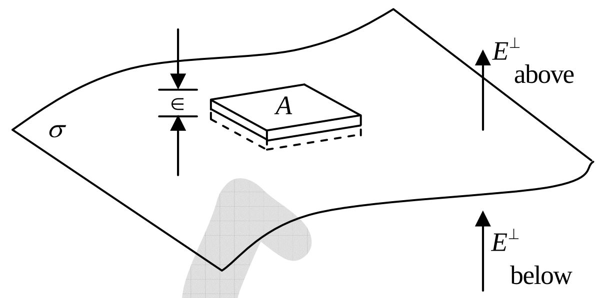
(k) Electrostatic Boundary Condition

The boundary between two medium is a thin sheet of surface charge σ. Consider a thin Gaussian pillbox, extending equally above and below the sheet as shown in figure below:

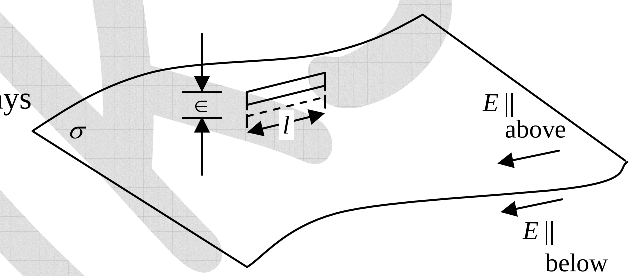


The Gauss's law states that $\oint_S \vec{E}.d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$.

$$\Rightarrow E^{\perp}_{above}A - E^{\perp}_{below}A = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}.$$

The normal component of $\vec{E}$ is discontinuous by an amount $\frac{\sigma}{\varepsilon_0}$ at any boundary. If there is no surface charge, $E^{\perp}$ is continuous.



The tangential component of $\vec{E}$ is always continuous.

Apply $\oint \vec{E}.d\vec{l} = 0$ to the thin rectangular loop,

$$E^{\|}_{above}l - E^{\|}_{below}l = 0 \Rightarrow E^{\|}_{above} = E^{\|}_{below}$$

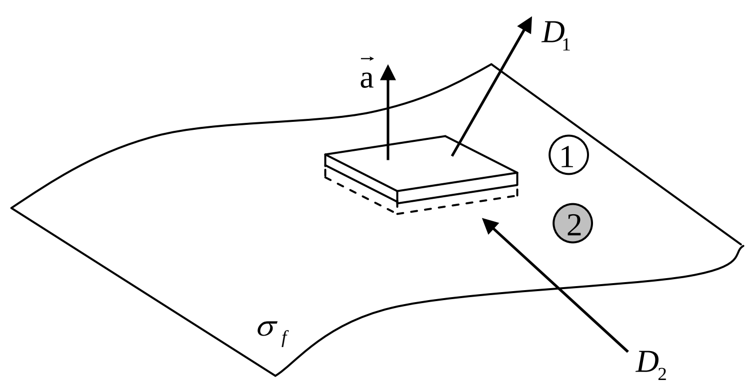
where $\vec{E}^{\|}$ stands for the components of $\vec{E}$ parallel to the surface.

The boundary conditions on $\vec{E}$ can be combined into single formula:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

where $\hat{n}$ is unit vector perpendicular to the surface, pointing upward.

The boundary between two medium is a thin sheet of free surface charge σ_f.



The Gauss's law states that

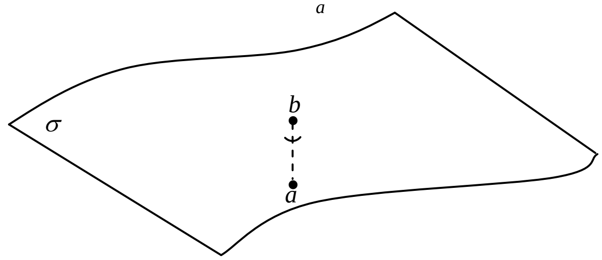
$$\oint_S \vec{D}.d\vec{a} = Q_{free}$$

$$\Rightarrow D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

Since $\vec{D} = \varepsilon_0\vec{E} + \vec{P} \Rightarrow \vec{\nabla}\times\vec{D} = \vec{\nabla}\times\vec{P}$

$$\Rightarrow \vec{D}^{\|}_{above} - \vec{D}^{\|}_{below} = \vec{P}^{\|}_{above} - \vec{P}^{\|}_{below} \quad \left(\because \vec{\nabla}\times\vec{E} = 0\right)$$

The potential is continuous across any boundary, since $V_{above} - V_{below} = -\int_a^b \vec{E}.d\vec{l}$; as the path shrinks to zero.



$\Rightarrow V_{above} = V_{below}$.

Since $\vec{E} = -\vec{\nabla}V \Rightarrow \vec{\nabla}V_{above} - \vec{\nabla}V_{below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$,

$$\Rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

where $\frac{\partial V}{\partial n} = \vec{\nabla}V \cdot \hat{n}$ denotes the normal derivative of V (that is the rate of change in the direction perpendicular to the surface.)

Because the field inside a conductor is zero, boundary condition $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ requires that the field immediately outside is

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}.$$

Force per unit area on the conductor is $\vec{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{n}$.

This amounts to an outwards *electrostatic pressure* on the surface, tending to draw the conductor into the field, regardless the sign of σ. Expressing the pressure in terms of the field just outside the surface,

$$P = \frac{\varepsilon_0}{2}E^2.$$

In terms of potential equation $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$ yields

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}.$$

These equations enable us to calculate the surface charge on a conductor, if we can determine $\vec{E}$ or V.