(b) Homogeneous Equation (Reduction to Separable Form)

A differential equation of the form

$$\frac{dy}{dx}=\frac{f(x,y)}{\phi(x,y)}$$

is called a homogeneous equation if each term of $f(x,y)$ and $\phi(x,y)$ is of the same degree.

Example: Find the solution of the differential equation $2xyy'=y^2-x^2$.

Solution: $2xyy'=y^2-x^2\Rightarrow\frac{dy}{dx}=\frac{y^2-x^2}{2xy}$

Put $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}\Rightarrow v+x\frac{dv}{dx}=\frac{v^2x^2-x^2}{2vx^2}=\frac{v^2-1}{2v}$

$$\Rightarrow x\frac{dv}{dx}=\frac{v^2-1}{2v}-v=\frac{-v^2-1}{2v}\Rightarrow\left(\frac{2v}{v^2+1}\right)dv=-\frac{1}{x}dx$$

$$\Rightarrow\log\left(v^2+1\right)=-\log|x|+c'\Rightarrow 1+v^2=\frac{c}{x}\Rightarrow 1+\frac{y^2}{x^2}=\frac{c}{x}\Rightarrow x^2+y^2=cx$$

Thus $\left(x-\frac{c}{2}\right)^2+y^2=\frac{c^2}{4}$

This general solution represents the family of circles with centers on the x-axis and all passing through origin.

Example: Find the solution of the differential equation $\left(2xy+x^2\right)\frac{dy}{dx}=3y^2+2xy$.

Solution: $\left(2xy+x^2\right)\frac{dy}{dx}=3y^2+2xy\Rightarrow\frac{dy}{dx}=\frac{3y^2+2xy}{2xy+x^2}$

Put $y=vx\Rightarrow\frac{dy}{dx}=v+x\frac{dv}{dx}\Rightarrow v+x\frac{dv}{dx}=\frac{3v^2x^2+2vx^2}{2vx^2+x^2}=\frac{3v^2+2v}{2v+1}$

$$\Rightarrow x\frac{dv}{dx}=\frac{3v^2+2v}{2v+1}-v=\frac{v^2+v}{2v+1}\Rightarrow\left(\frac{2v+1}{v^2+v}\right)dv=\frac{1}{x}dx\Rightarrow\log\left(v^2+v\right)=\log x+\log c$$

$$\Rightarrow v^2+v=cx\Rightarrow y^2+xy=cx^3$$

Equations Reducible to Homogeneous Form

The equations of the form $\frac{dy}{dx}=\frac{ax+by+c}{Ax+By+C}$ can be reduced to homogeneous form by the substitutions $x=X+h, y=Y+k$ (h,k being constants)

$$\Rightarrow \frac{dy}{dx}=\frac{dY}{dX}=\frac{aX+bY+ah+bk+c}{AX+BY+Ah+Bk+C}$$

Choose h,k so that $\left.\begin{matrix} ah+bk+c=0 \\ Ah+Bk+C=0 \end{matrix}\right\} \Rightarrow \frac{dy}{dx}=\frac{aX+bY}{AX+BY}$

Case of failure: $\frac{a}{A}=\frac{b}{B}=\frac{1}{m} \Rightarrow \frac{dy}{dx}=\frac{ax+by+c}{m(ax+by)+C}$

Now, put $ax+by=z$ and apply the method of separation of variables.

Example: Solve $\frac{dy}{dx}=\frac{x+2y-3}{2x+y-3}$

Solution: Put $x=X+h, y=Y+k$ (h,k being constants)

The given equation reduces to $\frac{dY}{dX}=\frac{(X+h)+2(Y+k)-3}{2(X+h)+(Y+k)-3}=\frac{X+2Y+(h+2k-3)}{2X+Y+(2h+k-3)}$

Now choose h,k so that $\left.\begin{matrix} h+2k-3=0 \\ 2h+k-3=0 \end{matrix}\right\} \Rightarrow h=k=1.$

$$\frac{dY}{dX}=\frac{X+2Y}{2X+Y}$$

Put $Y=vX$ so that $\frac{dY}{dX}=v+X\frac{dv}{dX} \Rightarrow v+X\frac{dv}{dX}=\frac{X+2vX}{2X+vX}=\frac{1+2v}{2+v}$

$$\Rightarrow X\frac{dv}{dX}=\frac{1+2v}{2+v}-v=\frac{1-v^2}{2+v} \Rightarrow \left(\frac{2+v}{1-v^2}\right)dv=\frac{dX}{X}$$

$$\Rightarrow \frac{1}{2}\frac{1}{(1+v)}dv+\frac{3}{2}\frac{1}{1-v}dv=\frac{dX}{X}$$

On integrating we have

$$\frac{1}{2}\log(1+v)-3/2\log(1-v)=\log X+\log c \Rightarrow \log\frac{1+v}{(1-v)^3}=\log c^2X^2 \Rightarrow \frac{1+v}{(1-v)^3}=c^2X^2$$

$$\frac{1+\frac{Y}{X}}{\left(1-\frac{Y}{X}\right)^3}=c^2X^2 \Rightarrow \frac{X+Y}{(X-Y)^3}=c^2 \Rightarrow X+Y=c^2(X-Y)^3$$

Put $X=x-1, Y=y-1 \Rightarrow x+y-2=a(x-y)^3$.

Example: Solve $\frac{dy}{dx}=\frac{x+2y-1}{x+2y+1}$

Solution: Put $x+2y=z \Rightarrow 1+2\frac{dy}{dx}=\frac{dz}{dx} \Rightarrow \frac{1}{2}\frac{dz}{dx}-\frac{1}{2}=\frac{z-1}{z+1} \Rightarrow \frac{z+1}{3z-1}dz=dx$

$$\Rightarrow \left(\frac{1}{3}+\frac{4}{3}\frac{1}{3z-1}\right)dz=dx \Rightarrow \frac{z}{3}+\frac{4}{9}\log(3z-1)=x+c \Rightarrow 3z+4\log(3z-1)=9x+9c$$

$$\Rightarrow 3(x+2y)+4\log(3x+6y-1)=9x+9c \Rightarrow 3x-3y+a=2\log(3x+6y-1)$$