrow T_1'=\frac{V_1}{V_0}T_0$$

$$Q'=C_P\left(T_1'-T_0\right)=C_P T_0\left(\frac{V_1}{V_0}-1\right)$$

$$PdV+VdP=RdT$$

$$PdV=RdT$$

$$dT_P=\frac{P}{R}dV=\frac{P_0}{R}\times\left(V_1-V_0\right)$$

$$dT_V=\frac{V}{R}dP=\frac{V_0}{R}\left(P_1-P_0\right)$$

$$C_V\times\frac{V_0}{R}\left(P_1-P_0\right)=C_P\times\frac{P_0}{R}\left(V_1-V_0\right)$$

$$\frac{C_P}{C_V}=\frac{V_0\left(P_1-P_0\right)}{P_0\left(V_1-V_0\right)}=\left(\frac{P_1-P_0}{V_1-V_0}\right)\times\frac{V_0}{P_0}$$

JEST-2019

Q43. Consider a system of N distinguishable particles with two energy levels for each particle, a ground state with energy zero and an excited state with energy $\varepsilon > 0$. What is the average energy per particle as the system temperature $T \to \infty$?

(a) 0 (b) $\frac{\varepsilon}{2}$ (c) ε (d) ∞

Ans. : (b)

Solution: $\langle E \rangle = \sum_i P_i E_i \Rightarrow P_i = \frac{e^{\beta E_i}}{z}$

$$\langle E \rangle = 0 \times \frac{01}{1+e^{-\beta\varepsilon}} + \varepsilon \times \frac{1}{1+e^{-\beta\varepsilon}}$$

$$= \frac{\varepsilon}{1+e^{-\varepsilon/k_BT}} = \frac{\varepsilon}{2} \text{ at } T \to \infty$$

Q44. Consider a diatomic molecule with an infinite number of equally spaced non-degenerate energy levels. The spacing between any two adjacent levels is ε and the ground state energy is zero. What is the single particle partition function Z?

(a) $Z = \frac{1}{1 - \frac{\varepsilon}{k_BT}}$ (b) $Z = \frac{1}{1 - e^{\frac{\varepsilon}{k_BT}}}$

(c) $Z = \frac{1}{1 - e^{\frac{2\varepsilon}{k_BT}}}$ (d) $Z = \frac{1 - \frac{\varepsilon}{k_BT}}{1 + \frac{\varepsilon}{k_BT}}$

Ans. : No option is matched

Solution: $Z = \sum_i g_i e^{-\beta\varepsilon_i}$

$$g_i = 1$$

$$Z = 1 + e^{-\beta\varepsilon} + e^{-2\beta\varepsilon} + \ldots\ldots$$

$$Z = \frac{1}{1 - e^{-\beta\varepsilon}}$$

Q45. Consider a grand ensemble of a system of one dimensional non-interacting classical harmonic oscillators (each of frequency ω). Which one of the following equations is correct? Here the angular bracket $\langle \cdot \rangle$ indicate the ensemble average. N, E and T represent the number of particles, energy and temperature, respectively. k_B is the Boltzmann constant.

(a) $\langle E \rangle = N \dfrac{k_B T}{2}$ (b) $\langle E \rangle = \langle N \rangle \dfrac{k_B T}{2}$

(c) $\langle E \rangle = N k_B T$ (d) $\langle E \rangle = \langle N \rangle k_B T$

Ans. : (d)

Solution: $E = K.E. + P.E. \Rightarrow E = \dfrac{P_x^2}{2m} + \dfrac{1}{2}kx^2 \,(1D)$

$E = \dfrac{1}{2}k_B T + \dfrac{1}{2}k_B T = k_B T$ (Equiportion)

$\langle E \rangle = \langle N \rangle k_B T$

Q46. Consider a non-relativistic two-dimensional gas of N electrons with the Fermi energy E_F. What is the average energy per particle at temperature $T = 0$?

(a) $\dfrac{3}{5}E_F$ (b) $\dfrac{2}{5}E_F$ (c) $\dfrac{1}{2}E_F$ (d) E_F

Ans. : (c)

Q47. The energy spectrum of a particle consists of four states with energies $0, \in, 2\in, 3\in$. Let $Z_B(T), Z_F(T)$ and $Z_C(T)$ denote the canonical partition functions for four non-interacting particles at temperature T. The subscripts B, F and C corresponds to bosons, fermions and distinguishable classical particles, respectively. Let $y = \exp\left(-\dfrac{\in}{k_B T}\right)$.

Which one of the following statements is true about $Z_B(T), Z_F(T)$ and $Z_C(T)$?

(a) They are polynomials in y of degree $12, 6$ and 12, respectively.

(b) They are polynomials in y of degree $16, 10$ and 16, respectively

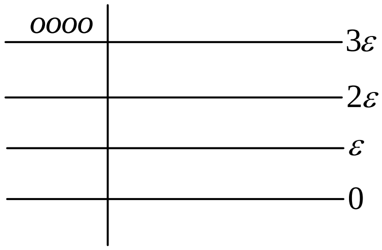
(c) They are polynomials in y of degree $9, 6$ and 12, respectively.

(d) They are polynomials in y of degree $12, 10$ and 16, respectively.

Ans. : (a)

Solution:

Bose



$$y = e^{-\varepsilon/k_BT}$$

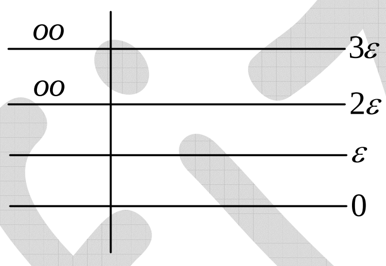
Number of particle $N = 4$ $\qquad \omega = \prod_i \frac{(n_i + g_i)!}{n_i! g_i!}$

Maximum energy $= 12\varepsilon$

$$Z_B = e^{-12\varepsilon/k_BT} + \cdots$$

$$= y^{12} + \cdots \qquad \text{degree } = 12$$

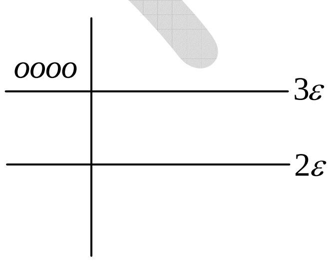
Fermions



Maximum energy $e^{-6\varepsilon/k_BT - 4\varepsilon/k_BT} + e + \cdots\cdots$

$Z_F = y^6 + \cdots$ degree $= 6$

Classical



$Z_C = y^{12} + \ldots..$

Q48. A diatomic ideal gas at room temperature, is expanded at a constant pressure P_0. If the heat absorbed by the gas is $Q = 14$ Joules, what is the maximum work in Joules that can be extracted from the system?

Ans. : 4

Solution: Diatomic gas has $C_v = \frac{5}{2}R,\ C_p = \frac{7}{2}R$

$$Q = C_p \Delta T \Rightarrow 14 = \frac{7}{2}R\Delta T$$

(Constant pressure process)

$$\Rightarrow \Delta T = \frac{14\times 2}{7\times 8.314} = 0.481\,^0c \text{ and } \Delta U = C_v\Delta T = \frac{5}{2}R\times\Delta T$$

$$= \frac{5}{2}\times 8.314\times 0.481 = 9.99\,J \text{ and } W_{\max} = Q - \Delta U$$

$$W_{\max} = 14 - 9.99 = 4\,J$$