2. Damped and Forced Oscillators

2.1 Damped Harmonic Oscillation

The majority of the oscillatory systems in everyday life suffer some sort of irreversible energy loss due to frictional or viscous heat generation while they are oscillating. Their amplitude of oscillation dies away with time. Such oscillation is called damped harmonic oscillation.

Consider the mass-spring system, body of mass m attached to spring with spring constant k is released from position x_0 (measured from equilibrium position) with velocity v; the mass is subject to a *frictional damping force* which opposes its motion, and is directly proportional to its instantaneous velocity $F_{res} = -bv$

The quantity v is a positive constant, whose value depends on the properties of the material providing the resistance. The minus sign indicates that the force resists the motion, so it is directed *opposite* to the velocity.

The total force on the body is the sum of the restoring force $F = -kx$, and the resistive force F_{res}: $F_{net} = F + F_{res} = -kx - bv$

Using Newton second law $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \Rightarrow \frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$

Here, we define the constant γ such that, $2\gamma = \frac{b}{m}$ called damping constant and

$\omega_0 = \sqrt{k/m}$ is the natural frequency of undamped oscillator.

Assuming a solution of the form $x(t) = Ce^{\alpha t}$

On differentiation, we get $\left\{(\alpha)^2 + 2\alpha\gamma + \omega_0^2\right\}Ce^{\alpha t} = 0$

The solution (for ω) from the quadratic formula is: $\alpha = \frac{-2\gamma \pm \sqrt{4\gamma^2 - 4\omega_0^2}}{2}$

$$\Rightarrow \alpha = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$$

Thus the solution is $x(t) = Ae^{-\gamma t}e^{\pm\sqrt{\gamma^2-\omega_0^2}t} = Ae^{-\gamma t}e^{\pm j\omega' t}$ where $\omega' = \sqrt{\omega_0^2 - \gamma^2}$.

If we consider the quantity under the square root sign, we see that there are three possibilities.

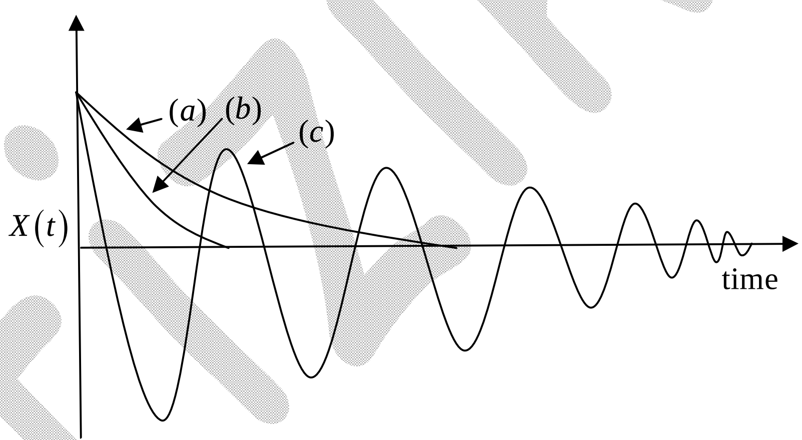
Case 1: Overdamped Case: If the damping coefficient is large, then $\gamma > \omega_0$ and ω' will be imaginary. Hence $x(t)$ will be a negative exponential function. It is shown in figure (a).

Case 2: Critically Damped Case: If $\omega_0 = \gamma$, then the square root vanishes. In this case, the solution is again a negative exponential function which goes to zero quicker than the overdamped case, as shown in figure (b).

Case 3: Underdamped Case: If $\omega_0 > \gamma$ then the quantity under the square root is positive and we have a real number for ω'. The solution for 'x' is then

$$x(t) = Ae^{-\gamma t}\sin(\omega' t + \phi)$$

This is the solution of the damped harmonic oscillator. The oscillatory motion is shown in figure (c).



Time Period: of the damped harmonic oscillator is: $T' = \dfrac{2\pi}{\omega'} = \dfrac{2\pi}{\sqrt{\omega_0^2 - \gamma^2}}$

This shows that due to damping the time period slightly increased.

Logarithmic Decrement: This measure the rate at which the amplitude decay

$$\lambda = \gamma T' = \frac{bT'}{2m}$$

Mean Life time (τ_m)**:** It is the time taken for the amplitude to decay to $\frac{1}{e}$ of the initial value. When $t = \tau_m = \frac{1}{\gamma}$, Amplitude $= \frac{A}{e}$ $\quad (\because Ae^{-\gamma t})$.

Energy of the Damped Oscillator

(i) Kinetic energy (K)

The displacement of a damped harmonic oscillator is $x(t) = Ae^{-\gamma t}\sin(\omega' t + \phi)$

The instantaneous velocity is

$$u = \frac{dx}{dt} = Ae^{-\gamma t}\left[-\gamma \sin(\omega' t + \phi) + \omega' \cos(\omega' t + \phi)\right] \approx Ae^{-\gamma t}\omega' \cos(\omega' t + \phi)$$

The approximation is done as $\gamma << \omega_0$

Thus, kinetic energy is $K = \frac{1}{2}mu^2 = \frac{1}{2}mA^2 e^{-2\gamma t}\omega'^2 \cos^2(\omega' t + \phi)$

(ii) Potential energy (U)

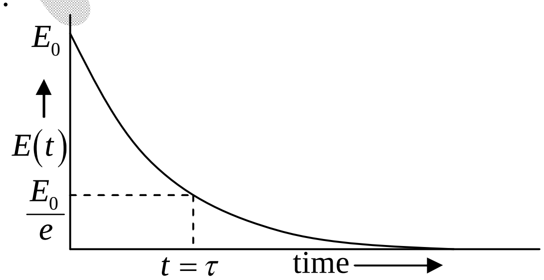
$$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 e^{-2\gamma t}\sin^2(\omega' t + \phi)$$

(iii) Total energy (E)

$$E = K + U = \frac{1}{2}mA^2 e^{-2\gamma t}\omega'^2 \cos^2(\omega' t + \phi) + \frac{1}{2}kA^2 e^{-2\gamma t}\sin^2(\omega' t + \phi)$$

$\Rightarrow E = \frac{1}{2}kA^2 e^{-2\gamma t}$ $\quad$ Since $\gamma << \omega_0$ thus $\omega' = \sqrt{\omega_0^2 - \gamma^2} \approx \omega_0 = \sqrt{\frac{k}{m}}$

This shows that the energy of the oscillator dcreases with time, the exponential decay of energy is show below.



Power Dissipation: It is the rate at which the energy is lost

$$P = -\frac{dE}{dt} = 2\gamma\left(\frac{1}{2}kA^2e^{-2\gamma t}\right) = 2\gamma E$$

Relaxation Time: It is the time taken for the total energy to decay to $\frac{1}{e}$ of its initial value E_0. If τ is the relaxation time, then at $t = \tau$, we shall have $E = \frac{E_0}{\tau}$.

Thus $\frac{E_0}{\tau} = E_0 e^{-2\gamma\tau}$. This gives relaxation time $\tau = \frac{1}{2\gamma}$

Thus we can also write power dissipated as $P = \frac{E}{\tau}$

and energy can be expressed as $E(t) = E_0 e^{-\frac{t}{\tau}}$.

Quality Factor (*Q*): It is defined as the 2π times the ratio of the energy stored in the system to the energy lost per period.

$$Q = 2\pi \frac{\text{energy stored in system}}{\text{energy loss per period}}$$

Energy stored in the system is E while the energy loss per period is PT, thus

$$Q = 2\pi\frac{E}{PT} = 2\pi\frac{E}{\left(\frac{E}{\tau}\right)T} = \frac{2\pi\tau}{T} = \omega_0\tau$$

This shows that, lower the damping, higher the value of Q.

Now energy can be written in term of Q as $E(t) = E_0 e^{-\frac{\omega_0 t}{Q}}$

It means that Q is related to the number of oscillation over which the energy fall to $\frac{1}{e}$ of its original value E_0, which is also called the relaxation time. This happens in time, $t = \tau$, where τ is given by

$$\frac{\omega_0\tau}{Q} = 1 \quad \Rightarrow \tau = \frac{Q}{\omega_0} = \frac{TQ}{2\pi}$$

In one period (T) number of oscillation is $=1$

In time τ the number of oscillation $=n$, then $n=\frac{\tau}{T}=\frac{Q}{2\pi}$

Thus the energy falls to $\frac{1}{e}$ of its original value after $n=\frac{Q}{2\pi}$ cycle of free oscillation.

Relation between relaxation time, mean time and Quality factor:

Let N is the number of oscillation in time $\tau_m=\frac{1}{\gamma}$

While n is the number of oscillation in time $\tau=\frac{1}{2\gamma}$

Thus, relation between mean and relaxation time is $\tau_m=2\tau$

In one period (T) number of oscillation is $=1$

In time τ_m the number of oscillation $=N$

$$N=\frac{\tau_m}{T}=\frac{2\tau}{T}=2\frac{Q}{2\pi} \quad \text{or} \quad N=2\frac{Q}{2\pi}=2n$$

2.2 Forced Oscillations and Resonance

When a body is made to oscillate under external periodic force, then in the beginning the body tries to oscillate with its natural frequency, but very soon these oscillation dies out and the body oscillate with the frequency of the applied force. Such oscillation is called forced oscillation.

Example:

1) Person swinging in a swing without anyone pushing → **Free or damped oscillation**

2) Someone pushes the swing periodically → **Forced or driven oscillations**

Natural angular frequency (ω_0) of the system when system oscillates freely after a sudden disturbance.

External frequency (p) of the system is the angular frequency of the external driving force causing the driven oscillations.

Resonance

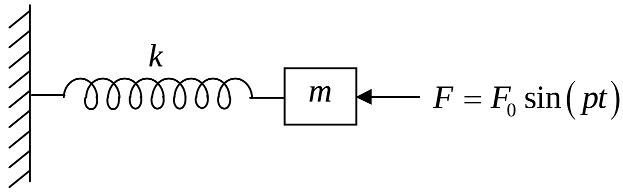
When the frequency of the external force is same as the natural frequency of the body, the amplitude of the oscillation is maximum. This phenomenon is called resonance.

Transient Effect

In the presence of external periodic force, initially body tries to oscillate with its natural frequency, while external force tries to impose its own frequency. Thus there is a tussel between external force and the body during which the amplitude rises and falls alternatively. This is the transient effect which soon dies out and body start oscillating with external frequency.

Equation of Forced Oscillation

Consider a mass m oscillating under external periodic force F. let x be the displacement at any instant. The forces acting on the mass are



(i) Restoring force proportional to the displacement. This is written as $F = -kx$

(ii) A frictional force proportional to the velocity. This is written as $F_{res} = -b\dfrac{dx}{dt}$

(iii) An external periodic force, which is $F = F_0 \sin(pt)$

Thus the total force on the body is $F = -kx - b\dfrac{dx}{dt} + F_0 \sin(pt)$

Using newton's third law, we can write $\dfrac{d^2x}{dt^2} + 2\gamma\dfrac{dx}{dt} + \omega_0^2 x = f_0 \sin(pt)$

where $2\gamma = \dfrac{b}{m}$, $\omega_0^2 = \dfrac{k}{m}$ and $f_0 = \dfrac{F_0}{m}$

This is the differential equation of the forced harmonic oscillator.

General solution of the differential equation: Once the transient effect dies out, body starts oscillating with frequency of the external force. So we can assume a solution of differential equation as $x = A\sin(pt - \theta)$

Differentiating and putting in differential equation we obtain the value of amplitude A

$$A = \frac{f_0}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}}$$

phase difference θ between the displacement and the driving force is $\tan\theta = \dfrac{2\gamma p}{\omega_0^2 - p^2}$

Thus solution of the forced oscillator becomes $x = \dfrac{f_0}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}}\sin(pt - \theta)$

Amplitude Resonance

The amplitude A of the forced oscillator depends on the constant f_0 and p of the driving force and the constant ω_0 and γ of the oscillator. At certain driving frequency amplitude A becomes maximum that is called amplitude resonance. Three different cases arise

Case 1: At very low driving frequency $(p << \omega_0)$. The amplitude A turns to be

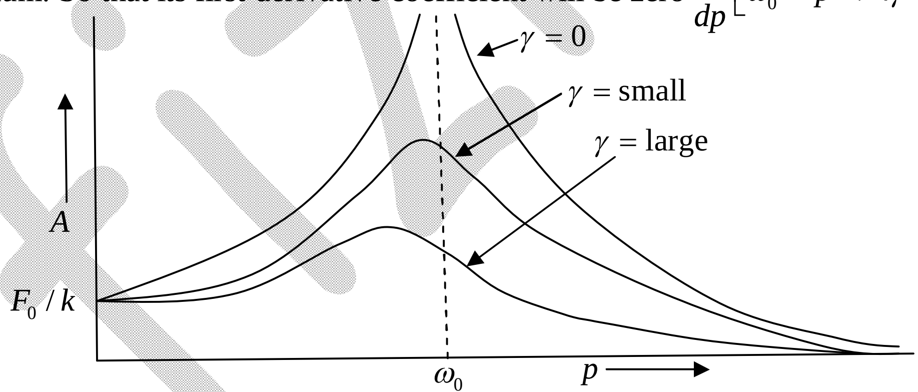
$$A = \frac{f_0}{\omega_0^2} \cong \frac{F_0}{k}$$

This shows that the amplitude depends only upon force constant and independent of mass, damping constant and driving frequency.

Case 2: At very high frequency $(p >> \omega_0)$, we get $A = \frac{f_0}{p^2} \cong \frac{F_0}{mp^2}$

This shows that amplitude now depends upon mass and driving frequency.

Case 3: Amplitude will be maximum when the denominator $\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}$ in A is minimum. So that its first derivative coefficient will be zero $\frac{d}{dp}\left[\omega_0^2 - p^2 + 4\gamma^2 p^2\right] = 0$



This gives the corresponding driving frequency i.e. amplitude resonance frequency at which A is maximum is $p = \sqrt{\left(\omega_0^2 - 2\gamma^2\right)}$.

At this frequency the amplitude is $A_{max} = \frac{f_0}{2\gamma\sqrt{\gamma^2 + p^2}} = \frac{f_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$

This shows that maximum amplitude depends upon the damping constant γ, smaller the damping larger is the amplitude. When, $\gamma = 0$, $A_{max} \to \infty$.

Dependence of the phase of displacement on the frequency of driving force

In steady state condition, the force equation and displacement equation is

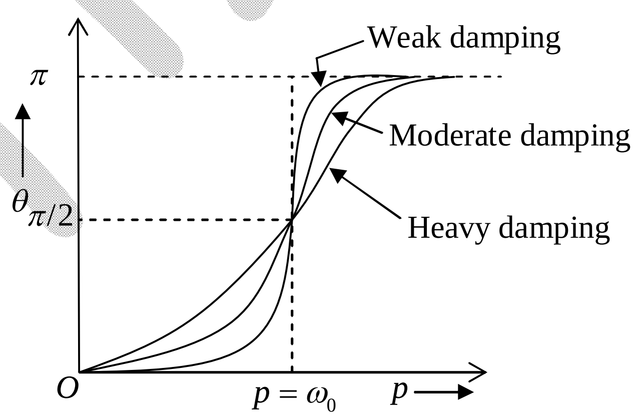
$F = F_0 \sin(pt)$ and $x = A\sin(pt-\theta)$, the phase factor is $\tan\theta = \dfrac{2\gamma p}{\omega_0^2 - p^2}$

This indicates that the displacement of the forced oscillation lags behind the driving force $F_0 \sin(pt)$ by an angle θ. The phase difference θ depends upon the damping γ and also on the difference between natural and driving frequency.

Case 1: If $p << \omega_0$, than $\tan\theta$ is positive and θ lies between 0 and $\frac{\pi}{2}$. In this condition (when $\gamma \approx 0$), θ is nearly zero i.e. displacement is in phase with driving force

Case 2: If $p = \omega_0$, than $\tan\theta$ is infinite and $\theta = \frac{\pi}{2}$. This is the resonance condition, displacement always lag behind the force by $\frac{\pi}{2}$. It means that at resonance the displacement is minimum when the driving force is maximum and vice versa.

Case 3: If $p >> \omega_0$, than $\tan\theta$ is negative and θ lies between $\frac{\pi}{2}$ and π. In this condition, θ is nearly π i.e. displacement almost opposite in phase with the driving force. The variation of θ with driving frequency p is given below.



Velocity Resonance

Velocity of the body also depends on the constant f_0 and p of the driving force and the constant ω_0 and γ of the oscillator. At certain driving frequency velocity amplitude becomes maximum that is called velocity resonance.

The instantaneous velocity of the body is

$$u = \frac{dx}{dt} = \frac{f_0 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}} \cos\left(pt - \theta\right)$$

The velocity will be maximum when $\cos\left(pt - \theta\right) = 1$. The maximum value is known as "velocity amplitude" u_0. Thus

$$u_0 = \frac{f_0 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}}$$

Three different cases arise.

Case 1: At very low driving frequency $\left(p << \omega_0\right)$. The velocity amplitude u_0 turns to be

$$u_0 = \frac{f_0 p}{\omega_0^2} = \frac{F_0 p}{k}$$

This indicates that the velocity amplitude depends on the spring constant k.

Case 2: At very high frequency $\left(p >> \omega_0\right)$, we get $u_0 = \frac{f_0}{p} = \frac{F_0}{mp}$

It shows that velocity amplitude depends on mass as well as driving frequency.

Case 3: At certain frequency the velocity amplitude becomes maximum, that frequency is called the velocity resonance frequency. The velocity amplitude can be written as

$$u_0 = \frac{f_0 p}{\sqrt{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}} = \frac{f_0}{\sqrt{\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2}}$$

The velocity amplitude u_0 will be maximum when the denominator $\sqrt{\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2}$ is minimum i.e. when

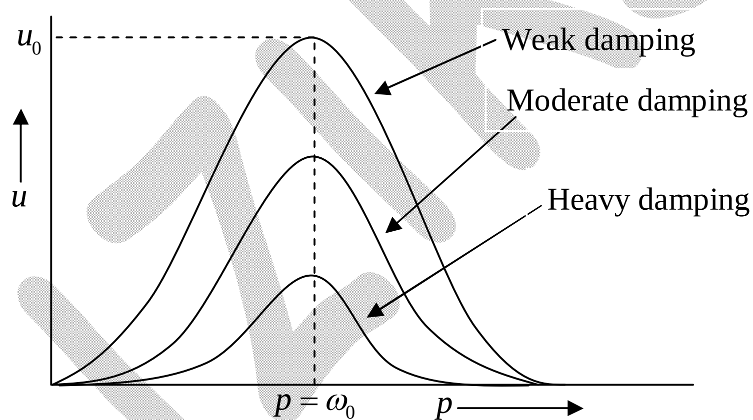
$$\left(\frac{\omega_0^2 - p^2}{p}\right) = 0 \Rightarrow p = \omega_0$$

This shows that the velocity resonance irrespective of damping value, always occurs when driving frequency is equal to the natural undamped frequency of the body.

The velocity amplitude at the resonance is $u_0 = \frac{f_0}{2\gamma} = \frac{F_0}{b}$

This shows that velocity amplitude at resonance only depends on the damping constant.

Dependence of the velocity of a forced oscillator on the driving frequency is shown as below



Dependence of the phase of velocity on the frequency of driving force:

In steady state condition, the force equation and velocity equation is

$$F = F_0 \sin(pt)$$

$$u = u_0 \cos(pt - \theta) = u_0 \sin\left(pt - \left(\theta - \frac{\pi}{2}\right)\right) = u_0 \sin(pt - \phi) \Rightarrow \phi = \theta - \frac{\pi}{2}$$

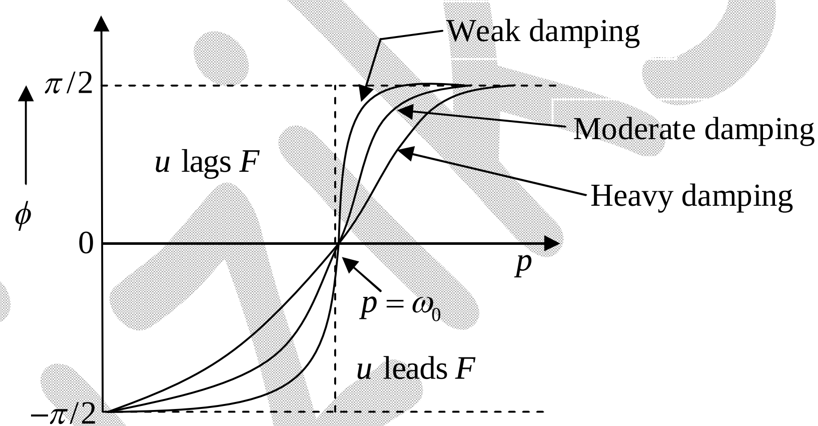
$$\Rightarrow \tan\phi = \cot\theta = \frac{\omega_0^2 - p^2}{2\gamma p} \qquad \because \tan\theta = \frac{2\gamma p}{\omega_0^2 - p^2}$$

Case 1: If $p << \omega_0$, than $\tan\theta$ is positive and θ lies between 0 and $\frac{\pi}{2}$. In this condition (when $\gamma \approx 0$), θ is nearly zero. Thus $\phi = -\frac{\pi}{2}$, this means velocity leads the driving force.

Case 2: If $p = \omega_0$, than $\tan\theta$ is infinite and $\theta = \frac{\pi}{2}$, and $\phi = 0$, this means that at resonance, the velocity is always in phase with the driving force.

Case 3: If $p >> \omega_0$, than $\tan\theta$ is negative and θ lies between $\frac{\pi}{2}$ and π, so that the ϕ is positive. This means that velocity lags behind the driving force.

The dependence of phase of velocity of the body on the frequency of the driving force is shown below.



Average power absorbed by oscillator (supplied by the driving force)

An oscillator absorbed energy from the driving force which is dissipated in doing work against the damping force present.

The instantaneous power P (i.e. rate at which work is done) absorbed by the oscillator is equal to energy per unit time.

$$P_{in} = \frac{\text{energy}}{\text{time}} = \frac{F.dx}{dt} = F.\frac{dx}{dt}$$

Thus P_{in} is equal to the product of the instantaneous driving force and the instantaneous velocity.

Thus

$$P_{in} = F_0 \sin(pt)\frac{dx}{dt} = \frac{mf_0^2 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2}} \sin(pt).\cos(pt - \phi)$$

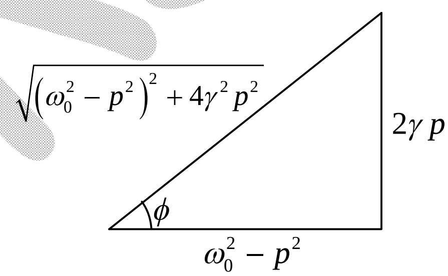
$$= \frac{mf_0^2 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2}} (\sin pt \cos pt \cos\phi) + \sin^2 pt \sin\phi$$

Now the average power absorbed is $\langle P_{in} \rangle = \frac{mf_0^2 p}{\sqrt{(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2}} \left(\frac{1}{2}\sin\phi\right)$

Note: the average values of the periodic function for the one period $T = \frac{2\pi}{p}$ is

$$\frac{1}{T}\int_0^T \sin pt \cos pt \, dt = 0 \quad \text{and} \quad \frac{1}{T}\int_0^T \sin p^2 t dt = \frac{1}{2}$$

Since, $\tan\theta = \frac{2\gamma p}{\omega_0^2 - p^2}$, thus using the vector model



$$\sin\phi = \frac{2\gamma p}{\sqrt{(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2}}$$

Therefore, the average power absorbed by the oscillator (average power supplied by the driving force) is

$$\langle P_{in} \rangle = \frac{mf_0^2 \gamma p^2}{(\omega_0^2 - p^2)^2 + 4\gamma^2 p^2}$$

The power supplied by the driving force is not stored in the system, but is dissipated as work done in moving the system against the force of friction.

Power Dissipated through frictional force

The instantaneous power dissipated through friction is given by

$$P_{dis} = |\text{instantaneous frictional force}| \times |\text{instantaneous velocity}|$$

$$= b\frac{dx}{dt}\times\frac{dx}{dt} = 2m\gamma\left(\frac{dx}{dt}\right)^2 = 2m\gamma\frac{f_0^2 p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}\cos^2\left(pt-\phi\right)$$

Now the average of $\cos^2\left(pt-\phi\right)$ for one full period is $\frac{1}{2}$. Therefore the average power dissipated is $\langle P_{dis}\rangle = \frac{mf_0^2\gamma p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}$

Thus $\langle P_{in}\rangle = \langle P_{dis}\rangle$

This shows that in the steady state the average power supplied by the driving force is equal to the average power dissipated by the frictional force.

Note: The instantaneous input power is not equal to the instantaneous power dissipated. Therefore at any instant of time the power stored in the oscillator is not constant.

Maximum power absorption

The average power can also be written as

$$\langle P_{in}\rangle = \frac{mf_0^2\gamma}{\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2}$$

$\langle P_{in}\rangle$ will be maximum when the denominator $\left(\frac{\omega_0^2 - p^2}{p}\right)^2 + 4\gamma^2$ is a minimum, this occurs when $\omega_0^2 - p^2 = 0$ or $p = \omega_0$

this is the condition of velocity resonance. Hence the power transferred from the driving force is maximum at the frequency of velocity resonance. The maximum power is

$$\langle P_{in}\rangle_{\text{maximum}} = \frac{mf_0^2}{4\gamma}$$

Sharpness of Resonance and Bandwidth

Sharpness of Resonance

The rapidity with which the power falls from its resonant value with change in a driving frequency is known as Sharpness of resonance.

The average power is maximum at certain frequency. As the driving frequency deviates either way from its resonant value, the power falls from its maximum value. If the fall in power with change in driving frequency from the resonant value is large, the resonance is said to be sharp, on the other hand, the fall is small, the resonant is said to be flat.

It is measured by the ratio of the resonant frequency ω_0 to the difference of two frequencies ω_1 and ω_2 at which the power falls to half of the resonant value.

$$\text{Sharpness of resonance} = \frac{\omega_0}{\omega_2 - \omega_1}$$

Bandwidth of Resonance

The difference in values of the driving frequency, at which the average power absorbed drops to half its maximum value, is called the band width of the resonance.

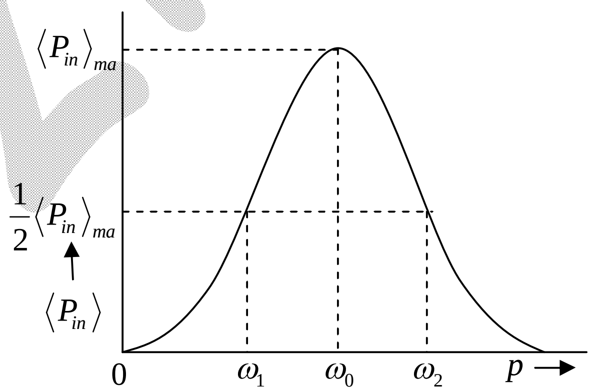
$\text{Bandwidth} = \omega_2 - \omega_1$

The average power absorbed is

$$\langle P_{in}\rangle = \frac{mf_0^2\gamma p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2}$$

The maximum average power is

$$\langle P_{in}\rangle_{\text{maximum}} = \frac{mf_0^2}{4\gamma}$$



The variation of the average power with driving frequency is shown in above figure also shows the half power frequencies.

The value of p at which the power goes half the maximum value obtained as

$$\langle P_{in}\rangle = \frac{1}{2}\langle P_{in}\rangle_{\text{maximum}} \Rightarrow \frac{mf_0^2\gamma 2p^2}{\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2} = \frac{1}{2}\frac{mf_0^2}{4\gamma}$$

or $\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2 = 8\gamma^2 p^2 \Rightarrow \left(\omega_0^2 - p^2\right)^2 = 4\gamma^2 p^2$

or $\omega_0^2 - p^2 = \pm 2\gamma p \Rightarrow p^2 = \omega_0^2 \pm 2\gamma p$

These are two quadratic equations in p,

$$p^2 + 2\gamma p - \omega_0^2 = 0 \quad \text{and} \quad p^2 - 2\gamma p - \omega_0^2 = 0$$

each has two roots one positive and other negative. Since negative frequency are not allowed, thus the allowed positive roots are

$$p_1\left(=\omega_1\right) = -\gamma + \sqrt{\gamma^2 + 2\gamma\omega_0^2} \quad \text{and} \quad p_2\left(=\omega_2\right) = \gamma + \sqrt{\gamma^2 + 2\gamma\omega_0^2}$$

The frequency difference between tow half power points i.e. bandwidth is

$$\text{Bandwidth} = \omega_2 - \omega_1 = 2\gamma = \frac{1}{\tau}$$

and sharpness of resonance $= \dfrac{\omega_0}{\omega_2 - \omega_1} = \omega_0 \tau$

It indicates that smaller the bandwidth, sharper is the resonance.

Quality Factor

The quality factor is defined as $Q = \dfrac{\omega_0}{\omega_2 - \omega_1} = \dfrac{\text{Resonance frequency}}{\text{bandwidth}} = \dfrac{\omega_0}{2\gamma} = \omega_0 \tau$

Quality factor also defined as

$$Q = 2\pi \frac{\text{average energy stored in one period}}{\text{average energy lost in one period}} = 2\pi \frac{E}{\langle P_{dis} \rangle T}$$

Where $\langle P_{dis} \rangle$ is the average power dissipated and T is the time period of oscillation. Thus $\langle P_{dis} \rangle \times T$ is the average energy lost in one period. On solving for energy we get the following expression of the quality factor.

$$Q = \frac{p^2 + \omega_0^2}{4\gamma p} = \frac{1}{2}\left(\frac{p^2 + \omega_0^2}{p^2}\right)\left(p\tau\right)$$

This is the exact expression for the quality factor of forced oscillator.

Near resonance, $p = \omega_0$, so we get $Q = \omega_0 \tau$

MCQ (Multiple Choice Questions)

Q1. A mass m attached to a spring is oscillating in water. If the spring constant is k and frictional force is $R\dot{\psi}$, which one of the following is correct?

(a) $m\ddot{\psi} = R\dot{\psi} + k\psi$ (b) $m\ddot{\psi} = R\dot{\psi} - k\psi$

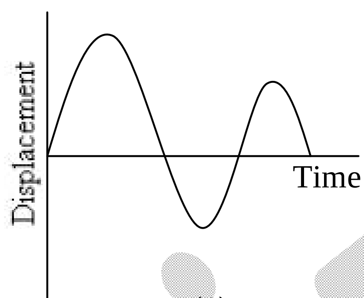
(c) $m\ddot{\psi} = -R\dot{\psi} + k\psi$ (d) $m\ddot{\psi} = -R\dot{\psi} - k\psi$

Q2. A lightly damped oscillator have a characteristic frequency ω. When operating frequency $\Omega << \omega$, the response of the oscillator is controlled by:

(a) damping coefficient (b) inertia of the mass

(c) oscillator frequency (d) spring constant

Q3.



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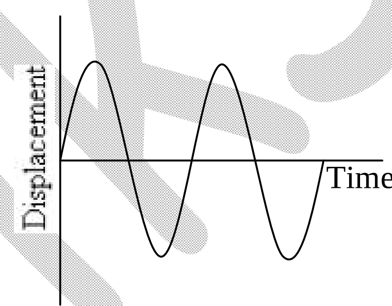
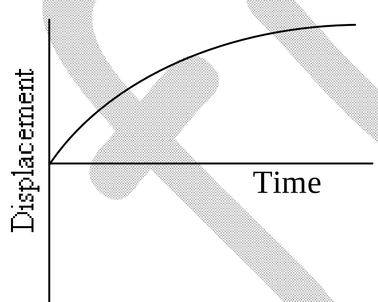


(2)



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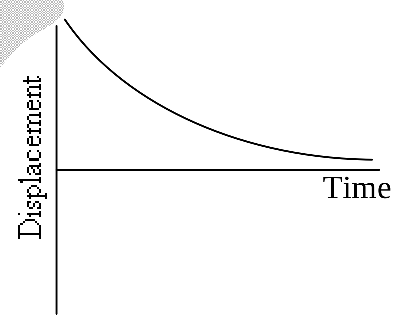


(4)

Which of the above figure(s) represent(s) damped simple harmonic motion?

(a) Figure 1 alone (b) Figure 2 alone

(c) Figure 4 alone (d) Figure 3 and 4

Q4. Which one of the following expressions correctly represents forced oscillation?

(a) $\frac{d^2y}{dt^2}+\omega^2 y=0$ (b) $\frac{d^2y}{dt^2}=0$

(c) $\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+\omega^2 y=0$ (d) $\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+\omega^2 y=F\sin pt$

Q5. A spring oscillating in water is acted upon by an external force $B\cos\omega t$. With the passage of time the frequency of the spring tends to be:

(a) greater than ω (b) less than ω

(c) equal to ω (d) decreasing exponentially

Q6. A damped simple harmonic oscillator of frequency f_1 is constantly driven by an external periodic force of frequency f_2. At the steady state, the oscillator frequency will be:

(a) f_1 (b) f_2 (c) f_1-f_2 (d) $\frac{f_1+f_2}{2}$

Q7. In the case of forced simple harmonic vibrations, the body generally vibrates with:

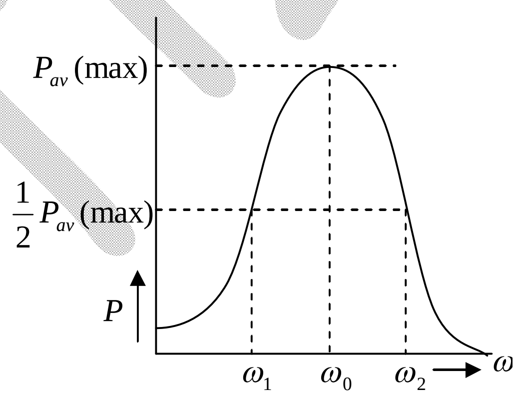
(a) Its natural frequency of vibration and its amplitude is small

(b) its natural frequency of vibration but its amplitude is large

(c) the frequency of the external force with a small amplitude

(d) the frequency of the external force with a large amplitude

Q8.



The above graph shows the average power P_{av} against the frequency of the driving force acting on an oscillator.

Consider the following statements:

(1) The sharpness of the peak at resonance is determined by the damping constant.

(2) The sharpness of resonance is defined by $\frac{\omega_2 - \omega_1}{\omega_0}$

(3) The peak occurs at the frequency of velocity resonance when the power absorbed by the oscillator from the driving force is maximum.

Which of these statements are correct?

(a) 1 and 2 (b) 2 and 3 (c) 1 and 3 (d) 1, 2 and 3

Q9. The maximum amplitude in the case of a forced oscillator occurs at the

(a) natural frequency of the oscillator

(b) frequency of the force

(c) frequency greater than the natural frequency of the oscillator

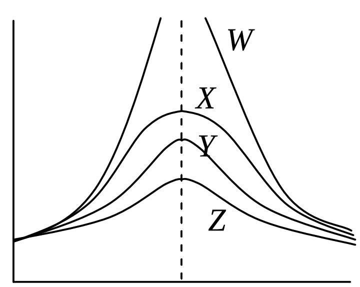
(d) frequency less than the natural frequency of the oscillator

Q10. If the differential equation given by $\frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = F_0 \sin pt$ describe the oscillatory motion of body in a dissipative medium under the influence of a periodic force, then the state of maximum amplitude of the oscillation is a measure of:

(a) free vibration (b) damped vibration

(c) forced vibration (d) resonance

Q11. The four curves shown in the given figure represent the variation of amplitude (A) with driving frequency (ω) for different values of damping constant. The curve which represents zero damping is the curve labeled:



(a) W (b) X

(c) Y (d) Z

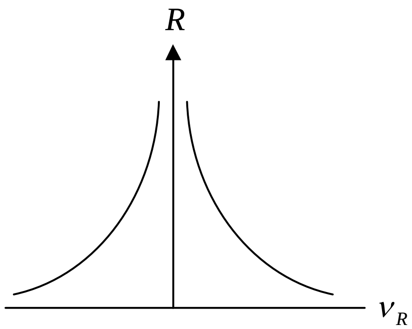
Q12. In the case of a forced vibration, the resonance wave becomes very sharp when the:

(a) damping force is small (b) restoring force is small

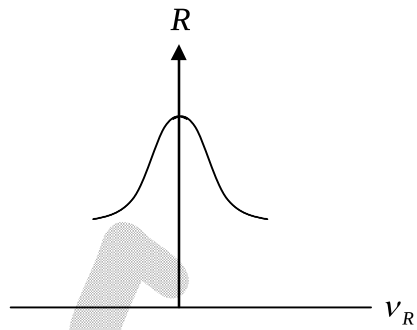
(c) applied oscillatory force is small (d) quality factor is small

Q13. The frequency response curve of a damped forced oscillator will be of the form ($R =$ amplitude, $\nu_R =$ frequency):

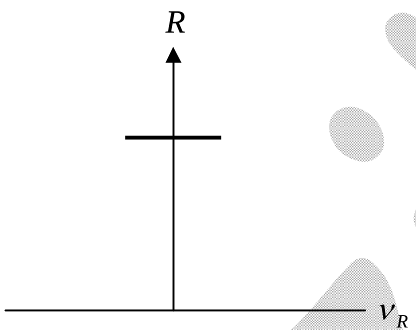
(a)



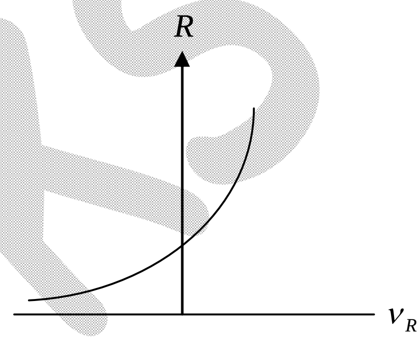
(b)



(c)



(d)



MSQ (Multiple Select Questions)

Q14. The equation of motion of an object of mass m attached with a spring of spring constant K is oscillating in medium of damping constant b is written as

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+Kx=0$$

if ω_0 is the natural angular frequency of the free oscillator thus which of the following are true?

(a) The critical damping occurs when $b=\sqrt{2mK}$

(b) Non-oscillatory, a periodic motion occurs when $b>2m\omega_0$

(c) Oscillations about equilibrium with an exponentially decaying amplitude occurs when $b<2m\omega_0$

(d) Non-oscillatory and faster return to equilibriums occurs when $b=2\sqrt{mK}$

Q15. A massless spring suspended from a rigid support, carries a flat disc of mass $200\ g$ at its lower end. It is observed that the system oscillates with a frequency of $20\ Hz$ and amplitude of the damped oscillation reduces to half of its undamped value in one minute. Which of the following statements are correct?

(a) Time period of oscillation decreases as amplitude decreases with time

(b) The damping constant b is $4.6\times10^{-3}\ Nsm^{-1}$

(c) The time in which amplitude decays to $\frac{1}{e}$ of itial value is 8.6 sec

(d) The time in which energy decays to $\frac{1}{e}$ of its itial value is 4.3 sec

Q16. A massless spring of spring constant $20\ N/m$ is suspended from a rigid support and carries a mass $0.2kg$ at its lower end. It is observed that the system performs damped oscillatory motion and its energy decays to $\frac{1}{e}$ of initial value in $50\sec$. Which of the following statements are correct?

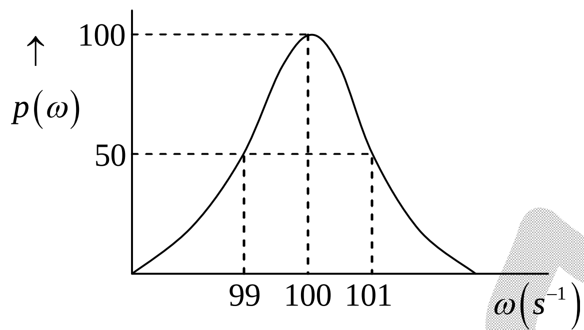
(a) The time in which amplitude decays to $\frac{1}{e}$ of initial value is 100 sec

(b) Damping constant b is $8\times10^{-3}\ Nsm^{-1}$

(c) Quality factor Q is 500

(d) Natural frequency of oscillation ω_0 is $10\ rad/\sec$

Q17. The graph shows the power resonance curve of a certain mechanical system which is driven by a force of constant magnitude but variable angular frequency ω of the following statements are correct?



(a) The resonance frequency ω_0 is 100 $rad/\sec$

(b) The full width at half maxima (FWHM) is $2\sec^{-1}$

(c) The Quality factor Q is 100

(d) The energy decays to $\frac{1}{e}$ of initial value in the absence of driving force in 0.5 sec

NAT (Numerical Answer Type)

Q18. A pendulum with a length 1.0 m is released from an initial angle of 15^0. After 1000 sec, its amplitude has been reduced by friction to 5.5^0. If amplitude $A = A_0 e^{-\gamma t}$ then the value of γ is ……..sec^{-1}

Q19. A 10.6 kg object oscillates at the end of a vertical spring that has a spring constant of $2.05\times10^4\ N/m$. The effect of air resistance is represented by damping co-efficient $b = 3N.s/m$. The frequency of the damped is …… Hz

Q20. A 2.0 kg object attached to a spring moves without friction and is driven by an external force given by $F = (3.0\,N)\sin(2\pi t)$. If the force constant of the spring is 20 N/m, then amplitude of the motion is ………. cm

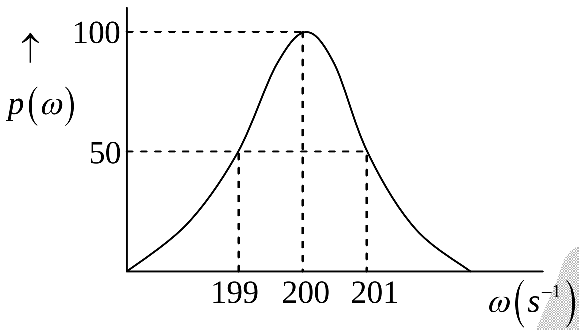
Q21. A weight of $40N$ is suspended from a spring that has a force constant of $20\frac{N}{m}$. The system is undamped and is subjected to a harmonic driving force of frequency 10 Hz resulting in a forced- motion amplitude of 2 cm. The maximum value of the driving force is……..… N

Q22. The Q-value of an underdamped harmonic oscillator of frequency 480 Hz is 80000. The number of oscillation does it take in time in which its amplitude decays to $\frac{1}{e}$ of its initial value is ……………..

Q23. The energy of a piano string of frequency 256 Hz reduces to half its initial values in 2sec. The Q-vlaue of the string is ……

Q24. The quality factor of a sonometer wire of frequency 500 Hz is 5000. The time in which its energy decays to $\frac{1}{e}$ of its initial value is ……. sec

Q25. The graph shows the power resonance curve of a certain mechanical system which is drives by a force of constant magnitude but variable frequency ω



The quality factor of the system is ....

Q26. An object of mass $2\ kg$ hangs from a spring of negligible mass and spring constant of $800\ N/m$. The spring is extended by $2.5\ cm$ when the object is attached. The top end of the spring is oscillated up and down in SHM with an amplitude of $2\ mm$. The damping constant is $0.5\ \sec^{-1}$. The amplitude of forced oscillations at $p = \omega_0$ is ...... cm

Solution-MCQ (Multiple Choice Questions)

Ans. 1: (d)

Solution: $-R\dot{\psi}$ is frictional force and is applied against the acceleration and if k is spring constant then $-k\psi$ represent restoring arise due to extension in string and act in opposite direction of motion. The total force acting on mass m is

$$F = F_1 + F_2 = -R\dot{\psi} - k\psi \Rightarrow m\ddot{\psi} = -R\dot{\psi} - k\psi$$

Ans. 2: (d)

Q2. If the angular frequency of vibration (ω') of a damped simple harmonic oscillator is related to ω_0

$$\omega_0^2 - \omega'^2 = 10^{-6}\omega_0^2$$

where ω_0 is the angular frequency of its vibration when there is no damping, the Q-factor will be equal to:

(a) 500 (b) 1,000 (c) 5,000 (d) 10000

Ans. 2: (a)

Quality factor is defined as $Q = \frac{\omega_0}{2\gamma}$ where $\omega' = \sqrt{\omega_0^2 - \gamma^2} \Rightarrow \omega_0^2 - \omega'^2 = \gamma^2$

Thus $\gamma^2 = 10^{-6}\omega_0^2 \Rightarrow \gamma = 10^{-3}\omega_0 \Rightarrow Q = \frac{\omega_0}{2\gamma} = \frac{\omega_0}{2\times 10^{-3}\omega_0} = 500$

Ans. 3: (a)

Solution: In damped simple harmonic motion the amplitude dies exponentially.

Ans. 4: (d)

Solution: If mass of a particle is acted up by a periodic force.

Given as force $= F_0 \sin pt$, then its equation is given as

$$m\frac{d^2y}{dt^2} + r\frac{dy}{dt} + cy = F_0 \sin \omega t$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{r}{m}\frac{dy}{dt} + \frac{c}{m}y = \frac{F_0}{m}\sin pt \Rightarrow \frac{d^2y}{dt^2} + 2k\frac{dy}{dt} + \omega^2 y = F \sin pt$$

Ans. 5: (c)

Solution: When a spring oscillates in water under periodic force $F = B\cos\omega t$, the oscillation is like forced oscillation and so after some time the spring oscillate with frequency equal to that of the force.

Ans. 6: (b)

Solution: When an external periodic force is switched on a harmonic oscillator, the motion of the oscillator in the beginning is irregular because the oscillator tries to oscillate with its own frequency where as the driving force gets success in imposing its own frequency. In this state, called the steady state, the oscillator oscillates with constant amplitude and with frequency equal to that of the driving force.

Ans. 7: (c)

Solution: In forced SHM the body oscillates with the frequency of external force and its amplitude is small.

Ans. 8: (c)

Solution: The quality factor Q measure sharpness and $Q = \frac{\omega_0}{\omega_2 - \omega_1}$

Sharpness = $Q = \frac{\omega_0}{\omega_2 - \omega_1}$

Hence statement (2) is wrong

Ans. 9: (d)

Solution: The equation of forced oscillation is given as

$$\ddot{x} + 2\gamma x + \omega_0^2 x = f_0 \sin pt$$

Amplitude of the motion is given as $A = \frac{f_0}{\left[\left(\omega_0^2 - p^2\right)^2 + 4\gamma^2 p^2\right]^{\frac{1}{2}}}$

This becomes maximum when $p^2 = \left(\omega_0^2 - 2\gamma^2\right)$

$\Rightarrow$ The maximum amplitude occurs when $p < \omega_0$

Ans. 10: (d) Solution: In forced SHM the state of maximum amplitude of oscillation is a measure of resonance.

Ans. 11: (a)

Solution: $\because A_{\max} = \dfrac{f_0}{2\gamma\sqrt{\omega_0^2 - \gamma^2}}$. Thus as damping decreases the response curve becomes sharper and so its amplitude increases gradually. Thus when damping is zero, amplitude becomes infinite. Thus the curve W represents the response curve having zero damping.

Ans. 12: (a)

Solution: If m be mass of a particle, then forced vibration equation is given as

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \frac{c}{m}x = F_0 \sin pt \qquad \text{-----(i)}$$

where F_0 is magnitude of applied force of frequency p.

Dividing equation (i) by m, we get

$$\frac{d^2x}{dt^2} + \frac{r}{m}\frac{dx}{dt} + \frac{c}{m}x = \frac{F_0}{m}\sin pt \text{ or } \frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega_0^2 x = f_0 \sin pt \text{ where } f_0 = \frac{F_0}{m}.$$

When steady state has been attained the oscillator has settled down to oscillate with forcing frequency $\dfrac{p}{2\pi}$ and a constant amplitude.

Let solution of equation (i) be given as $x = A\sin\left(pt - \theta\right)$

where θ is phase difference between the applied force and displacement of the oscillator.

Solution can be written as

$$A = \frac{f_0}{\left[\left(\omega_0^2 - p^2\right)^2 + 4k^2p^2\right]^{1/2}} \text{ where } \theta = \tan^{-1}\frac{2kp}{\left(\omega_0^2 - p^2\right)}$$

At resonance frequency amplitude is given as $A_{\max} = \dfrac{f_0}{2k\left(p^2 + k^2\right)^{1/2}}$

Sharpness is a measure of the rate of fall of amplitude from its maximum value at resonance frequency on either side of it. The sharper the fall in amplitude, the sharper the resonance.

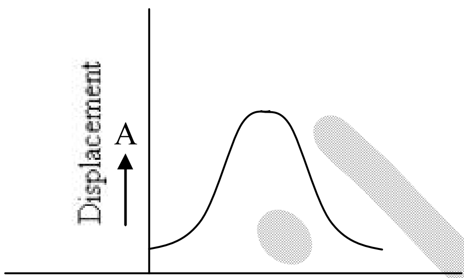
Ans. 13: (b)

Solution: The differential equation of forced oscillation is given as

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+\omega_0^2x=f_0\sin pt$$

solving this the amplitude is given as $$A=\frac{f_0}{\left[\left(\omega_0^2-p^2\right)+\left(2kp\right)^2\right]^{1/2}}$$

The curve given for this amplitude is given as



Solution-MSQ (Multiple Select Questions)

Ans. 14: (b), (c) and (d)

Solution: The damped frequency of the oscillator is $\omega=\sqrt{\omega_0^2-\frac{b^2}{4m^2}}$

(i) Critical damping occurs when $b=2m\omega_0=2\sqrt{mK}$. This provides for the faster return to equilibrium

(ii) Under damping (or light damping) occurs, when $b<2m\omega_0$

This entails oscillation about equilibrium with exponentially decaying amplitude

(iii) Over damping (or heavy damping) occurs when $b>2m\omega_0$.

This gives non-oscillatory, a periodic motion with a monatomic approach to equilibrium at late times, which slower than critical damping.

Ans. 15: (b) and (c)

Solution: (a) Time period of damped oscillation is independent of its amplitude

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}}$$

(b) The amplitude of the damped oscillator at an instant t is given by $A = A_0 e^{-\gamma t}$.

$$\Rightarrow \frac{1}{2} = e^{-\gamma\times 60} \qquad \because \frac{A}{A_0} = \frac{1}{2} \text{ when } t = 1 \text{ minute} = 60 \text{ sec}$$

$$e^{60\gamma} = 2 \qquad \text{or } \gamma = \frac{2.303\times 0.301}{60} = 0.116 s^{-1}$$

$\therefore$ The damping constant b is

$$b = 2\gamma m = 2\times 0.116 s^{-1}\times 200\times 10^{-3}\ kg = 4.6\times 10^{-3}\ Nsm^{-1}$$

(c) The time in which amplitude decay to $\frac{1}{e}$ of itial value is $\tau = \frac{1}{\gamma} = 8.6$ sec

(d) The time in which energy decays to $\frac{1}{e}$ of itial value is $\Gamma = 2\tau = 17.24$ sec

Ans. 16: (b), (c) and (d)

Solution: Given $\Gamma = 50\,\text{sec}$

(a) $\tau = \frac{\Gamma}{2} = \frac{50}{2} = 25$ sec

(b) $b = \frac{2m}{\tau} = \frac{2\times 0.2}{25} = 8\times 10^{-3}\ Nsm^{-1}$

(c) $Q = \omega\Gamma$, $\omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}} = \sqrt{\omega_0^2 - \left(\frac{1}{\tau}\right)^2} = \sqrt{(10)^2 - \left(\frac{1}{25}\right)^2} = \sqrt{99.9} = 9.99$

$\omega \cong 10\, rad/\text{sec} \qquad \therefore \quad Q = 10\times 50 = 500$

(d) $\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{20}{0.2}} = 10\, rad/\text{sec}$

Ans. 17: (a), (b) and (d)

Solution: (a) The maximum value of the mean power input occurs when the angular frequency ω of driving force is equal to the resonant angular frequency ω of the oscillator

$\therefore \quad \omega = \omega_0 = 100\, rad/\sec$

(b) $FWHM = \Delta\omega = 2\sec^{-1}$

(c) $Q = \frac{\omega_0}{\Delta\omega} = \frac{100}{101-99} = \frac{100}{2} = 50$

(d) $E = E_0 e^{-(\Delta\omega)t} = E_0 e^{-2t}$, when $t = \frac{1}{2} = 0.5\sec$, energy E decays to $\frac{1}{e}$ its initial value

Solution-<u>NAT (Numerical Answer Type)</u>

Ans. 18: 0.001

Solution: $\theta_0 = 15^0$ and $\theta(t = 1000) = 5.5^0$

Now, $A = A_0 e^{-\gamma t} \Rightarrow \frac{A}{A_0} = \frac{A_0 e^{-\gamma 1000}}{A_0} = \frac{5.5}{15} \Rightarrow \quad \ln\left(\frac{5.5}{15}\right) = -1000\gamma \therefore \quad \gamma = 0.001 \sec^{-1}$

Ans. 19: 7

Solution: The frequency of undamped oscillator is $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{2.05\times10^4\, N/m}{10.6\, kg}} = 44\ Hz$

The frequency of damped oscillation is

$$\omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2} = \sqrt{(44)^2 - \left(\frac{3}{2\times1.06}\right)^2} = \sqrt{1933.96 - 0.02} = 44 \sec^{-1}$$

$$\therefore \quad f = \frac{\omega}{2\pi} = \frac{44}{2\pi} = 7Hz$$

Ans. 20: 5.09

Solution: Given $F=(3.0N)\sin(2\pi t)$ and $k=20\frac{N}{m}$

$$\therefore \quad \omega=\frac{2\pi}{T}=2\pi\ rad/\sec \text{ also } \omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{20}{2}}=3.10\ rad\sec$$

The equation for the amplitude of a driven oscillator, with $b=0$, gives

$$A=\frac{F_0}{m}\left(\omega^2-\omega_0^2\right)^{-1}=\frac{3}{2}\left[4\pi^2-(3.16)^2\right]^{-1} \quad \Rightarrow A=0.0509\ m=5.09\ cm$$

Ans. 21: 318

Solution: Amplitude of a driven oscillator with no damping $A=\frac{\frac{F_0}{m}}{\sqrt{\left(\omega^2-\omega_0^2\right)^2}}$

where $\omega=2\pi f=20\pi\ \sec^{-1}$, $\omega_0^2=\frac{k}{m}=\frac{200}{\left(\frac{40}{9.8}\right)}=49\ \sec^{-2}$

$$\therefore \quad F_0=mA\left(\omega^2-\omega_0^2\right)=\left(\frac{40}{9.8}\right)\left(2\times10^{-2}\right)(3950-49)=318\ N \Rightarrow F_0=318\ N$$

Ans. 22: 25464.8

Solution: if τ_m is the mean time in which amplitude decays to $\frac{1}{e}$ of initial value. Then the number of oscillation in time τ_m is $N=\frac{Q}{\pi}=\frac{80000}{\pi}=25464.8$

Ans. 23: 4641

Solution: The average energy of the oscillation is $E(t)=E_0 e^{\frac{-\omega t}{Q_0}}$

$$\Rightarrow \frac{1}{2}=e^{\frac{-2\omega_0}{Q_0}} \quad \text{in } t=2\sec,\ E(t)=\frac{E_0}{2} \Rightarrow e^{\frac{2\omega_0}{Q_0}}=2$$

$$\Rightarrow \frac{2\omega_0}{Q_0}=\ln 2 \Rightarrow Q_0=\frac{2\omega_0}{\ln 2}=\frac{2\times 2\pi\nu_0}{\ln 2}=\frac{4\pi\nu_0}{\ln 2} \therefore Q_0=\frac{4\times\pi\times 256}{\ln 2}=\frac{3216.99}{0.693}=4641$$

Ans. 24: 1.6

Solution: The average energy of oscillation is $E(t)=E_0\ e^{\frac{-\omega_0 t}{Q_0}}$. It means that Q is related to the number of oscillation over which the energy fall to $\frac{1}{e}$ of its initial value E_0. This happens in time $t=\tau$ where $\Rightarrow\ \frac{\omega_0\tau}{Q}=1\Rightarrow\ \tau=\frac{Q}{\omega_0}=\frac{Q}{2\pi\nu_0}$

$$\therefore\ T=\frac{5000}{2\pi(500)}=\frac{10}{2\pi}=\frac{5}{\pi}=1.59=1.6\text{ sec}\qquad \because Q=5000,\ \nu_0=500\ Hz$$

Ans. 25: 100

Solution: quality factor is defined as $Q=\frac{\omega_0}{\Delta\omega}$, where $\omega_0=200\ rad/\sec$

and $\Delta\omega=201-199=2\ rad/\sec\Rightarrow Q=\frac{200}{2}=100$

Ans. 26: 4

Solution: The amplitude of forced oscillation is $A=\frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2-p^2\right)+4\gamma^2\omega^2}}$

At $p=\omega_0$, we get $A=\frac{\frac{F_0}{m}}{2\gamma\omega_0}$. To evaluate A, we need to know F_0, the amplitude of the driving force and γ which measures the damping of system.

$F_0=$ spring constant $\times$ displacement amplitude $=(800\ N/m)\left(2\times10^{-3}\ m\right)=1.6\ N$

where $\omega_0=\sqrt{\frac{k}{m}}=\sqrt{\frac{800\ N/m}{2}}\Rightarrow\ \omega_0=20\ rad/\sec$

$$A=\frac{\frac{1.6}{2}}{2\times0.5\times20}=\frac{0.8}{20}=0,04\ m\quad A=4\ cm$$