2. Nuclear Models

2.1 Liquid Drop Model of Nucleus (Semi-Empirical mass formula or Weizsacker formula)

(*A simple explanation for the binding-energy curve*)

Using semi-empirical approach (based on experimental results) Weizsacker showed that it is possible to achieve a quantitative and more basic understanding of binding energies of nuclei.

Assumptions

(i) Nucleus is modeled on a drop of liquid.

(ii) The nuclear interaction between protons and neutrons, between protons and protons, and between neutron and neutrons are identical

(iii) $N = Z = A/2$.

(iv) Nuclear forces are saturated.

Deduction

(i) Volume Energy term (B_v)

In a liquid drop, in which each molecule interacts only with its neighbors and number of neighboring molecules is independent of overall size of the liquid drop, the binding energy of liquid drop is $B = LM_m A$ where L=latent heat of liquid, M_m =mass of each molecule, A=number of molecules.

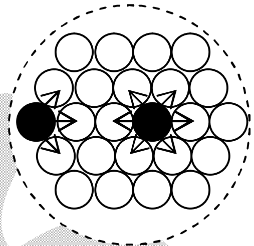
In analogy to the liquid drop, for nuclei we expect a volume term in the expression for binding energy.

Volume energy term, $B_v = a_v A$

where a_v : Volume coefficient (=14.1MeV) and A: Mass number

(ii) Surface Energy Term (B_s)

At the surface of the nucleus, there are nucleons which are not surrounded from all sides; consequently these surface nucleons are not bound as tightly as the nucleons in the interior and hence its binding energy is less. The larger the nucleus, the smaller the proportion of nucleons at the surface



Surface area of the nucleus= $4\pi R^2 = 4\pi R_o^2 A^{2/3}$.

Hence number of nucleons with fewer than maximum number of neighbor is proportional to $A^{2/3}$. So reducing the binding energy by introducing the term

$$B_s = -a_s A^{2/3}$$

where a_s : Surface energy coefficient (=13.0MeV).

It is most significant for lighter nuclei since a greater fraction of their nucleons are on the surface. Because natural systems always tend to evolve toward configurations of minimum potential energy, nuclei tend toward configurations of maximum binding energy. Hence a nucleus should exhibit the same surface-tension effects as a liquid drop, and in the absence of other effects it should be spherical, since a sphere has the least surface area for a given volume.

(iii) Coulomb Energy term (B_c)

The electric repulsion between each pair of proton in a nucleus also contributes towards decreasing its binding energy. The potential energy of protons $'r'$ apart is equal to

$$V = \frac{e^2}{4\pi\varepsilon_o r}.$$

Since there are $\frac{Z(Z-1)}{2}$ pair of protons, the coulomb energy $B_c = \frac{Z(Z-1)}{2}V$,

$$B_c = \frac{e^2}{4\pi\varepsilon_o}\frac{Z(Z-1)}{2}\left(\frac{1}{r}\right)_{av.}$$

Now, $\left(\frac{1}{r}\right)_{av.}$ is average value of $\left(\frac{1}{r}\right)$, averaged over all proton pairs. If the protons are uniformly distributed $\left(\frac{1}{r}\right)_{av.} \propto \frac{1}{R} \propto \frac{1}{A^{2/3}}$, thus $B_c = -a_c\frac{Z(Z-1)}{A^{1/3}}$

where a_c : Coulomb energy coefficient(=.595 MeV)

The coulomb energy is negative because it arises from an effect that opposes nuclear stability. So, the total Binding energy is $B = B_v + B_s + B_c = a_vA - a_sA^{2/3} - a_c\frac{Z(Z-1)}{A^{1/3}}$

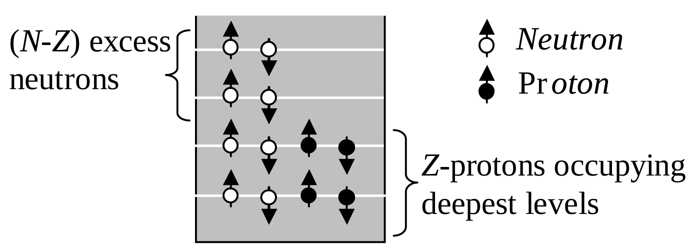
(iv) Corrections to the Formula

The above binding energy formula can be improved by taking into account two effects that do not fit into the simple liquid drop model but which make sense in terms of a model that provides for nuclear energy levels. The above result was improved by including two effects

(a) Asymmetry Effect (b) Pairing Effect

(a)Asymmetry Effect (B_a)

Asymmetry Energy Term, B_a depends on the neutron excess $(N-Z)$ and decreases the nuclear binding energy. So far, we have neglected the quantization of energy states of individual nucleons in the nucleus and the application of the Pauli Exclusion Principle.



If we put Z protons and N neutrons into the nuclear energy shells, the lowest Z energy levels are filled first. By Pauli Exclusion Principle, the excess $(N-Z)$ neutrons must go into previously unoccupied quantum states since the first Z quantum states are already filled up with protons and neutrons.

These $(N-Z)$ excess neutrons are occupying higher energy quantum states and are consequently less tightly bound than the first $2Z$ nucleons which occupy the deepest lying energy levels. Thus neutron asymmetry gives rise to a disruptive term in nuclear binding energy. Excess energy per nucleon $\alpha \frac{N-Z}{A}$

Since the total number of excess neutrons is $(N-Z)$, the total deficit in nuclear binding energy is proportional to product of these

$$\Rightarrow B_a = -a_a \frac{(N-Z)^2}{A} = -a_a \frac{(A-2Z)^2}{A},$$

where a_a : asymmetric energy coefficient(19.0 MeV).

(b) Pairing Effect

Since all the previous terms have involved a smooth variation of B whenever Z or N changes and does not account for the kinks which show an evidence for favored pairing.

In liquid drop model we have omitted the intrinsic spin of the nucleons and shell effects. This is corrected by adding a pairing energy term B_p to the nuclear binding energy.

$$B_p = (\pm, 0)\frac{a_p}{A^{+3/4}}, \quad a_p = \begin{cases} 0 \text{ for odd-even or even-odd} \\ \text{-ve for odd-odd} \\ \text{+ve for even-even} \end{cases} \quad \text{and } a_p = 33.5 \text{ MeV}$$

The final expression for binding energy is

$$B = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} (\pm, 0) \frac{a_p}{A^{3/4}}$$

Now, nuclear mass can be written as

$$M(Z, A) = AM_N - Z(M_N - M_P) - B/c^2 \qquad \text{(M \& B in mass units)}$$

$$M(Z, A) = AM_N - Z(M_N - M_P) + \left\{ -a_v A + a_s A^{2/3} + a_c \frac{Z(Z-1)}{A^{1/3}} + a_a \frac{\left((A-2Z)^2\right)}{A} (\mp, 0) a_p A^{-3/4} \right\} \frac{1}{c^2}$$

2.1.1 Most stable nuclei among members of Isobaric family

For a given *A*, we have to find the value of Z for which the binding energy *B* is a maximum, which corresponds to maximum stability, we must show $\left(\frac{dB}{dZ}\right)_{Z=Z_0} = 0$

Since $B = a_v A - a_s A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} (\pm, 0) \frac{a_p}{A^{3/4}}$

$$\Rightarrow \left(\frac{dB}{dZ}\right)_{Z=Z_0} = -\frac{a_c}{A^{1/3}}(2Z_0 - 1) - \frac{a_a}{A} 2(A - 2Z_0)(-2) = 0$$

$$\Rightarrow -\frac{a_c}{A^{1/3}}(2Z_0 - 1) + \frac{4a_a}{A}(A - 2Z_0) = 0 \qquad \Rightarrow -2Z_0\left(\frac{a_c}{A^{1/3}} + \frac{4a_a}{A}\right) = -\frac{a_c}{A^{1/3}} - 4a_a$$

$$\Rightarrow Z_0 = \frac{\left(4a_a + \frac{a_c}{A^{1/3}}\right)}{2\left(\frac{a_c}{A^{1/3}} + \frac{4a_a}{A}\right)} \Rightarrow Z_0 = \frac{4a_a + a_c A^{-1/3}}{2a_c A^{-1/3} + 8a_a A^{-1}}$$

Example: For *A* = 25 we get $Z_0 = \frac{4 \times 19 + 4 \times 0.595(25)^{-1/3}}{2 \times (0.595) \times (25)^{-1/3} + 8 \times 19 \times 25^{-1}} = \frac{76.81}{6.48} \approx 12$

should be the atomic number of the most stable isobar of *A* = 25. This nuclide is ${}^{25}_{12}\mathrm{Mg}$, which is in fact the only stable *A* = 25 isobar. The other isobars ${}^{25}_{11}\mathrm{Na}$ and ${}^{25}_{13}\mathrm{Al}$, are both radioactive.

Example: The atomic mass of the zinc isotope ${}^{64}_{30}\mathrm{Zn}$ is 63.9294. Compare its binding energy with the prediction of the liquid drop model.

Solution: B.E. = [30 × 1.007825 + 34 × 1.008665 – 63.929] × 931.49 = 559.1 MeV

From semi-empirical B.E. formula $\left(Z = 30, N = 34, A = 64\right)$

$$B = a_v A - a_c A^{2/3} - a_c \frac{Z(Z-1)}{A^{1/3}} - a_a \frac{(A-2Z)^2}{A} + \frac{a_p}{A^{3/4}} = 561.7 \text{ MeV}$$

Thus percentage difference = 0.5%.

2.1.2 Mass Parabola's

From the semi-empirical mass equation we have

$$M\left(Z,A\right) = AM_N - Z\left(M_N - M_P\right) - a_v A + a_s A^{2/3} + a_c \frac{Z(Z-1)}{A^{1/3}} + a_a \frac{(A-2Z)^2}{A}\left(\mp, 0\right) a_p A^{-3/4}$$

$$M\left(Z,A\right) = A\left[M_N - \left(a_v - a_a - \frac{a_s}{A^{1/3}}\right)\right] + Z\left[\left(M_P - M_N\right) - \frac{a_c}{A^{1/3}} - 4a_a\right] + Z^2\left[\frac{a_c}{A^{1/3}} + \frac{4a_a}{A}\right] \pm E_p$$

or

$$M\left(Z,A\right) = \alpha A + \beta Z + \gamma Z^2 \pm \delta$$

where $\alpha = M_N - \left(a_v - a_a - \frac{a_s}{A^{1/3}}\right), \beta = -4a_a - \left(M_n - M_p\right) - \frac{a_c}{A^{1/3}}$ and $\gamma = \left(\frac{4a_a}{A} + \frac{a_c}{A^{1/3}}\right)$.

δ is pairing energy (E_p) = + δ for even Z even N

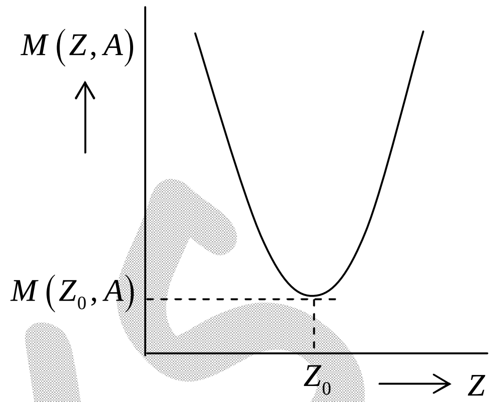
= 0 for odd Z even N or even N and odd Z

= - δ for odd Z odd N

When *A* is constant, the equation $M\left(Z,A\right) = \alpha A + \beta Z + \gamma Z^2 \pm \delta$ represents a parabola. Thus the plot of *M* and *Z* is parabolic with the "minimum" corresponding to that value of Z which gives the (hypothetical) "most stable" isobar in the isobaric family.

For Odd *A* ($\delta = 0$)

As either one of *N* or *Z* is even and the other one is odd (since odd + even = odd), so only one parabola implying that there is only one stable nucleus.



Consider the isobaric family for *A*,

$$\left(\frac{\delta M}{\delta Z}\right)_A = \beta + 2\gamma Z_0 = 0,$$

{Z_0 = Nuclear charge of "most stable nuclei},

$$\therefore Z_0 = \frac{-\beta}{2\gamma} \Rightarrow (\beta = -2\gamma Z_0).$$

So mass of the "most stable" isobar is

$$M(Z_0, A) = \alpha A - 2\gamma Z_0 Z_0 + \gamma Z_0^{\ 2} \quad (\because \beta = -2\gamma Z_0)$$

$$\therefore M(Z_0, A) = \alpha A - \gamma Z_0^{\ 2}$$

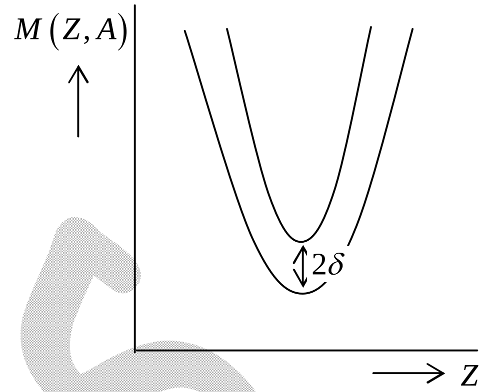
Also, $$M(Z, A) = \alpha A - 2\gamma Z_0 . Z + \gamma Z^2$$

The difference in masses for odd *A* is:

$$M(Z, A) - M(Z_0, A) = -2\gamma Z_0 Z + \gamma Z^2 + \gamma Z_0^{\ 2} = \gamma (Z - Z_0)^2 = \gamma (Z - Z_0)^2$$

Even A isobars ($\delta \neq 0$)

Here pairing term $\delta \neq 0$ since both odd-odd and even-even nuclei are included. So two parabolas,



$$\left.\begin{array}{ll} \text{For odd-odd:} & M(Z_0,A) = \alpha A - \gamma Z_0^2 - \delta \\ \text{For even-even:} & M(Z_0,A) = \alpha A - \gamma Z_0^2 + \delta \end{array}\right\}$$

where $Z_0 = -\dfrac{\beta}{2\gamma}$

The vertical separation between two parabolas is 2δ

$$M(Z,A) = \alpha A - 2\gamma Z_0 Z + \gamma Z^2 \pm \delta$$

2.1.3 β-decay stability

Prediction of stability against β-decay for members of an isobaric family (For Odd A and Even *A* isobars)

The β-decay process furnishes an isobaric pair which can be easily studied with the help of semi-empirical mass formula. There are two types of β-decay viz. β^+ and β^-. In the β^--decay, Z increases by 1-unit and in β^+ -decay Z decreases by 1-unit, while *A* remains constant.

Energy Released in β^- -decay $Q_{\beta^-} = M(Z,A) - M(Z+1,A)$; $\quad (Z \to Z+1)$

Energy released in β^+ -decay $Q_{\beta^+} = M(Z,A) - M(Z-1,A)$; $\quad (Z \to Z-1)$

(a)Odd *A* nuclei decay

Since only one parabola, there is only one minimum value Z_0. Therefore we expect that for odd-A nuclei there is only one β-stable nucleus.

Only β^- -decay along the left arm and only β^+-decay for the right arm of the parabola because nuclei are driven towards achieving more stable states.

Energy released in β-decay varies with Z. Hence different transitions in the same parabola may release different amount of energy.

Now, energy released in decay is given by β^- -decay,

$$Q_{\beta^-} = M(Z,A) - M(Z+1,A) = \left[M(Z,A) - M(Z_0,A)\right] - \left[M(Z+1,A) - M(Z_0,A)\right]$$

$$Q_{\beta^-} = \gamma(Z-Z_0)^2 - \gamma(Z+1-Z_0)^2 = \gamma\left[-2(Z-Z_0)-1\right] = 2\gamma\left(Z_0 - Z - \frac{1}{2}\right)$$

Thus $Q_{\beta^-} = 2\gamma\left(Z_0 - Z - \frac{1}{2}\right)$ and similarly $Q_{\beta^+} = 2\gamma\left(Z - Z_0 - \frac{1}{2}\right)$

(b)Even *A* nuclei decay

Here the pairing term $\delta \neq 0$ and since both odd-odd and even-even nuclei are included, we have two parabola, displaced in binding energy by 2δ or corresponding mass value.

The decay always terminates on the lower parabola because it represents greater stability. (An even-even nucleus makes the lower parabola). In each β- transformation an even-even nuclei changes to odd-odd nuclei and odd-odd nuclei changes to even-even. Hence in each β-transformation there will be jump from one parabola to the other parabola.

Example: For the family of Isobars with A = 91, estimate (i) nuclear charge of the most stable isobar, (ii) the energy released Q_{β^-} and Q_{β^+} for transitions leading to Z_0.

Solution: (i) The atomic number of most stable nucleus is given by

$$Z_0 = \frac{-\beta}{2\gamma}$$

where $\beta = -4a_a - (M_N - M_P) - \frac{a_c}{A^{1/3}} = -4\times 19 - 0.8 - \frac{0.595}{(91)^{1/3}}$ MeV=-77 MeV

$$\gamma = \left(\frac{4a_a}{A} + \frac{a_c}{A^{1/3}}\right) = \frac{4\times 19}{91} + \frac{0.595}{(91)^{1/3}} = 0.96 \text{ MeV} \Rightarrow Z_0 = \frac{-\beta}{2\gamma} = \frac{77}{2\times 0.96} = 40.104$$

(ii) $Q_{\beta^-} = 2\gamma\left(Z_0 - Z - \frac{1}{2}\right)$, $(Z{:}39 \to 40)$; $Z = 39$ and $Z_0 = 40$.

$$\therefore Q_{\beta^-} = 2\times 0.96\left(40 - 39 - \frac{1}{2}\right) = 0.96 \text{ MeV.}$$

And $Q_{\beta^+} = 2\gamma\left(Z - Z_0 - \frac{1}{2}\right)$, $(Z{:}\ 41 \to 40)$; $Z = 41$ and $Z_0 = 40$.

$$Q_{\beta^+} = 2\times 0.96\left(40 - 40 - \frac{1}{2}\right) = 0.96 \text{ MeV.}$$

Example: (i) For "mirror" nuclei which have N and Z differing by one unit, determine the mass difference (Consider A to be odd).

(ii) The masses of $^{15}_{7}\mathrm{N}$ and $^{15}_{8}\mathrm{O}$ are 15.000108 u and 15.003070 u respectively. Using this data, determine the coulomb coefficient a_c in the semi-empirical mass formula.

Solution: Mirror nuclei to be considered have the same odd value of A but the values of N *and* Z are interchanged such that they differ by one unit $\therefore N - Z = \pm 1$.

Now, from semi empirical mass formula we know

$$M(Z,A) = M_N A - (M_N - M_P)Z - a_v A + a_s A^{2/3} + a_c \frac{Z(Z-1)}{A^{1/3}} + a_a \frac{(A-2Z)^2}{A}(\pm, 0)a_p . \frac{1}{A^{3/4}}$$

Now to find mass difference between pair Mirror Nuclei are

$$M_{Z+1} - M_Z = M(Z+1, A) - M(Z,\ A)$$

But $A - 2Z = N + Z - 2Z = N - Z \ = +1 \ (Let\, N > Z)$

$$\Rightarrow M_{Z+1} - M_Z = -(M_N - M_P)[(Z+1) - Z] + \frac{a_c}{A^{1/3}}[Z(Z+1) - Z(Z-1)]$$

$$\Rightarrow M_{Z+1} - M_Z = (M_P - M_N) + \frac{a_c}{A^{1/3}}[A-1]$$

(ii) For the given nuclei,

$$\left(2.962\times 10^{-3}\right)4 = (-0.000844) + a_c \frac{14}{15^{1/3}}$$

$$\therefore a_c = \frac{3.542}{6.08} = 0.58\ \text{MeV} \qquad (\because \text{u=931.5MeV}).$$