9.12 Line Integral

Scalar functions: $F(x,y,z)$, $F(x,y)$, $f(x)$
Scalar potential: $u(x,y,z)$
Curves: C, C_1, C_2
Limits of integrations: a, b, α, β
Parameters: t, s
Polar coordinates: r, θ
Vector field: $\vec{F}(P,Q,R)$
Position vector: $\vec{r}(s)$
Unit vectors: $\vec{i}$, $\vec{j}$, $\vec{k}$, $\vec{\tau}$
Area of region: S
Length of a curve: L
Mass of a wire: m
Density: $\rho(x,y,z)$, $\rho(x,y)$
Coordinates of center of mass: $\bar{x}$, $\bar{y}$, $\bar{z}$
First moments: M_{xy}, M_{yz}, M_{xz}
Moments of inertia: I_x, I_y, I_z
Volume of a solid: V
Work: W
Magnetic field: $\vec{B}$
Current: I
Electromotive force: ε
Magnetic flux: ψ

1117. Line Integral of a Scalar Function

Let a curve C be given by the vector function $\vec{r} = \vec{r}(s)$, $0 \le s \le S$, and a scalar function F is defined over the curve C. Then

$$\int_0^S F(\vec{r}(s))ds = \int_C F(x,y,z)ds = \int_C F ds,$$

where ds is the arc length differential.

1118. $$\int_{C_1 \cup C_2} F\,ds = \int_{C_1} F\,ds + \int_{C_2} F\,ds$$

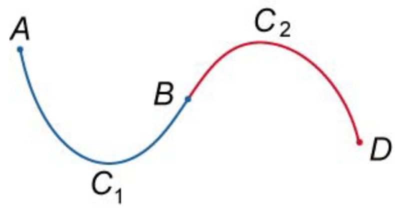

Figure 203.

1119. If the smooth curve C is parametrized by $\vec{r} = \vec{r}(t)$, $\alpha \le t \le \beta$, then

$$\int_C F(x,y,z)ds = \int_\alpha^\beta F(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt.$$

1120. If C is a smooth curve in the xy-plane given by the equation $y = f(x)$, $a \le x \le b$, then

$$\int_C F(x,y)ds = \int_a^b F(x,f(x))\sqrt{1 + (f'(x))^2}\,dx.$$

1121. Line Integral of Scalar Function in Polar Coordinates

$$\int_C F(x,y)ds = \int_\alpha^\beta F(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta,$$

where the curve C is defined by the polar function $r(\theta)$.

1122. Line Integral of Vector Field
Let a curve C be defined by the vector function $\vec{r} = \vec{r}(s)$, $0 \le s \le S$. Then

$$\frac{d\vec{r}}{ds} = \vec{\tau} = (\cos\alpha, \cos\beta, \cos\gamma)$$

is the unit vector of the tangent line to this curve.

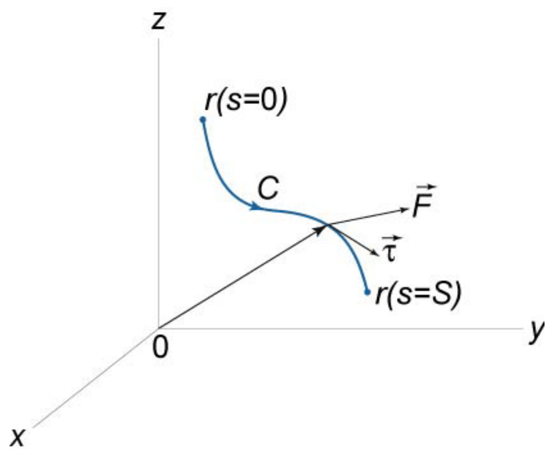

Figure 204.

Let a vector field $\vec{F}(P,Q,R)$ is defined over the curve C. Then the line integral of the vector field $\vec{F}$ along the curve C is

$$\int_C Pdx + Qdy + Rdz = \int_0^S (P\cos\alpha + Q\cos\beta + R\cos\gamma)ds.$$

1123. Properties of Line Integrals of Vector Fields

$$\int_{-C}\left(\vec{F}\cdot d\vec{r}\right)=-\int_{C}\left(\vec{F}\cdot d\vec{r}\right),$$

where -C denote the curve with the opposite orientation.

$$\int_{C}\left(\vec{F}\cdot d\vec{r}\right)=\int_{C_1\cup C_2}\left(\vec{F}\cdot d\vec{r}\right)=\int_{C_1}\left(\vec{F}\cdot d\vec{r}\right)+\int_{C_2}\left(\vec{F}\cdot d\vec{r}\right),$$

where C is the union of the curves C_1 and C_2.

1124. If the curve C is parameterized by $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$, $\alpha\le t\le\beta$, then

$$\int_{C} Pdx+Qdy+Rdz=$$

$$=\int_{\alpha}^{\beta}\left(P(x(t),y(t),z(t))\frac{dx}{dt}+Q(x(t),y(t),z(t))\frac{dy}{dt}+R(x(t),y(t),z(t))\frac{dz}{dt}\right)dt$$

1125. If C lies in the xy-plane and given by the equation $y=f(x)$, then

$$\int_{C} Pdx+Qdy=\int_{a}^{b}\left(P(x,f(x))+Q(x,f(x))\frac{df}{dx}\right)dx.$$

1126. Green's Theorem

$$\iint_{R}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\oint_{C} Pdx+Qdy,$$

where $\vec{F}=P(x,y)\vec{i}+Q(x,y)\vec{j}$ is a continuous vector function with continuous first partial derivatives $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ in a some domain R, which is bounded by a closed, piecewise smooth curve C.

1127. Area of a Region R Bounded by the Curve C

$$S = \iint_R dxdy = \frac{1}{2}\oint_C xdy - ydx$$

1128. Path Independence of Line Integrals

The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is said to be path independent, if and only if P, Q, and R are continuous in a domain D, and if there exists some scalar function $u = u(x,y,z)$ (a scalar potential) in D such that

$$\vec{F} = \text{grad}\, u, \text{ or } \frac{\partial u}{\partial x} = P, \frac{\partial u}{\partial y} = Q, \frac{\partial u}{\partial z} = R.$$

Then

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C Pdx + Qdy + Rdz = u(B) - u(A).$$

1129. Test for a Conservative Field

A vector field of the form $\vec{F} = \text{grad}\, u$ is called a conservative field. The line integral of a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is path independent if and only if

$$\text{curl}\, \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}.$$

If the line integral is taken in xy-plane so that

$$\int_C Pdx + Qdy = u(B) - u(A),$$

then the test for determining if a vector field is conservative can be written in the form

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

1130. Length of a Curve

$$L=\int_C ds=\int_\alpha^\beta\left|\frac{d\vec{r}}{dt}(t)\right|dt=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt,$$

where C ia a piecewise smooth curve described by the position vector $\vec{r}(t)$, $\alpha\le t\le\beta$.

If the curve C is two-dimensional, then

$$L=\int_C ds=\int_\alpha^\beta\left|\frac{d\vec{r}}{dt}(t)\right|dt=\int_\alpha^\beta\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt.$$

If the curve C is the graph of a function $y=f(x)$ in the xy-plane $(a\le x\le b)$, then

$$L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

1131. Length of a Curve in Polar Coordinates

$$L=\int_\alpha^\beta\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta,$$

where the curve C is given by the equation $r=r(\theta)$, $\alpha\le\theta\le\beta$ in polar coordinates.

1132. Mass of a Wire

$$m=\int_C\rho(x,y,z)ds,$$

where $\rho(x,y,z)$ is the mass per unit length of the wire.

If C is a curve parametrized by the vector function $\vec{r}(t)=\langle x(t),y(t),z(t)\rangle$, then the mass can be computed by the formula

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

If C is a curve in xy-plane, then the mass of the wire is given by

$$m = \int_C \rho(x, y)\, ds,$$

or

$$m = \int_{\alpha}^{\beta} \rho(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \text{ (in parametric form).}$$

1133. Center of Mass of a Wire

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \int_C x\rho(x, y, z)\, ds,$$

$$M_{xz} = \int_C y\rho(x, y, z)\, ds,$$

$$M_{xy} = \int_C z\rho(x, y, z)\, ds.$$

1134. Moments of Inertia

The moments of inertia about the x-axis, y-axis, and z-axis are given by the formulas

$$I_x = \int_C (y^2 + z^2)\rho(x, y, z)\, ds,$$

$$I_y = \int_C (x^2 + z^2)\rho(x, y, z)\, ds,$$

$$I_z = \int_C (x^2 + y^2)\rho(x, y, z)\, ds.$$

1135. Area of a Region Bounded by a Closed Curve

$$S = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx\,.$$

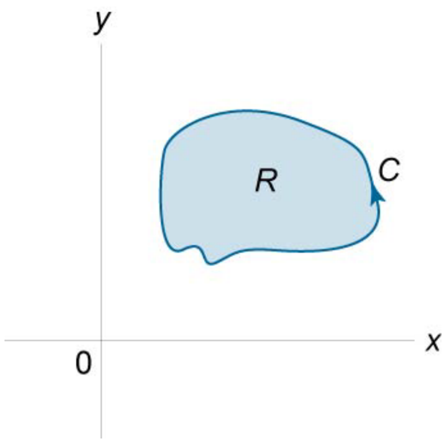

Figure 205.

If the closed curve C is given in parametric form $\vec{r}(t) = \langle x(t), y(t) \rangle$, then the area can be calculated by the formula

$$S = \int_\alpha^\beta x(t)\frac{dy}{dt}dt = -\int_\alpha^\beta y(t)\frac{dx}{dt}dt = \frac{1}{2}\int_\alpha^\beta \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt} \right)dt\,.$$

1136. Volume of a Solid Formed by Rotating a Closed Curve about the x-axis

$$V = -\pi\oint_C y^2 dx = -2\pi\oint_C xy\,dy = -\frac{\pi}{2}\oint_C 2xy\,dy + y^2 dx$$

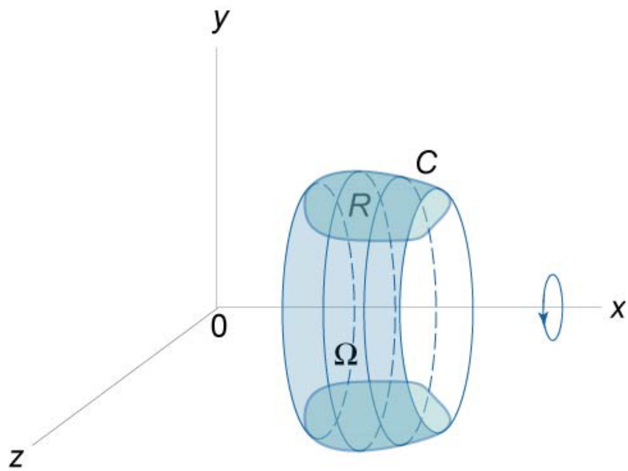

Figure 206.

1137. Work

Work done by a force $\vec{F}$ on an object moving along a curve C is given by the line integral

$$W = \int_C \vec{F} \cdot d\vec{r},$$

where $\vec{F}$ is the vector force field acting on the object, $d\vec{r}$ is the unit tangent vector.

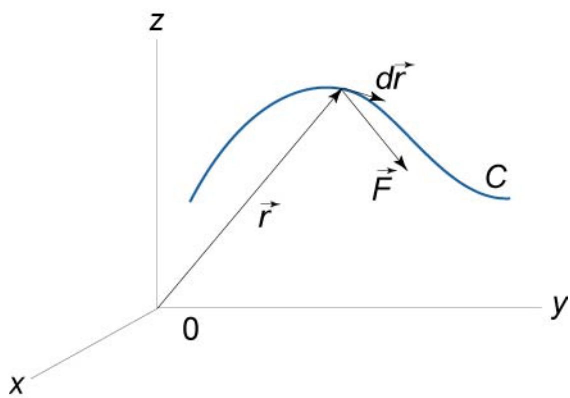

Figure 207.

If the object is moved along a curve C in the xy-plane, then

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C P dx + Q dy ,$$

If a path C is specified by a parameter t (t often means time), the formula for calculating work becomes

$$W = \int_\alpha^\beta \left[P(x(t), y(t), z(t)) \frac{dx}{dt} + Q(x(t), y(t), z(t)) \frac{dy}{dt} + R(x(t), y(t), z(t)) \frac{dz}{dt} \right] dt,$$

where t goes from α to β.

If a vector field $\vec{F}$ is conservative and $u(x, y, z)$ is a scalar potential of the field, then the work on an object moving from A to B can be found by the formula

$$W = u(B) - u(A).$$

1138. Ampere's Law

$$\oint_C \vec{B} \cdot d\vec{r} = \mu_0 I .$$

The line integral of a magnetic field $\vec{B}$ around a closed path C is equal to the total current I flowing through the area bounded by the path.

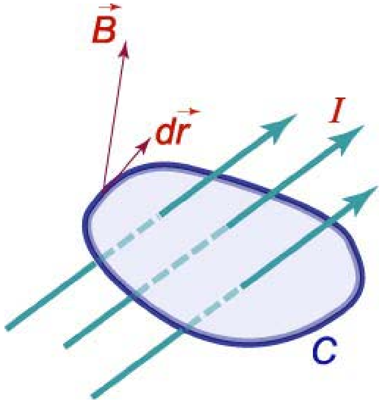

Figure 208.

1139. Faraday's Law

$$\varepsilon = \oint_C \vec{E} \cdot d\vec{r} = -\frac{d\psi}{dt}$$

The electromotive force (emf) ε induced around a closed loop C is equal to the rate of the change of magnetic flux ψ passing through the loop.

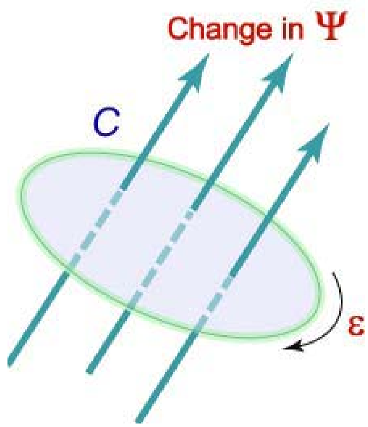

Figure 209.