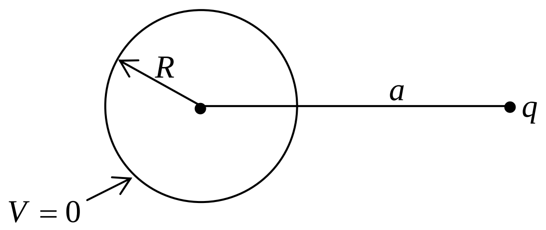
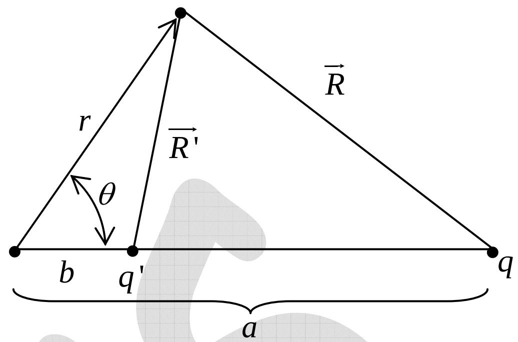
(n) Other Image Problem

The method just described is not limited to a single point charge; any stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image.





Let us examine the completely different configuration, consisting of the point charge q together with another point charge $q' = -\frac{R}{a}q$.

Placed at a distance $b = \frac{R^2}{a}$

to the right of the centre of sphere. No conductor, now-just two point charges. The potential of this configuration is $V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R} + \frac{q'}{R'}\right)$

where $R = \sqrt{r^2 + a^2 - 2ra\cos\theta}$, $R' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$

$$\Rightarrow V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2rb\cos\theta}}\right\}$$

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right\}$$

Clearly when $r = R, V \to 0$

Induced charge

$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$. In this case $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ at the point $r = R$.

$$\Rightarrow \sigma(\theta) = -\varepsilon_0 \left.\frac{\partial V(r,\theta)}{\partial r}\right|_{r=R}$$

$$\sigma(\theta)=-\frac{q}{4\pi}\left\{-(R^2+a^2-2Ra\cos\theta)^{-3/2}(R-a\cos\theta)+(R^2+a^2-2Ra\cos\theta)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\right\}$$

$$\sigma(\theta)=\frac{q}{4\pi}(R^2+a^2-2Ra\cos\theta)^{-3/2}\left[R-a\cos\theta-\frac{a^2}{R}+a\cos\theta\right]$$

$$\sigma(\theta)=\frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2}$$

$$q_{included}=q'=\int\sigma da=\frac{q}{4\pi R}(R^2-a^2)\int(R^2+a^2-2Ra\cos\theta)^{-3/2}R^2\sin\theta d\theta d\phi$$

$$q'=\frac{q}{4\pi R}(R^2-a^2)2\pi R^2\left[-\frac{1}{Ra}(R^2+a^2-2Ra\cos\theta)^{-1/2}\right]_0^{\pi}$$

$$q'=\frac{q}{2a}(a^2-R^2)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}}-\frac{1}{\sqrt{R^2+a^2-2Ra}}\right]$$

But $a>R$ (else q would be inside), so $\sqrt{R^2+a^2-2Ra}=a-R$

$$q'=\frac{q}{2a}(a^2-R^2)\left[\frac{1}{(a+R)}-\frac{1}{(a-R)}\right]=-\frac{q}{a}R\Rightarrow q'=-\frac{q}{a}R$$

Force

The force on q, due to the sphere, is the same as the force of the image charge q', thus:

$$F=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2}=\frac{1}{4\pi\varepsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{\left(a-\frac{R^2}{a}\right)^2}=-\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}$$

Energy

To bring q in from infinity to a, we do work

$$W=\frac{q^2R}{4\pi\varepsilon_0}\int_{\infty}^{a}\frac{\bar{a}}{(\bar{a}^2-R^2)^2}d\bar{a}=\frac{q^2R}{4\pi\varepsilon_0}\left[-\frac{1}{2}\frac{1}{(\bar{a}^2-R^2)}\right]_{\infty}^{a}=-\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{2(a^2-R^2)}$$

$$W=-\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{2(a^2-R^2)}$$