

9.13 Surface Integral

Scalar functions: $f(x, y, z)$, $z(x, y)$

Position vectors: $\vec{r}(u, v)$, $\vec{r}(x, y, z)$

Unit vectors: \vec{i} , \vec{j} , \vec{k}

Surface: S

Vector field: $\vec{F}(P, Q, R)$

Divergence of a vector field: $\text{div } \vec{F} = \nabla \cdot \vec{F}$

Curl of a vector field: $\text{curl } \vec{F} = \nabla \times \vec{F}$

Vector element of a surface: $d\vec{S}$

Normal to surface: \vec{n}

Surface area: A

Mass of a surface: m

Density: $\mu(x, y, z)$

Coordinates of center of mass: \bar{x} , \bar{y} , \bar{z}

First moments: M_{xy} , M_{yz} , M_{xz}

Moments of inertia: I_{xy} , I_{yz} , I_{xz} , I_x , I_y , I_z

Volume of a solid: V

Force: \vec{F}

Gravitational constant: G

Fluid velocity: $\vec{v}(\vec{r})$

Fluid density: ρ

Pressure: $p(\vec{r})$

Mass flux, electric flux: Φ

Surface charge: Q

Charge density: $\sigma(x, y)$

Magnitude of the electric field: \vec{E}

1140. Surface Integral of a Scalar Function

Let a surface S be given by the position vector

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k},$$

where (u, v) ranges over some domain $D(u, v)$ of the uv -plane.

The surface integral of a scalar function $f(x, y, z)$ over the surface S is defined as

$$\iint_S f(x, y, z) dS = \iint_{D(u, v)} f(x(u, v), y(u, v), z(u, v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv,$$

where the partial derivatives $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are given by

$$\frac{\partial \vec{r}}{\partial u} = \frac{\partial x}{\partial u}(u, v)\vec{i} + \frac{\partial y}{\partial u}(u, v)\vec{j} + \frac{\partial z}{\partial u}(u, v)\vec{k},$$

$$\frac{\partial \vec{r}}{\partial v} = \frac{\partial x}{\partial v}(u, v)\vec{i} + \frac{\partial y}{\partial v}(u, v)\vec{j} + \frac{\partial z}{\partial v}(u, v)\vec{k}$$

and $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ is the cross product.

- 1141.** If the surface S is given by the equation $z = z(x, y)$ where $z(x, y)$ is a differentiable function in the domain $D(x, y)$, then

$$\iint_S f(x, y, z) dS = \iint_{D(x, y)} f(x, y, z(x, y)) \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy.$$

- 1142.** Surface Integral of the Vector Field \vec{F} over the Surface S

- If S is oriented **outward**, then

$$\begin{aligned} \iint_S \vec{F}(x, y, z) \cdot d\vec{S} &= \iint_S \vec{F}(x, y, z) \cdot \vec{n} dS \\ &= \iint_{D(u, v)} \vec{F}(x(u, v), y(u, v), z(u, v)) \cdot \left[\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right] du dv. \end{aligned}$$

- If S is oriented **inward**, then

$$\begin{aligned} \iint_S \vec{F}(x, y, z) \cdot d\vec{S} &= \iint_S \vec{F}(x, y, z) \cdot \vec{n} dS \\ &= \iint_{D(u, v)} \vec{F}(x(u, v), y(u, v), z(u, v)) \cdot \left[\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u} \right] du dv. \end{aligned}$$

$d\vec{S} = \vec{n} dS$ is called the **vector element of the surface**. Dot means the scalar product of the appropriate vectors.

The partial derivatives $\frac{\partial \vec{r}}{\partial u}$ and $\frac{\partial \vec{r}}{\partial v}$ are given by

$$\begin{aligned} \frac{\partial \vec{r}}{\partial u} &= \frac{\partial x}{\partial u}(u, v) \cdot \vec{i} + \frac{\partial y}{\partial u}(u, v) \cdot \vec{j} + \frac{\partial z}{\partial u}(u, v) \cdot \vec{k}, \\ \frac{\partial \vec{r}}{\partial v} &= \frac{\partial x}{\partial v}(u, v) \cdot \vec{i} + \frac{\partial y}{\partial v}(u, v) \cdot \vec{j} + \frac{\partial z}{\partial v}(u, v) \cdot \vec{k}. \end{aligned}$$

1143. If the surface S is given by the equation $z = z(x, y)$, where $z(x, y)$ is a differentiable function in the domain $D(x, y)$, then

- If S is oriented **upward**, i.e. the k -th component of the normal vector is positive, then

$$\begin{aligned} \iint_S \vec{F}(x, y, z) \cdot d\vec{S} &= \iint_S \vec{F}(x, y, z) \cdot \vec{n} dS \\ &= \iint_{D(x, y)} \vec{F}(x, y, z) \cdot \left(-\frac{\partial z}{\partial x} \vec{i} - \frac{\partial z}{\partial y} \vec{j} + \vec{k} \right) dx dy, \end{aligned}$$

- If S is oriented **downward**, i.e. the k -th component of the normal vector is negative, then

$$\begin{aligned} \iint_S \vec{F}(x, y, z) \cdot d\vec{S} &= \iint_S \vec{F}(x, y, z) \cdot \vec{n} dS \\ &= \iint_{D(x, y)} \vec{F}(x, y, z) \cdot \left(\frac{\partial z}{\partial x} \vec{i} + \frac{\partial z}{\partial y} \vec{j} - \vec{k} \right) dx dy. \end{aligned}$$

1144.
$$\iint_S (\vec{F} \cdot \vec{n}) dS = \iint_S P dy dz + Q dz dx + R dx dy$$

$$= \iint_S (P \cos \alpha + Q \cos \beta + R \cos \gamma) dS,$$

where $P(x, y, z)$, $Q(x, y, z)$, $R(x, y, z)$ are the components of the vector field \vec{F} .

$\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the angles between the outer unit normal vector \vec{n} and the x -axis, y -axis, and z -axis, respectively.

1145. If the surface S is given in parametric form by the vector $\vec{r}(x(u,v), y(u,v), z(u,v))$, then the latter formula can be written as

$$\iint_S (\vec{F} \cdot \vec{n}) dS = \iint_S P dydz + Q dzdx + R dx dy = \iint_{D(u,v)} \begin{vmatrix} P & Q & R \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} du dv,$$

where (u,v) ranges over some domain $D(u,v)$ of the uv -plane.

1146. Divergence Theorem

$$\oiint_S \vec{F} \cdot d\vec{S} = \iiint_G (\nabla \cdot \vec{F}) dV,$$

where

$$\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

is a vector field whose components P , Q , and R have continuous partial derivatives,

$$\nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

is the **divergence** of \vec{F} , also denoted $\text{div} \vec{F}$. The symbol \oiint indicates that the surface integral is taken over a closed surface.

1147. Divergence Theorem in Coordinate Form

$$\oiint_S P dydz + Q dx dz + R dx dy = \iiint_G \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz.$$

1148. Stoke's Theorem

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S},$$

where

$$\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$$

is a vector field whose components P , Q , and R have continuous partial derivatives,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

is the **curl** of \vec{F} , also denoted $\text{curl } \vec{F}$.

The symbol \oint indicates that the line integral is taken over a closed curve.

1149. Stoke's Theorem in Coordinate Form

$$\begin{aligned} \oint_C P dx + Q dy + R dz \\ = \iint_S \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy \end{aligned}$$

1150. Surface Area

$$A = \iint_S dS$$

1151. If the surface S is parameterized by the vector

$$\vec{r}(u, v) = x(u, v)\vec{i} + y(u, v)\vec{j} + z(u, v)\vec{k},$$

then the surface area is

$$A = \iint_{D(u,v)} \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv,$$

where $D(u, v)$ is the domain where the surface $\vec{r}(u, v)$ is defined.

1152. If S is given explicitly by the function $z(x, y)$, then the surface area is

$$A = \iint_{D(x,y)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} dx dy,$$

where $D(x, y)$ is the projection of the surface S onto the xy -plane.

1153. Mass of a Surface

$$m = \iint_S \mu(x, y, z) dS,$$

where $\mu(x, y, z)$ is the mass per unit area (density function).

1154. Center of Mass of a Shell

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m},$$

where

$$M_{yz} = \iint_S x\mu(x, y, z) dS,$$

$$M_{xz} = \iint_S y\mu(x, y, z) dS,$$

$$M_{xy} = \iint_S z\mu(x, y, z) dS$$

are the first moments about the coordinate planes $x = 0$, $y = 0$, $z = 0$, respectively. $\mu(x, y, z)$ is the density function.

1155. Moments of Inertia about the xy -plane (or $z = 0$), yz -plane ($x = 0$), and xz -plane ($y = 0$)

$$I_{xy} = \iint_S z^2 \mu(x, y, z) dS,$$

$$I_{yz} = \iint_S x^2 \mu(x, y, z) dS,$$

$$I_{xz} = \iint_S y^2 \mu(x, y, z) dS.$$

1156. Moments of Inertia about the x-axis, y-axis, and z-axis

$$I_x = \iint_S (y^2 + z^2) \mu(x, y, z) dS,$$

$$I_y = \iint_S (x^2 + z^2) \mu(x, y, z) dS,$$

$$I_z = \iint_S (x^2 + y^2) \mu(x, y, z) dS.$$

1157. Volume of a Solid Bounded by a Closed Surface

$$V = \frac{1}{3} \left| \oiint_S x dy dz + y dx dz + z dx dy \right|$$

1158. Gravitational Force

$$\vec{F} = Gm \iint_S \mu(x, y, z) \frac{\vec{r}}{r^3} dS,$$

where m is a mass at a point $\langle x_0, y_0, z_0 \rangle$ outside the surface,

$$\vec{r} = \langle x - x_0, y - y_0, z - z_0 \rangle,$$

$\mu(x, y, z)$ is the density function,

and G is gravitational constant.

1159. Pressure Force

$$\vec{F} = \iint_S p(\vec{r}) d\vec{S},$$

where the pressure $p(\vec{r})$ acts on the surface S given by the position vector \vec{r} .

1160. Fluid Flux (across the surface S)

$$\Phi = \oiint_S \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{v}(\vec{r})$ is the fluid velocity.

1161. Mass Flux (across the surface S)

$$\Phi = \iint_S \rho \vec{v}(\vec{r}) \cdot d\vec{S},$$

where $\vec{F} = \rho \vec{v}$ is the vector field, ρ is the fluid density.

1162. Surface Charge

$$Q = \iint_S \sigma(x, y) dS,$$

where $\sigma(x, y)$ is the surface charge density.

1163. Gauss' Law

The **electric flux** through any closed surface is proportional to the charge Q enclosed by the surface

$$\Phi = \iint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0},$$

where

Φ is the electric flux,

\vec{E} is the magnitude of the electric field strength,

$\epsilon_0 = 8,85 \times 10^{-12} \frac{F}{m}$ is permittivity of free space.

Acceleration of gravity: g

Current: I

Resistance: R

Inductance: L

Capacitance: C