(d) Inverse of a Matrix

The inverse of a $n\times n$ matrix $A=\left[a_{ij}\right]$ is denoted by A^{-1} and is an $n\times n$ matrix such that

$$AA^{-1}=A^{-1}A=I$$

where I is the $n\times n$ unit matrix.

Note:

(i) If A has inverse, then A is called a **nonsingular matrix.**

(ii) If A has no inverse, then A is called a **singular matrix.**

(iii) If A has an inverse, the inverse is unique.

The inverse of a nonsingular $n\times n$ matrix $A=\left[a_{ij}\right]$ is given by

$$A^{-1}=\frac{1}{\det A}\left[A_{ij}\right]^T=\frac{1}{\det A}\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

where A_{ij} is the cofactor of a_{ij} in $\det A$. Note well that in A^{-1}, the cofactor A_{ij} occupies the same place as a_{ji} does in A.

Cofactor of a_{ij}

A determinant of order n is a scalar associated with an $n\times n$ matrix $A=\left[a_{ij}\right]$, which is written

$$D=\det A=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

and is defined for $n=1$ by $D=a_{11}$

and is defined for $n\geq 2$ by

$$D=a_{i1}C_{i1}+a_{i2}C_{i2}+....+a_{in}C_{in}\quad\left(i=1,2,........,\text{or } n\right)$$

where $C_{ij}=\left(-1\right)^{i+j}M_{ij}$ and M_{ij} is a determinant of order $\left(n-1\right)$, namely, the determinant of the submatrix of A obtained from A by deleting the row and column of the entry a_{ij}.

M_{ij} is called the **minor** of a_{ij} in D and C_{ij} is the **cofactor** of a_{ij} in D.

Example: Let $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then cofactor of A is $\left[A_{ij}\right]=\begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$.

$\Rightarrow \left[A_{ij}\right]^T=\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.

Hence, $A^{-1}=\dfrac{1}{\det A}\left[A_{ij}\right]^T=\dfrac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

Example: Let $A=\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, then cofactor of A is $\left[A_{ij}\right]=\begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}$.

$\Rightarrow \left[A_{ij}\right]^T=\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$.

Hence, $A^{-1}=\dfrac{1}{\det A}\left[A_{ij}\right]^T=\dfrac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}=\begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$