

8.8 Multivariable Functions

Functions of two variables: $z(x, y)$, $f(x, y)$, $g(x, y)$, $h(x, y)$

Arguments: x, y, t

Small changes in x, y, z , respectively: $\Delta x, \Delta y, \Delta z$.

846. First Order Partial Derivatives

The partial derivative with respect to x

$$\frac{\partial f}{\partial x} = f_x \quad (\text{also } \frac{\partial z}{\partial x} = z_x),$$

The partial derivative with respect to y

$$\frac{\partial f}{\partial y} = f_y \quad (\text{also } \frac{\partial z}{\partial y} = z_y).$$

847. Second Order Partial Derivatives

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = f_{xx},$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = f_{yy},$$

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy},$$

$$\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}.$$

If the derivatives are continuous, then

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

848. Chain Rules

If $f(x, y) = g(h(x, y))$ (g is a function of one variable h), then

$$\frac{\partial f}{\partial x} = g'(h(x, y)) \frac{\partial h}{\partial x}, \quad \frac{\partial f}{\partial y} = g'(h(x, y)) \frac{\partial h}{\partial y}.$$

If $h(t) = f(x(t), y(t))$, then $h'(t) = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

If $z = f(x(u, v), y(u, v))$, then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v}.$$

849. Small Changes

$$\Delta z \approx \frac{\partial f}{\partial x} \Delta x + \frac{\partial f}{\partial y} \Delta y$$

850. Local Maxima and Minima

$f(x, y)$ has a **local maximum** at (x_0, y_0) if $f(x, y) \leq f(x_0, y_0)$ for all (x, y) sufficiently close to (x_0, y_0) .

$f(x, y)$ has a **local minimum** at (x_0, y_0) if $f(x, y) \geq f(x_0, y_0)$ for all (x, y) sufficiently close to (x_0, y_0) .

851. Stationary Points

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0.$$

Local maxima and local minima occur at stationary points.

852. Saddle Point

A stationary point which is neither a local maximum nor a local minimum

853. Second Derivative Test for Stationary Points

Let (x_0, y_0) be a stationary point $(\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0)$.

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix}.$$

If $D > 0$, $f_{xx}(x_0, y_0) > 0$, (x_0, y_0) is a point of local minima.

If $D > 0$, $f_{xx}(x_0, y_0) < 0$, (x_0, y_0) is a point of local maxima.

If $D < 0$, (x_0, y_0) is a saddle point.

If $D = 0$, the test fails.

854. Tangent Plane

The equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

855. Normal to Surface

The equation of the normal to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}.$$