8.9 Differential Operators

Unit vectors along the coordinate axes: $\vec{i}$, $\vec{j}$, $\vec{k}$
Scalar functions (scalar fields): $f(x,y,z)$, $u(x_1, x_2, \ldots, x_n)$
Gradient of a scalar field: grad u, ∇u
Directional derivative: $\dfrac{\partial f}{\partial l}$
Vector function (vector field): $\vec{F}(P,Q,R)$
Divergence of a vector field: div $\vec{F}$, $\nabla \cdot \vec{F}$
Curl of a vector field: curl $\vec{F}$, $\nabla \times \vec{F}$
Laplacian operator: ∇^2

856. Gradient of a Scalar Function

$$\operatorname{grad} f = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right),$$

$$\operatorname{grad} u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n} \right).$$

857. Directional Derivative

$$\frac{\partial f}{\partial l} = \frac{\partial f}{\partial x}\cos\alpha + \frac{\partial f}{\partial y}\cos\beta + \frac{\partial f}{\partial z}\cos\gamma,$$

where the direction is defined by the vector
$\vec{l}(\cos\alpha, \cos\beta, \cos\gamma)$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

858. Divergence of a Vector Field

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

859. Curl of a Vector Field

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

860. Laplacian Operator

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

861. $\operatorname{div}\left(\operatorname{curl} \vec{F}\right) = \nabla \cdot \left(\nabla \times \vec{F}\right) \equiv 0$

862. $\operatorname{curl}\left(\operatorname{grad} f\right) = \nabla \times \left(\nabla f\right) \equiv 0$

863. $\operatorname{div}\left(\operatorname{grad} f\right) = \nabla \cdot \left(\nabla f\right) = \nabla^2 f$

864. $\operatorname{curl}\left(\operatorname{curl} \vec{F}\right) = \operatorname{grad}\left(\operatorname{div} \vec{F}\right) - \nabla^2 \vec{F} = \nabla\left(\nabla \cdot \vec{F}\right) - \nabla^2 \vec{F}$