

7.9 Plane

Point coordinates: $x, y, z, x_0, y_0, z_0, x_1, y_1, z_1, \dots$

Real numbers: $A, B, C, D, A_1, A_2, a, b, c, a_1, a_2, \lambda, p, t, \dots$

Normal vectors: $\vec{n}, \vec{n}_1, \vec{n}_2$

Direction cosines: $\cos\alpha, \cos\beta, \cos\gamma$

Distance from point to plane: d

675. General Equation of a Plane

$$Ax + By + Cz + D = 0$$

676. Normal Vector to a Plane

The vector $\vec{n}(A, B, C)$ is normal to the plane

$$Ax + By + Cz + D = 0.$$

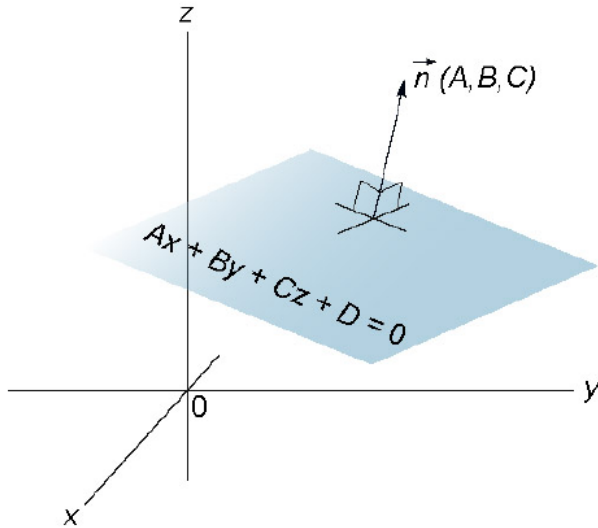


Figure 127.

677. Particular Cases of the Equation of a Plane

$$Ax + By + Cz + D = 0$$

If $A = 0$, the plane is parallel to the x -axis.

If $B = 0$, the plane is parallel to the y -axis.

If $C = 0$, the plane is parallel to the z -axis.

If $D = 0$, the plane lies on the origin.

If $A = B = 0$, the plane is parallel to the xy -plane.

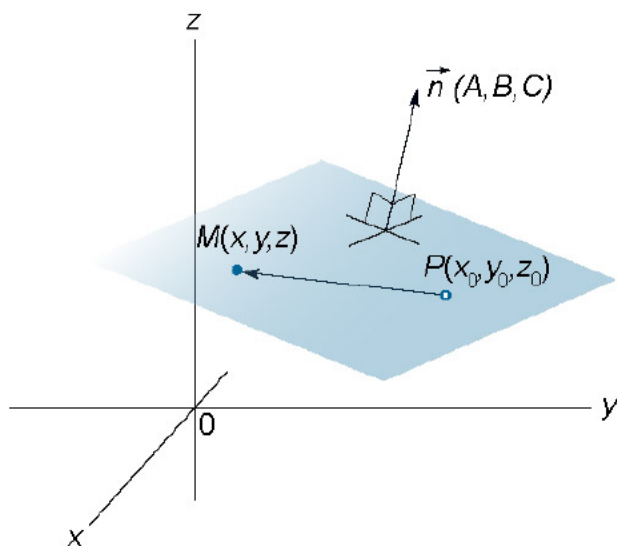
If $B = C = 0$, the plane is parallel to the yz -plane.

If $A = C = 0$, the plane is parallel to the xz -plane.

678. Point Direction Form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where the point $P(x_0, y_0, z_0)$ lies in the plane, and the vector (A, B, C) is normal to the plane.

**Figure 128.****679.** Intercept Form

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

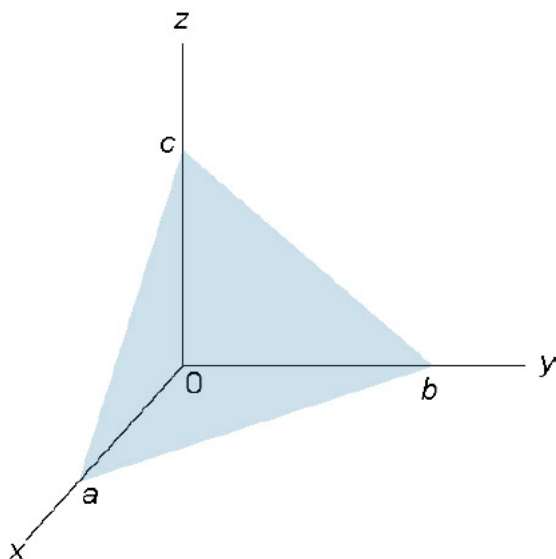


Figure 129.

680. Three Point Form

$$\begin{vmatrix} x-x_3 & y-y_3 & z-z_3 \\ x_1-x_3 & y_1-y_3 & z_1-z_3 \\ x_2-x_3 & y_2-y_3 & z_2-z_3 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

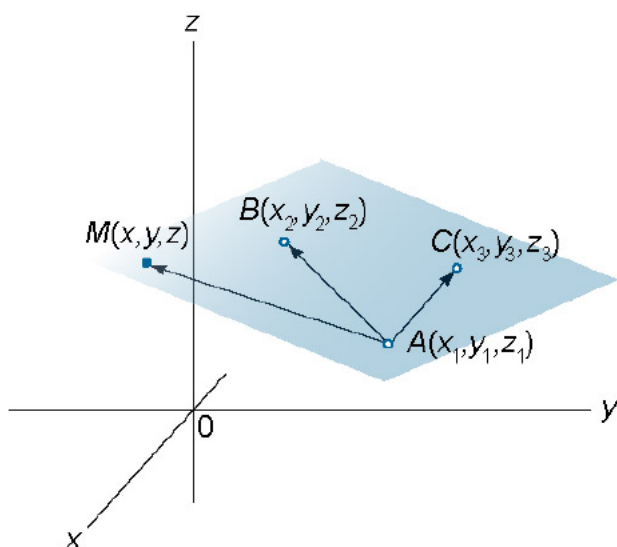


Figure 130.

681. Normal Form

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0,$$

where p is the perpendicular distance from the origin to the plane, and $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of any line normal to the plane.

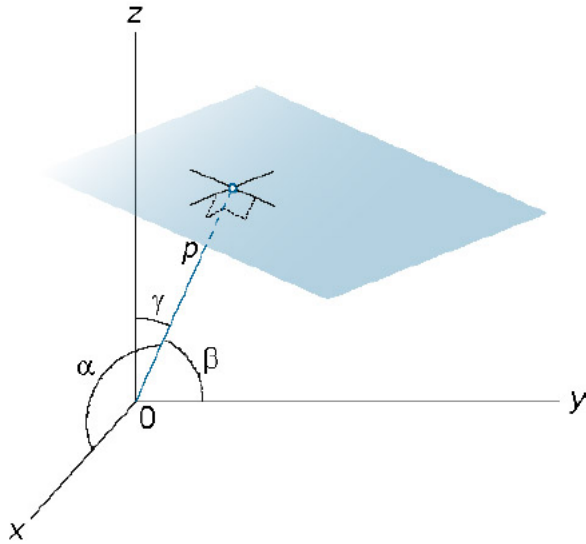


Figure 131.

682. Parametric Form

$$\begin{cases} x = x_1 + a_1s + a_2t \\ y = y_1 + b_1s + b_2t, \\ z = z_1 + c_1s + c_2t \end{cases}$$

where (x, y, z) are the coordinates of any unknown point on the line, the point $P(x_1, y_1, z_1)$ lies in the plane, the vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) are parallel to the plane.

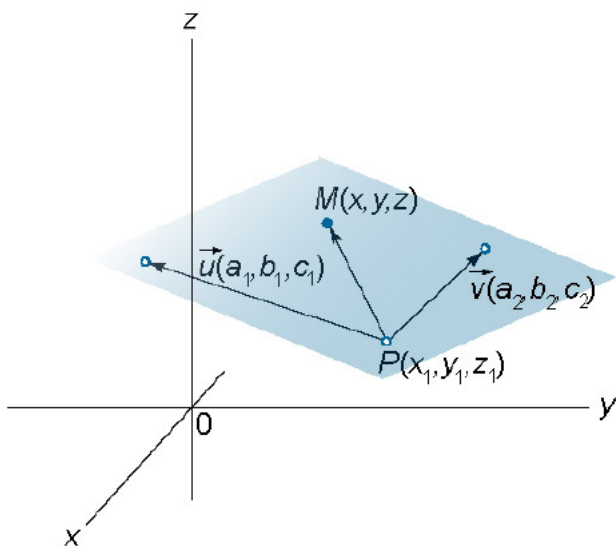


Figure 132.

683. Dihedral Angle Between Two Planes

If the planes are given by

$$A_1x + B_1y + C_1z + D_1 = 0,$$

$$A_2x + B_2y + C_2z + D_2 = 0,$$

then the dihedral angle between them is

$$\cos \varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

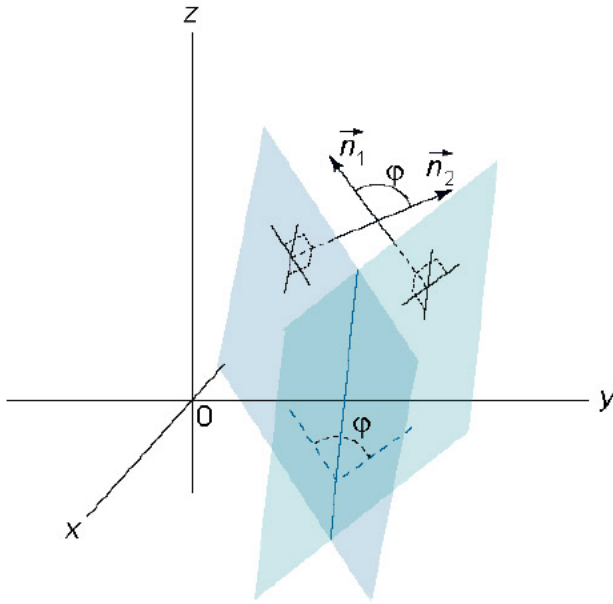


Figure 133.

684. Parallel Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are parallel if

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}.$$

685. Perpendicular Planes

Two planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are perpendicular if $A_1A_2 + B_1B_2 + C_1C_2 = 0$.

686. Equation of a Plane Through $P(x_1, y_1, z_1)$ and Parallel To the Vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) (Fig.132)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

- 687.** Equation of a Plane Through $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$, and Parallel To the Vector (a, b, c)

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a & b & c \end{vmatrix} = 0$$

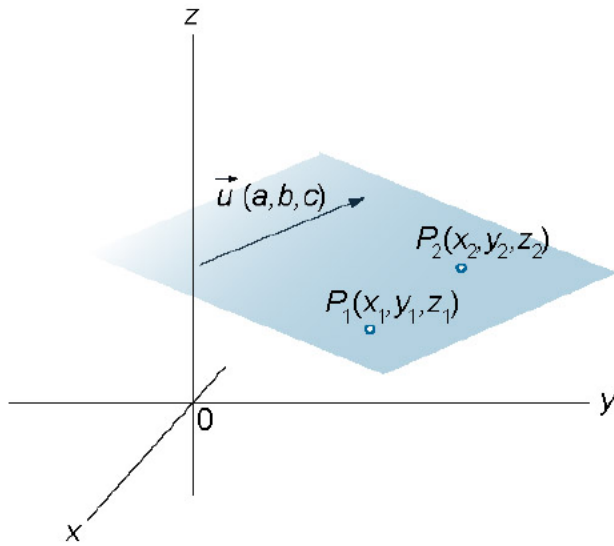


Figure 134.

- 688.** Distance From a Point To a Plane
The distance from the point $P_1(x_1, y_1, z_1)$ to the plane $Ax + By + Cz + D = 0$ is

$$d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

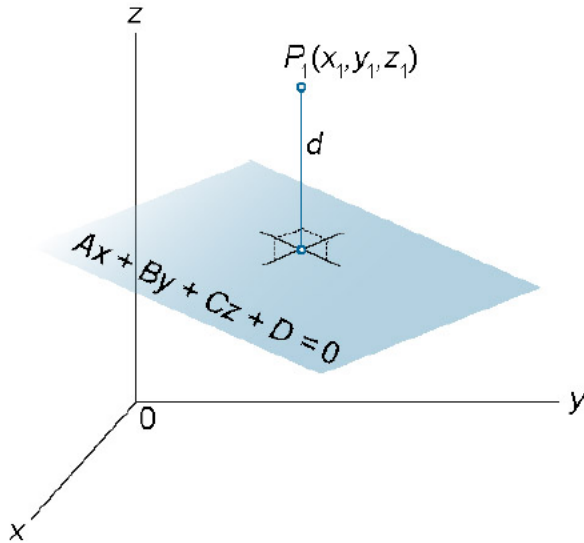


Figure 135.

689. Intersection of Two Planes

If two planes $A_1x + B_1y + C_1z + D_1 = 0$ and

$A_2x + B_2y + C_2z + D_2 = 0$ intersect, the intersection straight line is given by

$$\begin{cases} x = x_1 + at \\ y = y_1 + bt, \\ z = z_1 + ct \end{cases}$$

or

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c},$$

where

$$a = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}, \quad b = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}, \quad c = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix},$$

$$x_1 = \frac{b \begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix} - c \begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$y_1 = \frac{c \begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix} - a \begin{vmatrix} D_1 & C_1 \\ D_2 & C_2 \end{vmatrix}}{a^2 + b^2 + c^2},$$

$$z_1 = \frac{a \begin{vmatrix} D_1 & B_1 \\ D_2 & B_2 \end{vmatrix} - b \begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}}{a^2 + b^2 + c^2}.$$