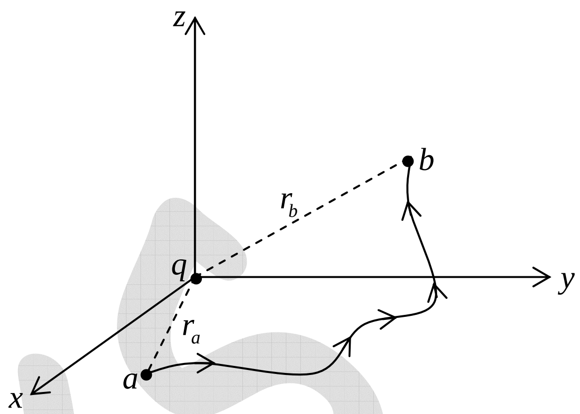
(c) Electric Potential

Curl of Electric field

Consider a point charge at the origin, then electric field at a distance r is given by

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.$$



Now we will calculate the line integral of this field from some point a to some other point b: $\int_a^b \vec{E}.d\vec{l}$.

In spherical coordinates, $d\vec{l} = dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta\,\hat{\phi}$

$$\Rightarrow \vec{E}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr \quad \Rightarrow \int_a^b \vec{E}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}dr,$$

$$\Rightarrow \int_a^b \vec{E}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}q\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$

where r_a is the distance from the origin to point a and r_b is the distance to b.

Then integral around a closed path is zero i.e. $\oint \vec{E}.d\vec{l} = 0 \ (\because r_a = r_b)$

This line integral is independent of path. It depends on two end points.

Applying stokes theorem, we get $\vec{\nabla}\times\vec{E} = 0$.

The electric field is not just any vector but only those vector whose curl is zero.

If we have many charges, the principle of superposition states that the total field is the vector sum of their individual fields:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \ldots\ldots\ldots\ldots$$

So, $$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\vec{E}_1 + \vec{\nabla}\times\vec{E}_2 + \ldots\ldots\ldots\ldots = 0$$

Since $\oint \vec{E}.d\vec{l} = 0$, the line integral is independent of path.

So, we can define a function

$$V(r) = -\int_{\vartheta}^{P} \vec{E}.d\vec{l}$$ where ϑ is some standard reference point.

V then depends only on the point r. It is called the **electric potential.**

Evidently, the potential difference between two points a and b is

$$V(b)-V(a)=-\int_{\vartheta}^{b}\vec{E}.d\vec{l}+\int_{\vartheta}^{a}\vec{E}.d\vec{l}\Rightarrow V(b)-V(a)=-\int_{\vartheta}^{b}\vec{E}.d\vec{l}-\int_{a}^{\vartheta}\vec{E}.d\vec{l}$$

$$\Rightarrow V(b)-V(a)=-\int_{a}^{b}\vec{E}.d\vec{l}.$$

Now, the fundamental theorem for gradients states that

$$V(b)-V(a)=\int_{a}^{b}\left(\vec{\nabla}.V\right).d\vec{l}=-\int_{a}^{b}\vec{E}.d\vec{l}$$

$$\Rightarrow \vec{E}=-\vec{\nabla}V.$$

Potential obeys the superposition principle.

Potential of localized charges

Potential of a point charge q is $V=\frac{1}{4\pi\varepsilon_0}\frac{q}{R}$ where R is the distance from the charge.

The potential of a collection of point charge is $V=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i}$.

For continuous volume charge distribution $V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')}{R}d\tau'$

The potential of line and surface charges are

$$V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(r')}{R}dl' \quad \text{and} \quad V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(r')}{R}da'.$$

Example: Which one of these is an impossible electrostatic field?

(a) $\vec{E}=k[xy\hat{x}+2yz\hat{y}+3xz\hat{z}]$;

(b) $\vec{E}=k[y^2\hat{x}+(2xy+z^2)\hat{y}+2yz\hat{z}]$.

Solution:

(a). $$\vec{\nabla}\times\vec{E}=k\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 2yz & 3zx\end{vmatrix}=k\left[\hat{x}(0-2y)+\hat{y}(0-3z)+\hat{z}(0-x)\right]\neq 0$$

So, $\vec{E}$ is an impossible electrostatic field.

(b). $\vec{\nabla}\times\vec{E}=0\Rightarrow$ so $\vec{E}$ is a possible electrostatic field.

Example: Find the potential inside and outside a spherical shell of radius R and charge q.

Solution: From Gauss's law the field $\vec{E} = \begin{cases} \vec{E}_1 = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}; & \text{outside the sphere } (r > R) \\ \vec{E}_2 = 0; & \text{inside the sphere } (r < R) \end{cases}$

Potential outside $(r > R)$ is: $V(r) = -\int_{\infty}^{r} \vec{E}_1 . d\vec{l} = -\int_{\infty}^{r} \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}.dr\hat{r} = \dfrac{q}{4\pi\varepsilon_0 r}$

Potential inside $(r < R)$ is: $V(r) = -\int_{\infty}^{r} \vec{E}.d\vec{l} = -\int_{\infty}^{R} \vec{E}_1 . d\vec{l} - \int_{R}^{r} \vec{E}_2 . d\vec{l} = \dfrac{q}{4\pi\varepsilon_0 R}$

So potential inside the spherical shell is constant.

Thus $V(r) = \dfrac{q}{4\pi\varepsilon_0 R};\ r \le R$ and $V(r) = \dfrac{q}{4\pi\varepsilon_0 r};\quad r > R$.