9.10 Double Integral

Functions of two variables: $f(x,y)$, $f(u,v)$, ...

Double integrals: $\iint\limits_R f(x,y)dxdy$, $\iint\limits_R g(x,y)dxdy$, ...

Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(u_i, v_j)\Delta x_i \Delta y_j$

Small changes: Δx_i, Δy_j

Regions of integration: R, S

Polar coordinates: r, θ

Area: A

Surface area: S

Volume of a solid: V

Mass of a lamina: m

Density: $\rho(x,y)$

First moments: M_x, M_y

Moments of inertia: I_x, I_y, I_0

Charge of a plate: Q

Charge density: $\sigma(x,y)$

Coordinates of center of mass: $\bar{x}$, $\bar{y}$

Average of a function: μ

1078. Definition of Double Integral

The double integral over a rectangle $[a, b]\times[c, d]$ is defined to be

$$\iint\limits_{[a,\,b]\times[c,\,d]} f(x,y)dA = \lim_{\substack{\max \Delta x_i \to 0 \\ \max \Delta y_j \to 0}} \sum_{i=1}^{m}\sum_{j=1}^{n} f(u_i, v_j)\Delta x_i \Delta y_j \ ,$$

where (u_i, v_j) is some point in the rectangle $(x_{i-1}, x_i)\times(y_{j-1}, y_j)$, and $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$.

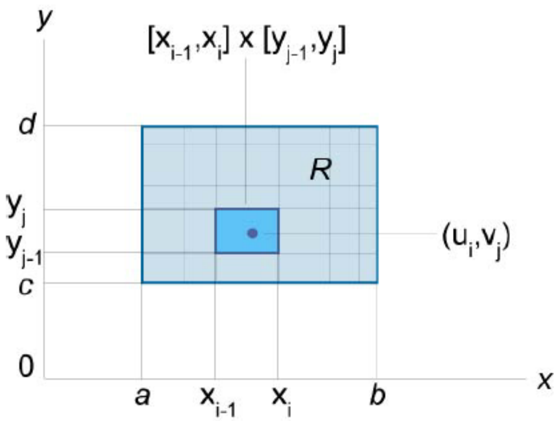

Figure 189.

The double integral over a general region R is

$$\iint_R f(x,y)\,dA = \iint_{[a,\,b]\times[c,\,d]} g(x,y)\,dA,$$

where rectangle $[a,\ b]\times[c,\ d]$ contains R,

$g(x,y) = f(x,y)$ if $f(x,y)$ is in R and $g(x,y) = 0$ otherwise.

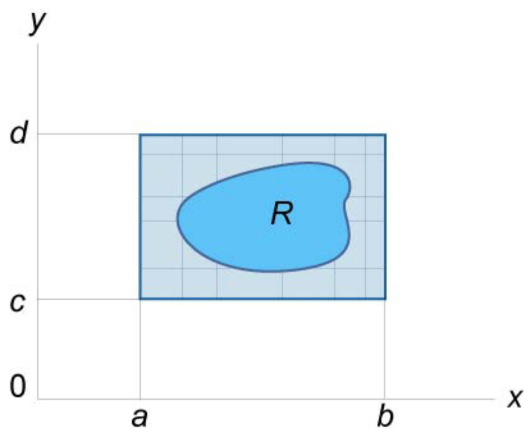

Figure 190.

1079. $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

1080. $\iint_R [f(x,y)-g(x,y)]\,dA = \iint_R f(x,y)\,dA - \iint_R g(x,y)\,dA$

1081. $\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA$,

where k is a constant.

1082. If $f(x,y) \le g(x,y)$ on R, then $\iint_R f(x,y)\,dA \le \iint_R g(x,y)\,dA$.

1083. If $f(x,y) \ge 0$ on R and $S \subset R$, then

$$\iint_S f(x,y)\,dA \le \iint_R f(x,y)\,dA .$$

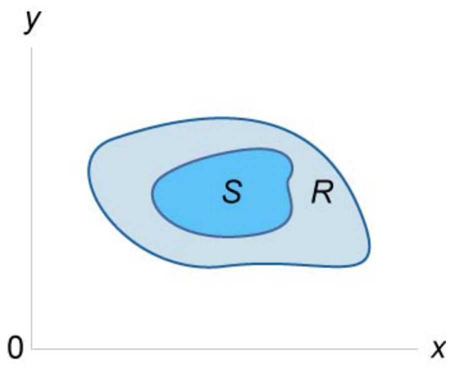

Figure 191.

1084. If $f(x,y) \ge 0$ on R and R and S are non-overlapping regions, then $\iint_{R\cup S} f(x,y)\,dA = \iint_R f(x,y)\,dA + \iint_S f(x,y)\,dA$.
Here $R \cup S$ is the union of the regions R and S.

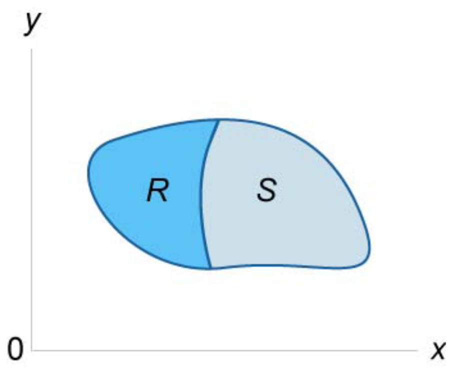

Figure 192.

1085. **Iterated Integrals and Fubini's Theorem**

$$\iint_R f(x,y)dA = \int_a^b \int_{p(x)}^{q(x)} f(x,y)dydx$$

for a region of type I,

$R = \{(x,y) \mid a \le x \le b, p(x) \le y \le q(x)\}$.

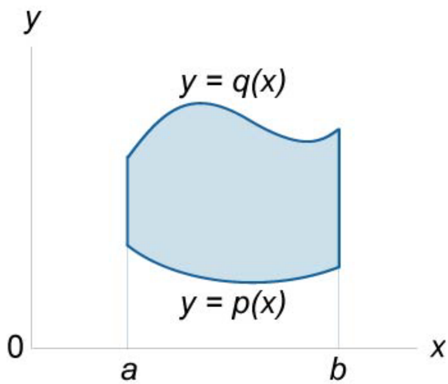

Figure 193.

$$\iint_R f(x,y)dA = \int_c^d \int_{u(y)}^{v(y)} f(x,y)dxdy$$

for a region of type II,

$R = \{(x,y) \mid u(y) \le x \le v(y), c \le y \le d\}$.

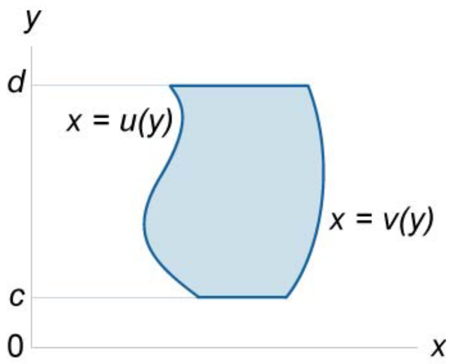

Figure 194.

1086. Double Integrals over Rectangular Regions

If R is the rectangular region $[a,b]\times[c,d]$, then

$$\iint\limits_R f(x,y)dxdy = \int_a^b\left(\int_c^d f(x,y)dy\right)dx = \int_c^d\left(\int_a^b f(x,y)dx\right)dy\,.$$

In the special case where the integrand $f(x,y)$ can be written as $g(x)h(y)$ we have

$$\iint\limits_R f(x,y)dxdy = \iint\limits_R g(x)h(y)dxdy = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

1087. Change of Variables

$$\iint\limits_R f(x,y)dxdy = \iint\limits_S f[x(u,v),y(u,v)]\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv\,,$$

where $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} \neq 0$ is the jacobian of the transformations $(x,y)\to(u,v)$, and S is the pullback of R which can be computed by $x = x(u,v)$, $y = y(u,v)$ into the definition of R.

1088. Polar Coordinates
$x = r\cos\theta$, $y = r\sin\theta$.

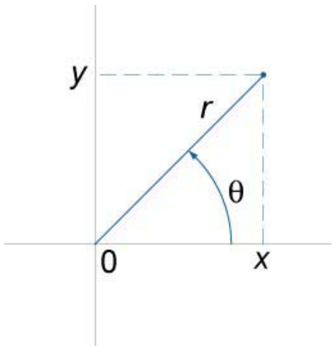

Figure 195.

1089. Double Integrals in Polar Coordinates

The Differential dxdy for Polar Coordinates is

$$dxdy = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| drd\theta = rdrd\theta .$$

Let the region R is determined as follows:
$0 \le g(\theta) \le r \le h(\theta)$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$.
Then

$$\iint\limits_R f(x,y)dxdy = \int\limits_{\alpha}^{\beta}\int\limits_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta) rdrd\theta .$$

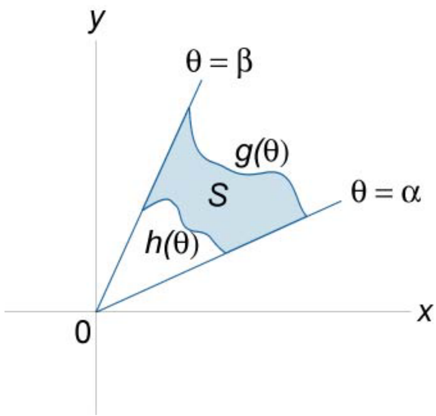

Figure 196.

If the region R is the polar rectangle given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $\beta - \alpha \le 2\pi$, then

$$\iint_R f(x, y)\,dxdy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta .$$

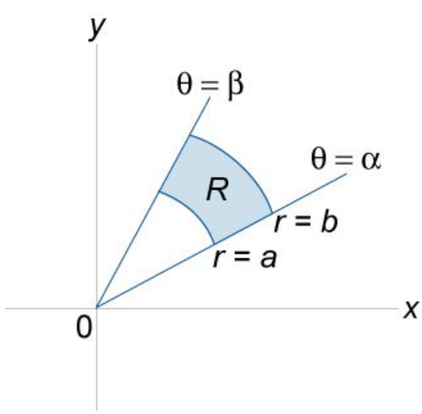

Figure 197.

1090. Area of a Region

$$A = \int_a^b \int_{g(x)}^{f(x)} dydx \quad \text{(for a type I region)}.$$

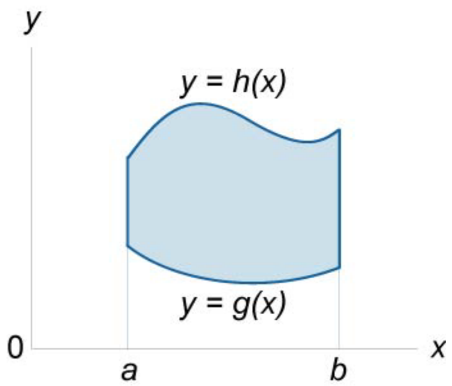

Figure 198.

$$A = \int_c^d \int_{p(y)}^{q(y)} dxdy \quad \text{(for a type II region)}.$$

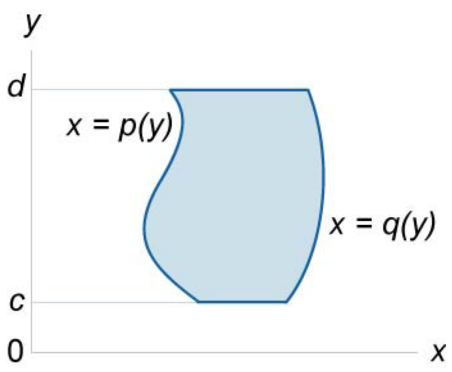

Figure 199.

1091. Volume of a Solid

$$V = \iint_R f(x, y)\,dA.$$

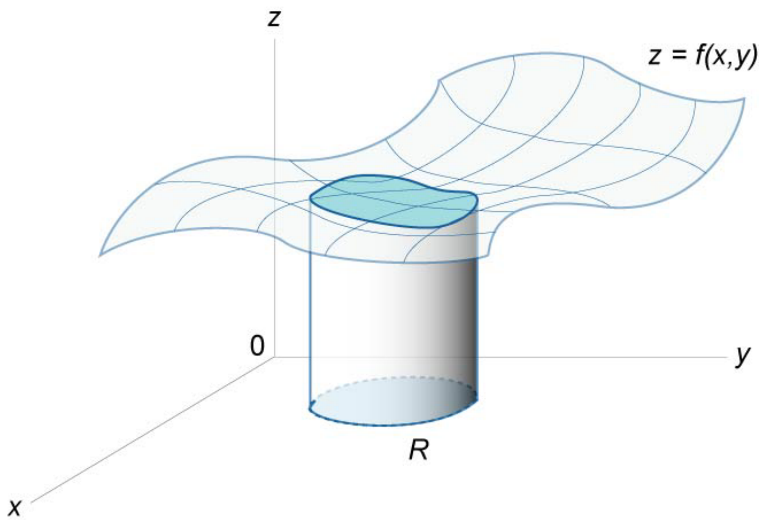

Figure 200.

If R is a type I region bounded by $x = a$, $x = b$, $y = h(x)$, $y = g(x)$, then

$$V = \iint_R f(x, y)\,dA = \int_a^b \int_{h(x)}^{g(x)} f(x, y)\,dy\,dx.$$

If R is a type II region bounded by $y = c$, $y = d$, $x = q(y)$, $x = p(y)$, then

$$V = \iint_R f(x, y)\,dA = \int_c^d \int_{p(y)}^{q(y)} f(x, y)\,dx\,dy.$$

If $f(x,y) \geq g(x,y)$ over a region R, then the volume of the solid between $z_1 = f(x,y)$ and $z_2 = g(x,y)$ over R is given by

$$V = \iint_R [f(x,y) - g(x,y)] dA.$$

1092. Area and Volume in Polar Coordinates

If S is a region in the xy-plane bounded by $\theta = \alpha$, $\theta = \beta$, $r = h(\theta)$, $r = g(\theta)$,

then

$$A = \iint_S dA = \int_\alpha^\beta \int_{h(\theta)}^{g(\theta)} r dr d\theta,$$

$$V = \iint_S f(r,\theta) r dr d\theta.$$

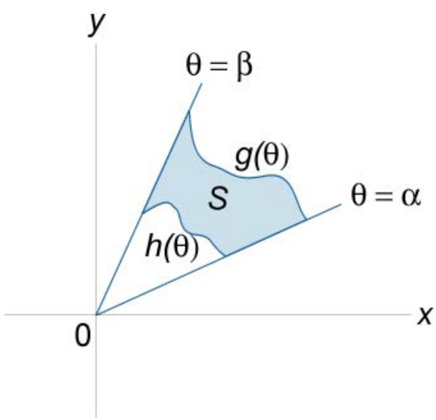

Figure 201.

1093. Surface Area

$$S = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dxdy$$

1094. Mass of a Lamina

$$m = \iint_R \rho(x,y)\,dA\,,$$

where the lamina occupies a region R and its density at a point (x,y) is $\rho(x,y)$.

1095. Moments

The moment of the lamina about the x-axis is given by formula

$$M_x = \iint_R y\rho(x,y)\,dA\,.$$

The moment of the lamina about the y-axis is

$$M_y = \iint_R x\rho(x,y)\,dA\,.$$

The moment of inertia about the x-axis is

$$I_x = \iint_R y^2\rho(x,y)\,dA\,.$$

The moment of inertia about the y-axis is

$$I_y = \iint_R x^2\rho(x,y)\,dA\,.$$

The polar moment of inertia is

$$I_0 = \iint_R \left(x^2 + y^2\right)\rho(x,y)\,dA\,.$$

1096. Center of Mass

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_R x\rho(x,y)\,dA = \frac{\iint_R x\rho(x,y)\,dA}{\iint_R \rho(x,y)\,dA}\,,$$

$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_R y\rho(x,y)dA = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA}.$$

1097. Charge of a Plate

$$Q = \iint_R \sigma(x,y)dA,$$

where electrical charge is distributed over a region R and its charge density at a point (x,y) is $\sigma(x,y)$.

1098. Average of a Function

$$\mu = \frac{1}{S}\iint_R f(x,y)dA,$$

where $S = \iint_R dA$.