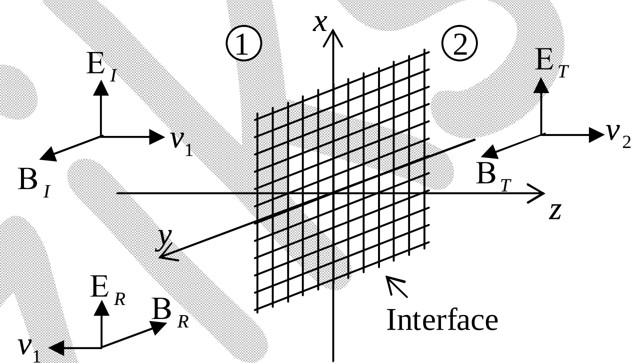
Chapter -7

Applications of Electromagnetic waves

7.1 Reflection and Refraction at Dielectric Interface

7.1.1 Normal Incidence

Suppose xy plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z-direction and polarized in the x direction, approaches the interface from the left then



Incident Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_I(z,t) &= \tilde{E}_{0I}e^{i(k_1z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_I(z,t) &= \frac{\tilde{E}_{0I}}{v_1}e^{i(k_1z-\omega t)}\hat{y}\end{aligned}\right\}$$

Reflected Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_R(z,t) &= \tilde{E}_{0R}e^{i(-k_1z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_R(z,t) &= -\frac{\tilde{E}_{0R}}{v_1}e^{i(-k_1z-wt)}\hat{y}\end{aligned}\right\}$$

Transmitted Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_T(z,t) &= \tilde{E}_{0T}e^{i(k_2z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_T(z,t) &= \frac{\tilde{E}_{0T}}{v_2}e^{i(k_2z-\omega t)}\hat{y}\end{aligned}\right\}$$

At $z=0$, the combined field on the left $\tilde{E}_I+\tilde{E}_R$ *and* $\tilde{B}_I+\tilde{B}_R$, must join the fields on the right $\tilde{E}_T$ & $\tilde{B}_T$, in accordance with the ***boundary conditions***

(i) $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ (ii) $B_1^{\perp} = B_2^{\perp}$ (iii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$ (iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$

In this case there are no electric component perpendicular to the surface, so (i) & (ii) are trivial. However (iii) gives

$$\tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}$$

While (iv) gives, $\dfrac{\tilde{E}_{0I}}{\mu_1 v_1}+\dfrac{\left(-\tilde{E}_{0R}\right)}{\mu_1 v_1}=\dfrac{\tilde{E}_{0T}}{\mu_2 v_2}$ or $\tilde{E}_{0I}-\tilde{E}_{0R}=\beta\tilde{E}_{0T}$

where $\beta=\dfrac{\mu_1 v_1}{\mu_2 v_2}=\dfrac{\mu_1 n_2}{\mu_2 n_1}$.

Solving above two equations we get $\tilde{E}_{0R}=\left(\dfrac{1-\beta}{1+\beta}\right)\tilde{E}_{0I}$, $\tilde{E}_{0T}=\left(\dfrac{2}{1+\beta}\right)\tilde{E}_{0I}$.

If $\mu_1=\mu_2=\mu_0 \Rightarrow \quad \beta=\dfrac{v_1}{v_2}=\dfrac{n_2}{n_1}$ (For non-magnetic medium)

$$\Rightarrow \tilde{E}_{0R}=\left(\frac{v_2-v_1}{v_2+v_1}\right)\tilde{E}_{0I} \quad , \quad \tilde{E}_{0T}=\left(\frac{2v_2}{v_1+v_2}\right)\tilde{E}_{0I}$$

Note: Reflected wave is *in phase* if $v_2>v_1$ *or* $n_2<n_1$ and *out of phase* if $v_2<v_1$ *or* $n_2>n_1$.

In terms of indices of refraction the real amplitudes are

$$E_{0R}=\left|\frac{n_1-n_2}{n_1+n_2}\right|E_{0I} \quad , \quad E_{0T}=\left|\frac{2n_1}{n_1+n_2}\right|E_{0I}.$$

Since Intensity $I=\frac{1}{2}\varepsilon v E_0^2$, then the ratio of the reflected intensity to the incident intensity is the *Reflection coefficient* $R=\dfrac{I_R}{I_I}=\left(\dfrac{E_{0R}}{E_{0I}}\right)^2=\left(\dfrac{n_1-n_2}{n_1+n_2}\right)^2$.

The ratio of the transmitted intensity to the incident intensity is the *Transmission coefficient*

$$T=\frac{I_T}{I_I}=\frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2=\frac{4n_1n_2}{\left(n_1+n_2\right)^2} \qquad \Rightarrow R+T=1$$

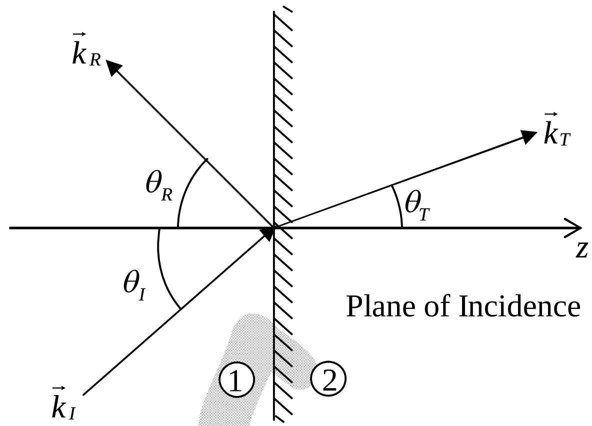
Example: Calculate the reflection coefficient for light at an air-to-dielectric interface $\left(\mu_1=\mu_2=\mu_0,\ n_1=1,\ n_2=1.5\right)$ at optical frequency $\omega=4\times10^{15}s^{-1}$.

Solution: Reflection coefficient $R=\left(\dfrac{n_1-n_2}{n_1+n_2}\right)^2=\left(\dfrac{1-1.5}{1+1.5}\right)^2=0.04$ *or* 4%

Thus only 4% of light is reflected and 96% is transmitted.

7.1.2 Oblique Incidence

In oblique incidence an incoming wave meets the boundary at an arbitrary angle θ_I. Of course, normal incidence is really just a special case of oblique incidence with $\theta_I = 0$. Suppose that a monochromatic plane wave of frequency ω, approaches the interface from the left then



Incident Wave

$$\vec{\tilde{E}}_I(\vec{r},t) = \vec{\tilde{E}}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}, \quad \vec{\tilde{B}}_I(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_I \times \vec{\tilde{E}}_I\right)$$

Reflected Wave

$$\vec{\tilde{E}}_R(\vec{r},t) = \vec{\tilde{E}}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)}, \quad \vec{\tilde{B}}_R(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_R \times \vec{\tilde{E}}_R\right)$$

Transmitted Wave

$$\vec{\tilde{E}}_T(\vec{r},t) = \vec{\tilde{E}}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}, \quad \vec{\tilde{B}}_T(\vec{r},t) = \frac{1}{v_2}\left(\hat{k}_T \times \vec{\tilde{E}}_T\right)$$

All three waves have the same frequency ω. The three wave numbers are related by ($\omega = kv$) as

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \text{ or } k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

The combined field in medium (1), $\vec{\tilde{E}}_I + \vec{\tilde{E}}_R$ *and* $\vec{\tilde{B}}_I + \vec{\tilde{B}}_R$, must join the fields $\vec{\tilde{E}}_T$ & $\vec{\tilde{B}}_T$ in medium (2), using the ***boundary conditions***

(i) $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ (ii) $B_1^{\perp} = B_2^{\perp}$ (iii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$ (iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$

First Law (Plane of Incidence)

The incident, reflected and transmitted wave vectors form a plane (called the plane of incidence), which also includes normal to the surface.

Second law (Law of Reflection)

The angle of incidence is equal to the angle of reflection i.e.

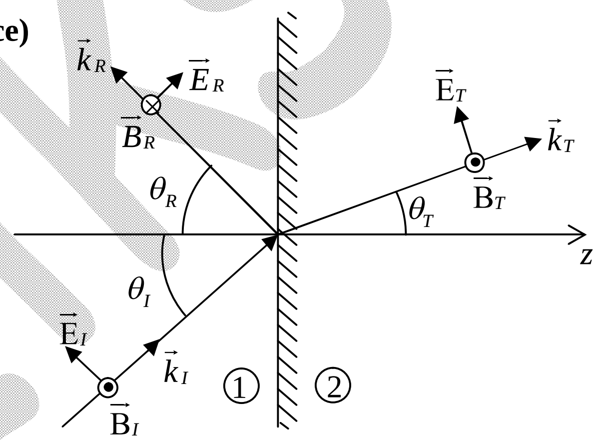
$$\theta_I = \theta_R$$

Third Law: (Law of Refraction, or Snell's law)

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

7.1.3 Fresnel's Relation (Parallel and Perpendicular Polarization)

Case-I: (Polarization in the Plane of Incidence)



Applying Boundary conditions, we get

Reflected and transmitted amplitudes

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \quad \text{and} \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I}$$

where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ *and* $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

These are known as **Fresnel's equations**.

Notice that transmitted wave is always in phase with the incident one; the reflected wave is either in phase , if $\alpha > \beta$, or 180^0 out phase if $\alpha < \beta$.

The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because α is a function of θ_I:

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(n_1/n_2\right)^2\sin^2\theta_I}}{\cos\theta_I}$$

Brewster's Angle

At *Brewster's angle* (θ_B) reflected light is completely extinguished when $\alpha = \beta$, or

$$\sin^2\theta_B = \frac{1-\beta^2}{\left(n_1/n_2\right)^2 - \beta^2}$$

For non-magnetic medium ($\mu_1 \cong \mu_2$), so $\beta \cong \frac{n_2}{n_1}$, $\sin^2\theta_B \cong \frac{\beta^2}{1+\beta^2}$, and hence

$$\tan\theta_B \approx \frac{n_2}{n_1} \quad \textit{and} \quad \theta_T + \theta_B = 90^o$$

Thus at Brewster angle $(\theta_I = \theta_B)$ reflected and transmitted rays are perpendicular to each other.

Critical Angle

When light enters from denser to rarer medium ($n_1 > n_2$) then after a ***critical angle*** (θ_C) there is total internal reflection.

$$\frac{\sin 90^0}{\sin\theta_c} = \frac{n_1}{n_2} \quad \Rightarrow \quad \sin\theta_c = \frac{n_2}{n_1} \quad at \ \theta_C, \ \theta_T = 90^o$$

Reflection and Transmission Coefficient

The power per unit are striking the interface is $\vec{S}\,\hat{z}$. Thus the incident intensity is

$$I_I = \frac{1}{2}\varepsilon_1 v_1 E_{0I}^2 \cos\theta_I,$$

while reflected and transmitted intensities are

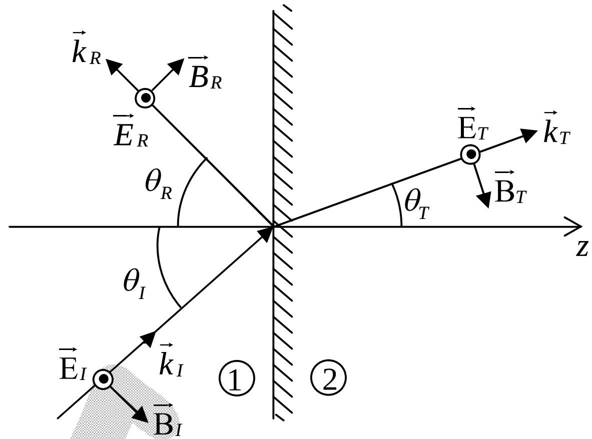
$$I_R = \frac{1}{2}\varepsilon_1 v_1 E_{0R}^2 \cos\theta_R \ \text{ and } \ I_T = \frac{1}{2}\varepsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

Reflection coefficient $\quad R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$

Transmission coefficient $\quad T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$

$$\Rightarrow R + T = 1$$

Case-II: (Polarization Perpendicular to plane of Incidence



Applying Boundary conditions, we get

Reflected and transmitted amplitudes

$$\hat{E}_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\hat{E}_{0I} \quad and \quad \hat{E}_{0T} = \left(\frac{2}{1+\alpha\beta}\right)\hat{E}_{0I}$$

where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ *and* $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$

In this case *Brewster's angle* (θ_B) is not possible i.e reflected light is never completely extinguished (since $\alpha\beta = 1$ is not possible).

Reflection and Transmission coefficient

The power per unit are striking the interface is $\vec{S}\cdot\hat{z}$. Thus the incident intensity is

$$I_I = \frac{1}{2}\varepsilon_1 v_1 E_{0I}^2 \cos\theta_I,$$

while reflected and transmitted intensities are

$$I_R = \frac{1}{2}\varepsilon_1 v_1 E_{0R}^2 \cos\theta_R \text{ and } I_T = \frac{1}{2}\varepsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

Reflection coefficient $R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2$

Transmission coefficient $T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$

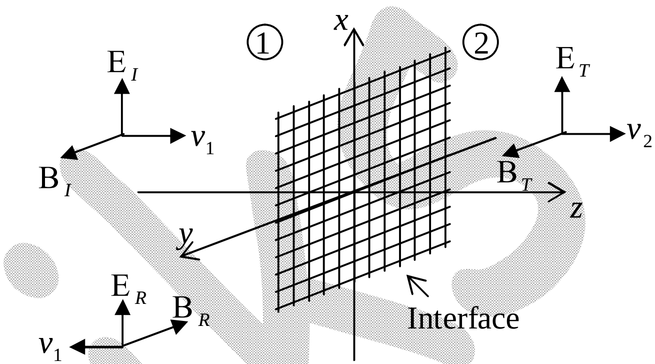
$$\Rightarrow R + T = 1$$

7.2 Reflection at Conducting Surface (Normal Incidence)

Suppose xy plane forms the boundary between a non-conducting linear medium (1) and a conductor (2). A plane wave of frequency ω, traveling in the z-direction and polarized in the x direction, approaches the interface from the left then

Incident Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_I(z,t) &= \tilde{E}_{0I}e^{i(k_1z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_I(z,t) &= \frac{\tilde{E}_{0I}}{v_1}e^{i(k_1z-\omega t)}\hat{y}\end{aligned}\right\}$$



Reflected Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_R(z,t) &= \tilde{E}_{0R}e^{i(-k_1z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_R(z,t) &= -\frac{\tilde{E}_{0R}}{v_1}e^{i(-k_1z-wt)}\hat{y}\end{aligned}\right\}$$

Transmitted Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_T(z,t) &= \tilde{E}_{0T}e^{i(\tilde{k}_2z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_T(z,t) &= \frac{\tilde{k}_2}{\omega}\tilde{E}_{0T}e^{i(\tilde{k}_2z-\omega t)}\hat{y}\end{aligned}\right\}$$

where $\tilde{k}_2 = k_2 + i\kappa_2$ where k_2 and κ_2 are real and imaginary part of $\tilde{k}_2$.

$$k_2 = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \quad and \quad \kappa_2 = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

At $z=0$, the combined field on the left $\tilde{E}_I + \tilde{E}_R$ *and* $\tilde{B}_I + \tilde{B}_R$, must join the fields on the right $\tilde{E}_T$ & $\tilde{B}_T$, in accordance with the ***boundary conditions***

(i) $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ (ii) $B_1^{\perp} = B_2^{\perp}$ (iii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$ (iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$

In this case there are no electric component perpendicular to the surface, so (i) & (ii) are trivial.

However (iii) gives $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

While (iv) gives, $\dfrac{\tilde{E}_{0I}}{\mu_1 v_1}+\dfrac{\left(-\tilde{E}_{0R}\right)}{\mu_1 v_1}=\dfrac{\tilde{k}_2}{\mu_2\omega}\tilde{E}_{0T}$ or $\tilde{E}_{0I}-\tilde{E}_{0R}=\tilde{\beta}\tilde{E}_{0T}$

where $\tilde{\beta}=\dfrac{\mu_1 v_1}{\mu_2\omega}\tilde{k}_2$.

Solving above two equations we get $\tilde{E}_{0R}=\left(\dfrac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$, $\tilde{E}_{0T}=\left(\dfrac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$.

Note: (i) For a perfect conductor $(\sigma=\infty)$, $k_2=\infty \Rightarrow \tilde{\beta}=\infty$. Thus

$$\tilde{E}_{0R}=-\tilde{E}_{0I},\ \tilde{E}_{0T}=0.$$

In this case wave is totally reflected, with a 180^0 phase shift.

(ii) For good conductor $(\sigma \gg \omega\varepsilon)$, $k_2 \cong \kappa_2=\sqrt{\dfrac{\sigma\omega\mu_2}{2}}$.

$$\tilde{\beta}=\frac{\mu_1 v_1}{\mu_2\omega}\sqrt{\frac{\sigma\omega\mu_2}{2}}(1+i)=\mu_1 v_1\sqrt{\frac{\sigma}{2\omega\mu_2}}(1+i)\Rightarrow \tilde{\beta}=\gamma(1+i) \quad \text{where } \gamma=\mu_1 v_1\sqrt{\frac{\sigma}{2\omega\mu_2}}.$$

Reflection Coefficient

$$R=\frac{I_R}{I_I}=\left|\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right|^2=\left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2=\left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right)=\frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}$$

Example: Calculate the reflection coefficient for light at an air-to-silver interface $\left(\mu_1=\mu_2=\mu_0, \varepsilon_1=\varepsilon_0, \sigma=6\times10^7\,\Omega^{-1}m^{-1}\right)$ at optical frequency $\omega=4\times10^{15}\,s^{-1}$.

Solution: $\gamma=\mu_0 c\sqrt{\dfrac{\sigma}{2\omega\mu_0}}=c\sqrt{\dfrac{\sigma\mu_0}{2\omega}}=\left(3\times10^8\right)\sqrt{\dfrac{\left(6\times10^7\right)\left(4\pi\times10^{-7}\right)}{2\left(4\times10^{15}\right)}}=29$

Reflection coefficient

$$R=\frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}=\frac{(28)^2+29^2}{(30)^2+29^2}=0.93 \quad or \quad 93\%.$$

Thus 93% of light is reflected and only 7% is transmitted.