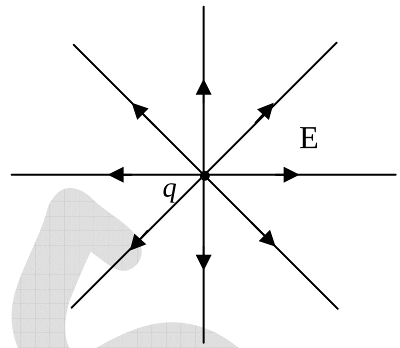
(b) Gauss's Law

Field Lines and Electric Flux

Consider that a point charge q is situated at the origin:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$



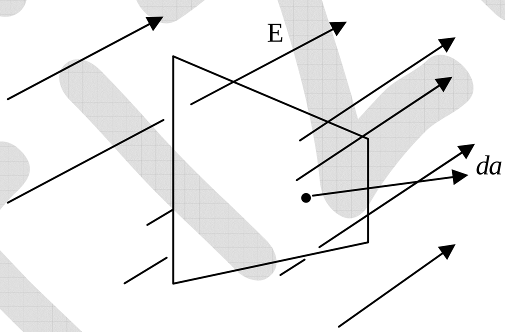
This field is represented by the **field line** as shown in figure below. The magnitude of the field is indicated by the density of the field lines: it's strong near the center where the field line are close together, and weak farther out, where they are relatively far apart.

The **field strength** (E) is proportional to the number of field lines per unit area (area perpendicular to the lines).

The **flux of** E through a surface S,

$$\phi_E = \int_S \vec{E}.d\vec{a}$$

is a measure of the "number of field lines" passing through S.



For the case of point charge at the origin, the flux of $\vec{E}$ through a sphere of radius r is

$$\oint \vec{E}.d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right).\left(r^2\sin\theta d\theta d\phi\hat{r}\right) = \frac{1}{\varepsilon_0}q.$$

Note that, any surface whatever its shape, would trap the same number of field lines. So the flux through any surface enclosing the charge is $\frac{q}{\varepsilon_0}$.

Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about. According to the principle of superposition, the total field is simply the (vector) sum of all the individual fields: $\vec{E} = \sum_{i=1}^{n} \vec{E}_i$.

The flux through any surface that encloses them all, then, is

$$\oint \vec{E}.d\vec{a} = \sum_{i}^{n}\left(\oint \vec{E}_i.d\vec{a}\right) = \sum_{i}^{n}\left(\frac{1}{\varepsilon_0}q_i\right).$$

A charge outside the surface would contribute nothing to the total flux, since its field lines go in one side and out the other. It follows, then, that for any closed surface,

$$\oint \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc}$$

where Q_{enc} is the total charge enclosed within the surface.

This is **Gauss's law in integral form.**

We can convert Gauss's law in **integral form** to **differential form**, for continuous charge distributions, by applying the divergence theorem:

$$\oint_{surface} \vec{E}.d\vec{a} = \int_{volume} \left(\vec{\nabla}.\vec{E}\right) d\tau .$$

We can write Q_{enc} in terms of the charge density ρ, we have $Q_{enc} = \int_{volume} \rho d\tau$.

So Gauss's law becomes $\int_{volume} \left(\vec{\nabla}.\vec{E}\right) d\tau = \int_{volume} \left(\frac{1}{\varepsilon_0}\rho\right) d\tau$.

Since this holds for any volume, the integrands must be equal:

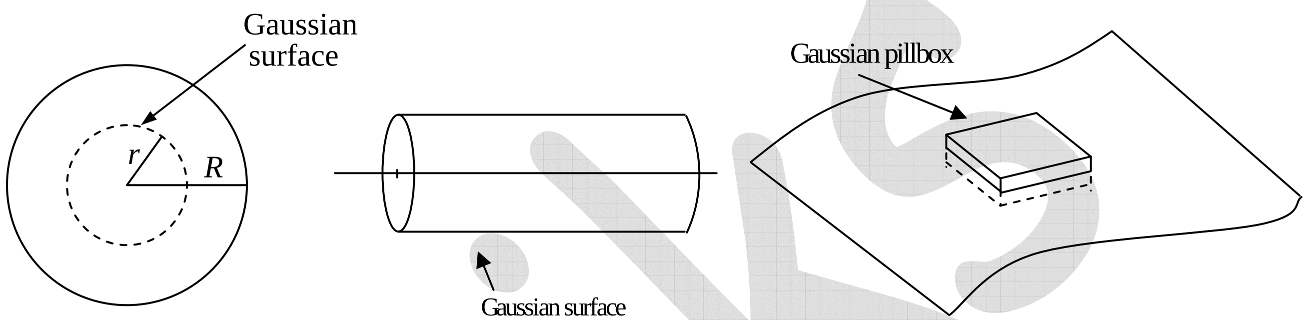
$$\vec{\nabla}.\vec{E} = \frac{1}{\varepsilon_0}\rho .$$

This is **Gauss's law in differential form.**

Applications of Gauss's Law

Gauss's law is always true, but it is not always useful. Gauss's law is useful for only three kinds of symmetry:

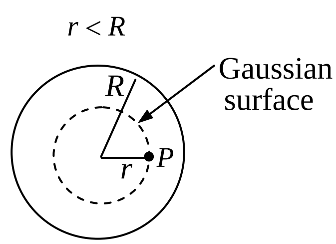
1. **Spherical Symmetry**. Make your Gaussian surface a concentric sphere.
2. **Cylindrical Symmetry**. Make your Gaussian surface a coaxial cylinder.
3. **Plane Symmetry**. Make your Gaussian surface a "pillbox," which extends equally above and below the surface.



Example: Find the electric field inside and outside a spherical shell of radius R, which carries a uniform surface charge density σ.

Solution: Draw a spherical surface of radius $r < R$, which is called as "Gaussian surface".

According to Gauss's Law

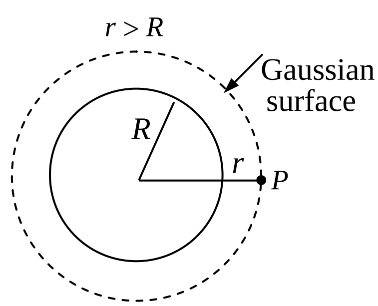


$$\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc} = 0$$

$$\Rightarrow \vec{E} = 0$$

For outside point, draw a spherical surface of radius $r > R$,

According Gauss's Law

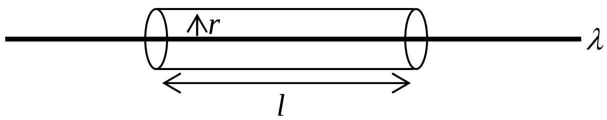


$$\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

$$\Rightarrow \left|\vec{E}\right| \times 4\pi r^2 = \frac{1}{\varepsilon_0}\left(\sigma \times 4\pi R^2\right) \Rightarrow \vec{E} = \frac{\sigma R^2}{\varepsilon_0 r^2}\hat{r}$$

Example: Find the electric field a distance *r* from an infinitely long straight wire, which carries a uniform line charge λ.

Solution: Draw a Gaussian cylinder of length l and radius r.

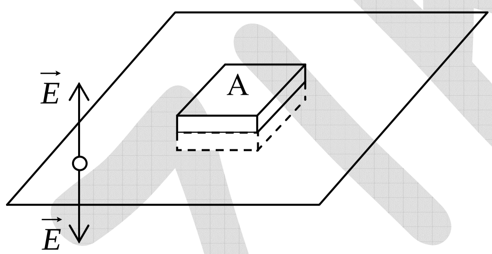


For this surface, Gauss's Law state: $\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc} \Rightarrow E \times 2\pi r l = \frac{1}{\varepsilon_0}\lambda l$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Example: An infinite plane carries a uniform surface charge σ. Find its electric field.

Solution: Draw a "Gaussian pill box", extending equal distances above and below the plane.



Apply Gauss's Law to this surface: $\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

In this case, $Q_{enc} = \sigma A$, where A is the area of the pill box. By symmetry, $\vec{E}$ points away from the plane (upward for the points above, downward for points below).

Thus $\int \vec{E}.d\vec{a} = 2A \times |\vec{E}|$ whereas sides contribute nothing. Thus $2A \times |\vec{E}| = \frac{1}{\varepsilon_0}\sigma A$

$\Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$ where $\hat{n}$ is the unit vector pointing away from the surface