(c) Exact Differential Equation

If a function $u(x, y)$ has continuous partial derivatives, its differential is

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy .$$

From this, it follows that if $u(x, y) = c = \text{constant}$, then $du = 0$.

For example, if $u = x + x^2y^3 = c$, then

$$du = \left(1+2xy^3\right)dx + 3x^2y^2dy = 0 \Rightarrow y' = \frac{dy}{dx} = -\frac{\left(1+2xy^3\right)}{3x^2y^2}$$

A differential equation that we can solve by going backward.

A first-order differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0 \qquad \text{....(1)}$$

is called an **exact differential equation** if **differentia form** $M(x, y)dx + N(x, y)dy$ is **exact,** that is, this form is the differential

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad \text{....(2)}$$

of some function $u(x, y)$.

Then the differential equation (1) can be written

$$du = 0 .$$

By integrating we obtain general solution of (1) in the form

$$u(x, y) = c \qquad \text{....(3)}$$

Comparing (1) and (2), we see that (1) is an exact differential if there is some function $u(x, y)$ such that

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N \qquad \text{....(4)}$$

Thus, $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This condition is not only necessary but also sufficient for (1) to be an exact differential equation.

$$\because \frac{\partial u}{\partial x} = M \quad \Rightarrow u = \int M dx + k(y)$$

In this integration, y is to be regarded as a constant, and $k(y)$ plays the role of a "constant" of integration. To determine $k(y)$, use $\frac{\partial u}{\partial y} = N$ and find $\frac{dk}{dy}$, then integrate it to get k.

Similarly, $\because \frac{\partial u}{\partial y} = N \quad \Rightarrow u = \int N dy + l(x)$

In this integration, x is to be regarded as a constant, and $l(x)$ plays the role of a "constant" of integration. To determine $l(x)$, use $\frac{\partial u}{\partial x} = M$ and find $\frac{dl}{dx}$, then integrate it to get k.

Equations Reducible to the Exact Form

Consider the equation

$$-ydx + xdy = 0.$$

$$\Rightarrow M = -y, \quad N = x \Rightarrow \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1.$$

Hence equation is not exact. But if we multiply it by $\frac{1}{x^2}$, we get an exact equation,

$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0.$$

$$\Rightarrow M = -\frac{y}{x^2}, \quad N = \frac{1}{x} \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -\frac{1}{x^2}.$$

All we have done was the multiplication of a given non-exact equation, say

$$P(x, y)dx + Q(x, y)dy = 0$$

by a function $F(x, y)$. The result was an equation

$$FPdx + FQdy = 0$$

That is exact. The function $F = F(x, y)$ is then called an integrating factor of

$$P(x, y)dx + Q(x, y)dy = 0.$$

How to Find Integrating Factors

For $FPdx + FQdy = 0$ to be exact

$$\frac{\partial}{\partial y}(FP) = \frac{\partial}{\partial x}(FQ) \Rightarrow F_y P + FP_y = F_x Q + FQ_x$$

In the general case, this would be complicated and useless.

(a) For simplification let $F = F(x)$, $F_y = 0, \quad F_x = F' = dF/dx$

$$\Rightarrow FP_y = F'Q + FQ_x \Rightarrow F'Q = FP_y - FQ_x \Rightarrow \frac{1}{F}\frac{dF}{dx} = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = R(x)$$

Integrating Factor $F(x) = e^{\int R(x)dx}$.

(b) If $F = F(y)$, $F_x = 0, \quad F_y = F' = dF/dy$

$$\Rightarrow F_y P + FP_y = F_x Q + FQ_x \Rightarrow F'P + FP_y = FQ_x \Rightarrow F'P = FQ_x - FP_y$$

$$\Rightarrow \frac{1}{F}\frac{dF}{dy} = \frac{1}{P}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = \tilde{R}(y)$$

Integrating Factor $F(y) = e^{\int \tilde{R}(y)dy}$.