(d) Linear Differential Equations

A first-order differential equation is said to be linear if it can be written as

$$\frac{dy}{dx}+p(x)y=r(x)$$

where p and r are function of *x* (but not *y*) or constant.

If $r(x)=0$, the equation is said to be **homogeneous**; i.e.

$$\frac{dy}{dx}+p(x)y=0$$

$$\Rightarrow \frac{dy}{y}=-p(x)dx \Rightarrow \ln|y|=-\int p(x)dx+c' \Rightarrow y=ce^{\int p(x)dx}$$

If $r(x)\neq 0$, the equation is said to be **non-homogeneous**; i.e.

$$\frac{dy}{dx}+p(x)y=r(x) \Rightarrow (py-r)dx+dy=0$$

Compare with $Pdx+Qdy=0$, thus $P=(py-r),\quad Q=1$

$$\Rightarrow \frac{1}{F}\frac{dF}{dx}=\frac{1}{Q}\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=p(x)$$

Integrating Factor $F(x)=e^{\int pdx}$.

Multiplying $\frac{dy}{dx}+p(x)y=r(x)$ by $F(x)=e^{\int pdx}$

$$e^{\int pdx}(y'+py)=\left(e^{\int pdx}y\right)'=e^{\int pdx}r$$

$$\Rightarrow e^{\int pdx}y=\int e^{\int pdx}rdx+c \text{ or } I.F\times y=\int (I.F)rdx+c$$

Equation Reducible to Linear Form

A differential equation of the form $\frac{dy}{dx}+py=qy^n$

where p and q are function of *x* (but not *y*) or constant can be reduced to the linear form

on dividing by y^n and substituting $\frac{1}{y^{n-1}}=z$. $\Rightarrow \frac{1}{y^n}\frac{dy}{dx}+p\frac{1}{y^{n-1}}=q$

$\because \frac{1}{y^{n-1}} = z \Rightarrow \frac{(1-n)}{y^n}\frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{1}{1-n}\frac{dz}{dx} + pz = q$

$\Rightarrow \frac{dz}{dx} + p(1-n)z = q(1-n)$ which is a linear differential equation.