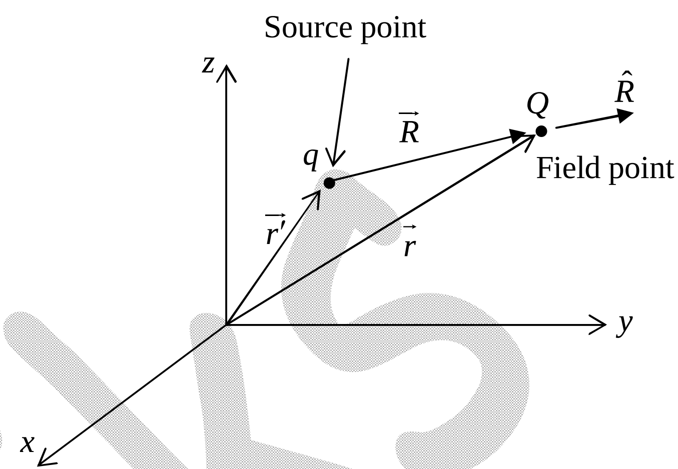
1. Coulomb's Law and Superposition Principle

The electric field at any point due to stationary source charges is called as electrostatic field.

The electric force on a test charge Q due to a single point charge q, which is at rest and a distance R apart is given by Coulomb's law

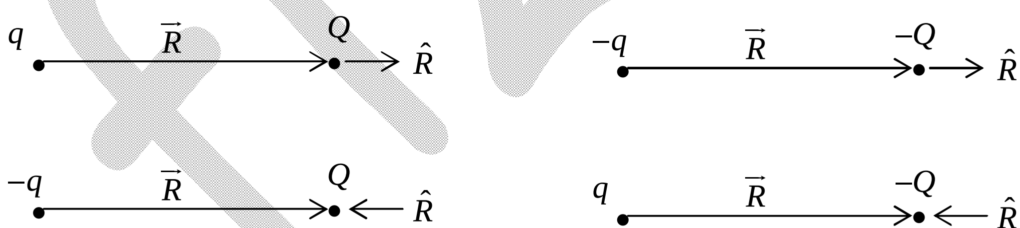


$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R^2}\hat{R}.$$

The constant ε_0 is called the permittivity of free space.

In mks units, $\varepsilon_0 = 8.85\times10^{-12}\dfrac{C^2}{N.m^2}$.

$\vec{R}$ is the separation vector from $\vec{r}'$ (the location of q) to $\vec{r}$ (the location of Q): $\vec{R} = \vec{r} - \vec{r}'$; R is its magnitude, and $\hat{R}$ is its direction. The force points along the line from q to Q; it is repulsive if q and Q have the same sign, and attractive if their signs are opposite.



The electric field is force per unit charge experienced by source charge Q

$$\vec{E} = \frac{\vec{F}}{Q} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{R}$$

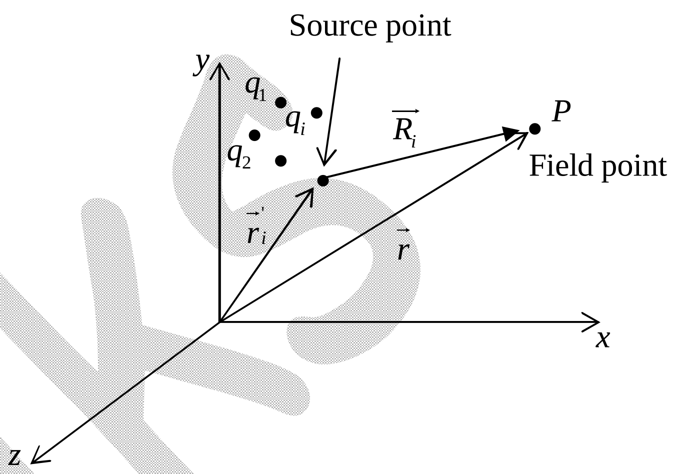
1.1.1 Electric Field

If we have many point charges $q_1, q_2, ...$ at distances $R_1, R_2, R_3......$ from test charge Q, then according to the **principle of superposition** the total force on Q is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + ... = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{R_1^2}\hat{R}_1 + \frac{q_2}{R_2^2}\hat{R}_2 + ...\right)$$

$$\Rightarrow \vec{F} = Q\vec{E}$$

where $\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i^2}\hat{R}_i$



$\vec{E}$ is called the **electric field** of the source charges. Physically $\vec{E}(P)$ is the force per unit charge that would be exerted on a test charge placed at P.

If **charge is distributed continuously** over some region, then

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{line}\frac{1}{R^2}\hat{R}dq$$ (See figure (a) shown below)

The electric field of a line charge is $(dq = \lambda dl')$ (See figure (b) shown below)

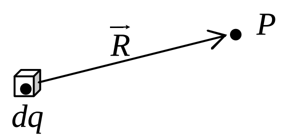
$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{line}\frac{\lambda(r')}{R^2}\hat{R}dl'$$ where λ is charge per unit length.

For surface charge $(dq = \sigma da')$ (See figure (c) shown below)

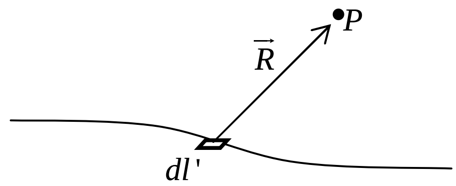
$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{surface}\frac{\sigma(r')}{R^2}\hat{R}da'$$ where σ is charge per unit area.

For a volume charge $(dq = \rho d\tau')$ (See figure (d) shown below)

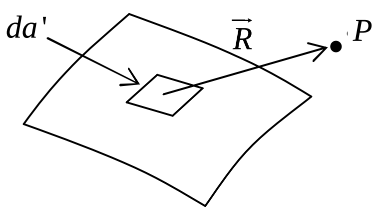
$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\rho(r')}{R^2}\hat{R}d\tau'$$ where ρ is charge per unit volume.



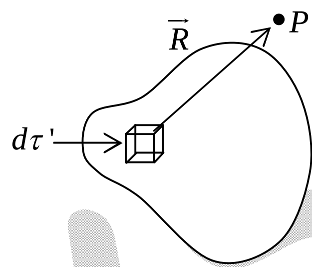
(a) Continuous distribution



(b) Line charge, λ



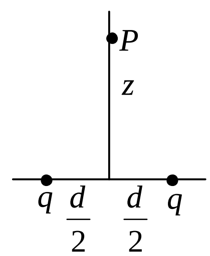
(c) Surface charge, σ



(d) Volume charge, ρ

Example: (a) Find the electric field a distance z above the mid point between two equal charges, q, a distance d apart.

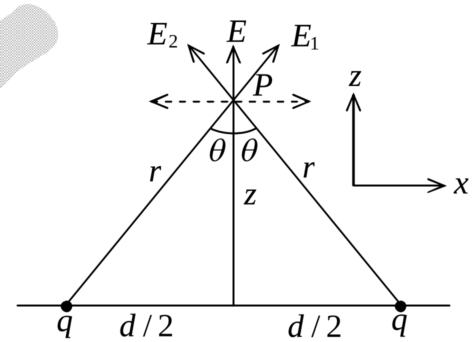
(b) Repeat part (a) after replacing right hand charge to $-q$.



Solution: (a) $E_1 = E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$

Horizontal components cancels and

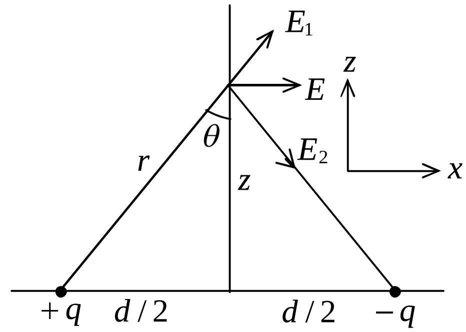
$$E_z = E_1\cos\theta + E_2\cos\theta = 2E_1\cos\theta$$



Since $r^2 = z^2 + \frac{d^2}{4}, \cos\theta = \frac{z}{r}$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left[z^2 + \frac{d^2}{4}\right]^{3/2}}\hat{z}$$

When $z >> d$, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2q}{z^2}$ (looks like a single charge $2q$).



(b) $E_1 = E_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$

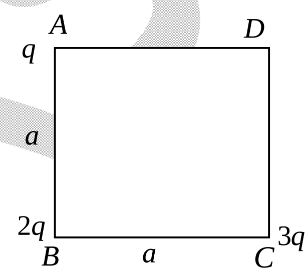
Component along z-direction cancel out.

Thus $\vec{E} = 2E_1 \sin\theta\, \hat{x}$, $\sin\theta = \frac{d}{2r}$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left[z^2 + \frac{d^2}{4}\right]^{3/2}}\hat{x}$$

When $z >> d$, $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{z^3}\hat{x}$ (field of a dipole)

Example: Three charges q, $2q$ and $3q$ are placed at the three corners $(A, B \text{ and } C)$ of a square of side a. Then the electric field at the fourth corner D.



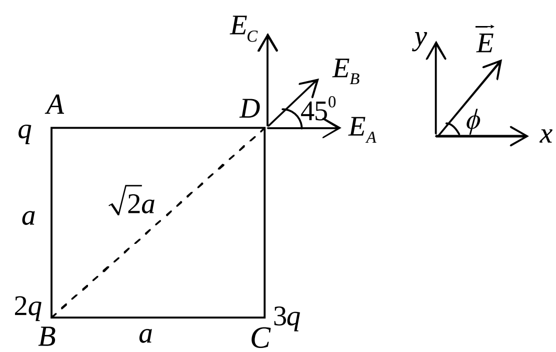
Solution:

Electric field due to charge at A,

$$E_A = \frac{q}{4\pi\varepsilon_0 a^2} \text{ along } AD$$

Electric field due to charge at B,

$$E_B = \frac{2q}{4\pi\varepsilon_0\left(\sqrt{2}a\right)^2} = \frac{q}{4\pi\varepsilon_0 a^2} = E_A \text{ along } BD$$



Electric field due to charge at C,

$$E_C = \frac{3q}{4\pi\varepsilon_0 a^2} \text{ along } CD$$

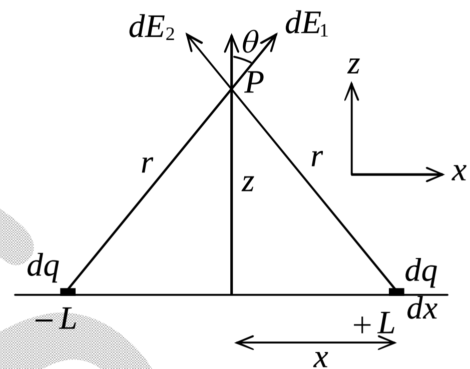
Thus resultant field $\vec{E} = \left(E_A + E_B\cos 45^0\right)\hat{x} + \left(E_C + E_B \sin 45^0\right)\hat{y}$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 a^2}\left(1 + \frac{1}{\sqrt{2}}\right)\hat{x} + \frac{q}{4\pi\varepsilon_0 a^2}\left(3 + \frac{1}{\sqrt{2}}\right)\hat{y} \text{ where } \left|\vec{E}\right| = \sqrt{E_x^2 + E_y^2} \text{ and } \phi = \tan^{-1}\left(\frac{E_y}{E_x}\right)$$

Example: Find the electric field a distance z above the midpoint of a straight line segment of length $2L$, which carries a uniform line charge λ.

Solution: Horizontal components of two field cancels and the field of the two segment is

$$dE_1 = dE_2 = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$



Net field is $d\vec{E} = 2dE_1\cos\theta\hat{z} = 2\frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda dx}{r^2}\right)\cos\theta\hat{z}$

Here, $\cos\theta = \frac{z}{r}, r = \sqrt{z^2 + x^2} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{2\lambda z}{\left[z^2 + x^2\right]^{3/2}}dx$

Thus $E = \frac{2\lambda z}{4\pi\varepsilon_0}\left[\frac{x}{z^2\sqrt{z^2 + x^2}}\right]_0^L \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2 + L^2}}\hat{z}$

For $z >> L$, $E \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z^2}$

and when $L \to \infty$, $E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$