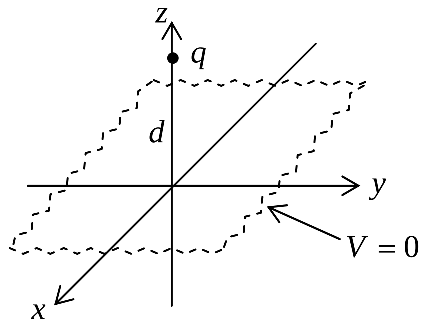
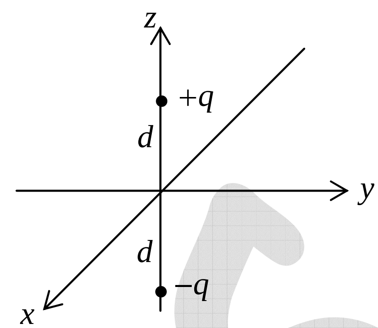
(m) The Classic Image Problem

Suppose a point charge q is held a distance d above an infinite grounded conducting plane. We can find out what is the potential in the region above the plane.





Forget about the actual problem; we are going to study a *complete different situation*.

The new problem consists of two point charges $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$ and no conducting plane. For this configuration we can easily write down the potential:

$$V(x,y,z)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

(The denominators represent the distances from (x,y,z) to the charges $+q$ and $-q$, respectively.) It follows that

1. $V=0$, when $z=0$ and
2. $V\to 0$ for $x^2+y^2+z^2>>d^2$,

and the only charge in the region $z>0$ is the point charge $+q$ at $(0,0,d)$. Thus the second configuration produces exactly the same potential as the first configuration, in the upper region $z\geq 0$.

Induced Surface Charge

The surface charge density σ induced on the conductor surface can be calculated by

$$\sigma=-\varepsilon_0\frac{\partial V}{\partial n},$$

where $\frac{\partial V}{\partial n}$ is the normal derivative of V at the surface. In this case the normal direction is the z-direction, so

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0} \quad \Rightarrow \frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{-q(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} + \frac{q(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$\Rightarrow \sigma(x,y) = \frac{-qd}{2\pi\left(x^2+y^2+d^2\right)^{3/2}}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

The total induced charge $Q = \int \sigma da$.

This integral, over the xy-plane, could be done in Cartesian coordinates, with $da = dx\,dy$, but its easier to use polar coordinates (r,ϕ), with $r^2 = x^2 + y^2$ and $da = rdrd\phi$.

Then

$$\sigma(R) = \frac{-qd}{2\pi\left(r^2+d^2\right)^{3/2}}$$

and $Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi\left(r^2+d^2\right)^{3/2}} rdrd\phi = \left.\frac{qd}{\sqrt{r^2+d^2}}\right|_0^{\infty} = -q$

Force and Energy

The charge q is attracted towards the plane, because of the negative induced surface charge. The force: $\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$.

One can determine the energy by calculating the work required to bring q in from infinity.

$$W = \int_{\infty}^{d} \vec{F}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}dz = \frac{1}{4\pi\varepsilon_0}\left.\left(-\frac{q^2}{4z}\right)\right|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$