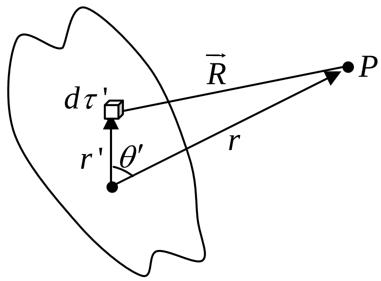
(l) Multipole Expansions (Approximate Potential at Large Distances)

Approximate potential at large distances due to arbitrary localized charge distribution is given by



$$V(r)=\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int\rho(r')d\tau'+\frac{1}{r^2}\int r'\cos\theta'\rho(r')d\tau'+\frac{1}{r^3}\int(r')^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)\rho(r')d\tau'+...\right]$$

The first term $(n=0)$ is the monopole contribution (it goes like $\frac{1}{r}$).

The second term $(n=1)$ is the dipole term (it goes like $\frac{1}{r^2}$).

The third term is quadrupole; the fourth octopole and so on.

The lowest nonzero term in the expansion provides the approximate potential at large r and the successive terms tell us how to improve the approximation if greater precision is required.

The Monopole and Dipole Terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{mon}(r)=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}.$$

where $Q=\int\rho d\tau$ is the total charge of the configuration.

If the total charge is zero, the dominant term in the potential will be the dipole (unless, of course, it also vanishes):

$$V_{dip}(r)=\frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\cos\theta'\rho(r')d\tau'=\frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}.\int\vec{r'}\rho(r')d\tau'=\frac{1}{4\pi\varepsilon_0}\frac{\hat{r}.\vec{p}}{r^2},$$

where ***dipole moment*** $\vec{p}=\int\vec{r'}\rho(r')d\tau'$

For surface and line charges ***dipole moment***

$$\vec{p}=\int\vec{r'}\sigma(r')da' \quad\text{and}\quad \vec{p}=\int\vec{r'}\lambda(r')dl'.$$

The dipole moment is determined by the geometry (size, shape and density) of the charge distribute.

The dipole moment of a collection of point charge is

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r_i'}$$

where $\vec{r_i'} \equiv$ position vector of source charge at i^{th} location.

Note: Ordinarily, the dipole moment does change when we shift the origin, but there is an important exception: If the total charge is zero, then the dipole moment is independent of the choice of origin.

(a) <u>Monopole term</u>: $Q = \int \rho d\tau = kR \int \left[\frac{1}{r^2}(R-2r)\sin\theta \right] r^2 \sin\theta dr\, d\theta\, d\phi = 0$

Since the r integral is $\int_0^R (R-2r)dr = 0$.

<u>Dipole term</u>:

$$\vec{p} = \int \vec{r'} \rho\left(r'\right) d\tau' = \int r\cos\theta\, \rho d\tau = KR \int (r\cos\theta) \left[\frac{1}{r^2}(R-2r)\sin\theta \right] r^2 \sin\theta dr\, d\theta\, d\phi = 0,$$

Since the integral is $\int_0^\pi \sin^2\theta \cos\theta d\theta = \left.\frac{\sin^3\theta}{3}\right|_0^\pi = 0$.

(b) <u>Quadrupole term</u>:

$$\int r^2 \left(\frac{3}{2}\cos^2\theta - \frac{1}{2} \right) \rho d\tau = \frac{1}{2} kR \int r^2 \left(3\cos^2\theta - 1 \right) \left[\frac{1}{r^2}\left(R-2r\right)\sin\theta \right] r^2 \sin\theta dr d\theta$$