

10.2 Second Order Ordinary Differential Equations

1173. Homogeneous Linear Equations with Constant Coefficients

$$y'' + py' + qy = 0.$$

The characteristic equation is

$$\lambda^2 + p\lambda + q = 0.$$

If λ_1 and λ_2 are distinct real roots of the characteristic equation, then the general solution is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}, \text{ where}$$

C_1 and C_2 are integration constants.

If $\lambda_1 = \lambda_2 = -\frac{p}{2}$, then the general solution is

$$y = (C_1 + C_2 x) e^{-\frac{p}{2}x}.$$

If λ_1 and λ_2 are complex numbers:

$$\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i, \text{ where}$$

$$\alpha = -\frac{p}{2}, \beta = \frac{\sqrt{4q - p^2}}{2},$$

then the general solution is

$$y = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x).$$

1174. Inhomogeneous Linear Equations with Constant Coefficients

$$y'' + py' + qy = f(x).$$

The general solution is given by

$$y = y_p + y_h, \text{ where}$$

y_p is a particular solution of the inhomogeneous equation and y_h is the general solution of the associated homogeneous equation (see the previous topic 1173).

If the right side has the form

$$f(x) = e^{\alpha x} (P_1(x) \cos \beta x + P_2(x) \sin \beta x),$$

then the particular solution y_p is given by

$$y_p = x^k e^{\alpha x} (R_1(x) \cos \beta x + R_2(x) \sin \beta x),$$

where the polynomials $R_1(x)$ and $R_2(x)$ have to be found by using the [method of undetermined coefficients](#).

- If $\alpha + \beta i$ is not a root of the characteristic equation, then the power $k = 0$,
- If $\alpha + \beta i$ is a simple root, then $k = 1$,
- If $\alpha + \beta i$ is a double root, then $k = 2$.

1175. Differential Equations with y Missing

$$y'' = f(x, y').$$

Set $u = y'$. Then the new equation satisfied by v is

$$u' = f(x, u),$$

which is a first order differential equation.

1176. Differential Equations with x Missing

$$y'' = f(y, y').$$

Set $u = y'$. Since

$$y'' = \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} = u \frac{du}{dy},$$

we have

$$u \frac{du}{dy} = f(y, u),$$

which is a first order differential equation.

1177. Free Undamped Vibrations

The motion of a Mass on a Spring is described by the equation

$$m\ddot{y} + ky = 0,$$

where

m is the mass of the object,

k is the stiffness of the spring,

y is displacement of the mass from equilibrium.

The general solution is

$$y = A \cos(\omega_0 t - \delta),$$

where

A is the amplitude of the displacement,

ω_0 is the fundamental frequency, the period is $T = \frac{2\pi}{\omega_0}$,

δ is phase angle of the displacement.

This is an example of simple harmonic motion.

1178. Free Damped Vibrations

$m\ddot{y} + \gamma\dot{y} + ky = 0$, where

γ is the damping coefficient.

There are 3 cases for the general solution:

Case 1. $\gamma^2 > 4km$ (overdamped)

$$y(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t},$$

where

$$\lambda_1 = \frac{-\gamma - \sqrt{\gamma^2 - 4km}}{2m}, \quad \lambda_2 = \frac{-\gamma + \sqrt{\gamma^2 - 4km}}{2m}.$$

Case 2. $\gamma^2 = 4km$ (critically damped)

$$y(t) = (A + Bt)e^{\lambda t},$$

where

$$\lambda = -\frac{\gamma}{2m}.$$

Case 3. $\gamma^2 < 4km$ (underdamped)

$$y(t) = e^{-\frac{\gamma}{2m}t} A \cos(\omega t - \delta), \text{ where}$$

$$\omega = \sqrt{4km - \gamma^2}.$$

1179. Simple Pendulum

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0,$$

where θ is the angular displacement, L is the pendulum length, g is the acceleration of gravity.

The general solution for small angles θ is

$$\theta(t) = \theta_{\max} \sin \sqrt{\frac{g}{L}}t, \text{ the period is } T = 2\pi \sqrt{\frac{L}{g}}.$$

1180. RLC Circuit

$$L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \frac{1}{C}I = V'(t) = \omega E_0 \cos(\omega t),$$

where I is the current in an RLC circuit with an ac voltage source $V(t) = E_0 \sin(\omega t)$.

The general solution is

$$I(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + A \sin(\omega t - \varphi),$$

where

$$r_{1,2} = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L},$$

$$A = \frac{\omega E_0}{\sqrt{\left(L\omega^2 - \frac{1}{C}\right)^2 + R^2\omega^2}},$$

$$\varphi = \arctan\left(\frac{L\omega}{R} - \frac{1}{RC\omega}\right),$$

C_1, C_2 are constants depending on initial conditions.