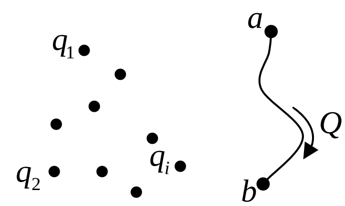
(e) Work and Energy in Electrostatics

The work done in moving a test charge Q in an external field $\vec{E}$, from point a to b is

$$W=\int_a^b \vec{F}\cdot d\vec{l}=-Q\int_a^b \vec{E}\cdot d\vec{l}=Q\left[V(b)-V(a)\right]$$



If $a=\infty$ and $b=r$

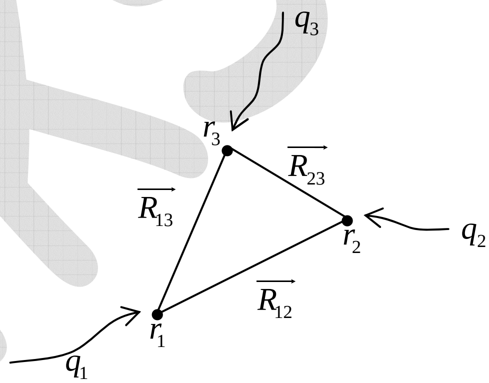
$$\Rightarrow W=Q\left[V(r)-V(\infty)\right]=QV(r) \qquad \text{since } V(\infty)=0$$

In this sense **potential** is potential energy (the work it takes to create the system) per unit charge (just as the field is the force per unit charge).

The Energy of Point Charge Distribution

When the first charge q_1 is placed, no work has been done.

When q_2 is placed work done $W_2=q_2V_1$ where V_1 is the potential due to q_1 so,



$$W_2=\frac{1}{4\pi\varepsilon_0}q_2\left(\frac{q_1}{R_{12}}\right).$$

Similarly when third charge q_3 is placed

$$W_3=\frac{1}{4\pi\varepsilon_0}q_3\left(\frac{q_1}{R_{13}}+\frac{q_2}{R_{23}}\right)$$

The work necessary to assemble the first three charges is $W=\dfrac{1}{4\pi\varepsilon_0}\left(\dfrac{q_1q_2}{R_{12}}+\dfrac{q_1q_3}{R_{13}}+\dfrac{q_2q_3}{R_{23}}\right)$

In general, $W=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{i=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\frac{q_iq_j}{R_{ij}}=\frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_iq_j}{R_{ij}}=\frac{1}{2}\sum_{i=1}^{n}q_iV(r_i)$,

where $V(r_i)=\dfrac{1}{4\pi\varepsilon_0}\displaystyle\sum_{j=1}^{n}\frac{q_j}{R_{ij}}$ is the potential at point r_i (the position of q_i) due to all other charges.

Energy of Continuous Charge Distribution

In general for discrete charges, $W=\dfrac{1}{2}\displaystyle\sum_{i=1}^{n}q_iV(r_i)$,

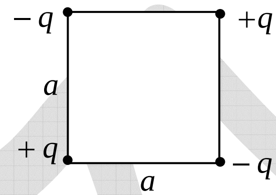
For a volume charge density $W=\dfrac{1}{2}\displaystyle\int \rho V d\tau$, Since $\rho = \varepsilon_0 \left(\vec{\nabla}.\vec{E}\right) \Rightarrow W = \frac{\varepsilon_0}{2}\int \left(\vec{\nabla}.\vec{E}\right) V d\tau \Rightarrow W = \frac{\varepsilon_0}{2}\left[-\int_V \vec{E}.\left(\vec{\nabla}V\right)d\tau + \int_V \vec{\nabla}.\left(V\vec{E}\right)d\tau\right]$

$$\Rightarrow W = \frac{\varepsilon_0}{2}\left[\int_V E^2 d\tau + \oint_S V\vec{E}.d\vec{a}\right] \qquad \text{Since } \vec{E} = -\vec{\nabla}V$$

The above equation gives the correct energy W, whatever volume we use as long as it encloses all the charges, but the contribution from the volume integral goes up, and that of the surface integral goes down, as we take larger and larger volumes. In particular, if we integrate over all space, then the surface integral goes to zero, and we have

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau$$

Example: Four charges are situated at the corners of a square (side a) as shown in figure. How much work does it take to assemble the whole configuration of four charges?



Solution: Work done in placing first charge ($-q$ charge upper left corner) $W_1 = 0$

Work done in placing second charge ($+q$ charge lower left corner) $W_2 = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{a}\right)$

Work done in placing third charge ($-q$ charge lower right corner)

$$W_3 = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a}\right)$$

Potential at fourth corner ($+q$ charge upper right corner)

$$V = \frac{1}{4\pi\varepsilon_0}\sum\frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{a} + \frac{q}{\sqrt{2}a} - \frac{q}{a}\right) = \frac{q}{4\pi\varepsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow W_4 = qV = \frac{q^2}{4\pi\varepsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$$

Total work done $= W_1 + W_2 + W_3 + W_4 = \frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a}\left(-2 + \frac{1}{\sqrt{2}}\right) = \frac{q^2}{2\pi\varepsilon_0 a}\left(-2 + \frac{1}{\sqrt{2}}\right)$

Example: Energy of a uniformly charged spherical shell of total charge q and radius R.

Solution: $\vec{E}_{inside} = 0, \qquad \vec{E}_{outside} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau = \frac{\varepsilon_0}{2}\int_0^R E_{in}^2 d\tau + \frac{\varepsilon_0}{2}\int_R^\infty E_{out}^2 d\tau = \frac{\varepsilon_0}{2(4\pi\varepsilon_0)^2}\int_{outside}\left(\frac{q^2}{r^4}\right)(r^2 \sin\theta dr d\theta d\phi)$$

$$W = \frac{1}{32\pi^2\varepsilon_0} q^2 4\pi \int_R^\infty \frac{1}{r^2} dr$$

$$\Rightarrow W = \frac{q^2}{8\pi\varepsilon_0 R}.$$