(e) Matrix Eigen value Problems

Let $A=\left[a_{ij}\right]$ be a given $n\times n$ square matrix and consider the equation

$$AX=\lambda X \qquad \text{....(1)}$$

Here X is an unknown vector and λ an unknown scalar and we want to determine both.

A value of λ for which (1) has a solution $X\neq 0$ is called **eigenvalue** of the matrix A. The corresponding solutions $X\neq 0$ of (1) are called **eigenvevtors** of A corresponding to that eigenvalue λ.

In matrix notation, $$\left(A-\lambda I\right)X=0 \qquad \text{....(2)}$$

This homogeneous linear system of equations has a nontrivial solution if and only if the corresponding determinant of the coefficients is zero

$$D(\lambda)=\det\left(A-\lambda I\right)=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}=0 \qquad \text{....(3)}$$

$D(\lambda)$ is called the characteristic determinant. The equation is called the characteristic equation of the matrix A. By developing $D(\lambda)$, we obtain a polynomial of n^{th} degree in λ. This is called the **characteristic polynomial** of A.

Note:

- The eigenvalues of a square matrix A are the roots of the characteristic equation (3) of A. Hence an $n\times n$ matrix has at least one eigenvalue and at most n numerically different eigenvalues.
- Once the eigenvalues are known, corresponding eigenvectors are obtained.
- Repeated eigenvalues are said to be degenerate eigenvalues. For degenerate eigenvalues there are different eigenvectors for same eigenvalues.
- Non repeated eigenvalues are non-degenerate eigenvalues. For non-degenerate eigenvalues there are different eigenvectors for different eigenvalues.
- Sum of eigenvalues are equal to trace of matrix $\sum_i \lambda_i = trace(A)=\sum_i^n a_{ii}$. Trace of matrix is sum of diagonal element.

- Product of eigenvalues are equal to determinant of matrix $\prod_{i}^{n} \lambda_i = |A|$
- Eigenvectors correspond to different eigenvalues are always independent.
- Eigenvectors corresponds to same eigenvalue may or may not be independent.

Orthonormality Condition

$$X_i^T . X_j = \delta_{ij}$$

If $i = j$, then $\delta_{ij} = 1$ (Normalisation Condition)

and if $i \neq j$, then $\delta_{ij} = 0$ (Orthogonal Condition)

Linear independence and dimensionality of a vector space

A set of N vectors $X_1, X_2, ..., X_N$ is said to be linearly independent if $\sum_{i=1}^{N} a_i X_i = 0$ is satisfied when $a_1 = a_2 = a_3 = a_4 = ... = 0$ otherwise it is said to be linear dependent.

The dimension of a space vector is given by the maximum number of linearly independent vectors the space can have.

The maximum number of linearly independent vectors a space has is $N\left(X_1, X_2, ..., X_N\right)$. This space is said to be N dimensional. In this case any vector Y of the vector space can be expressed as linear combination $Y = \sum_{i=1}^{N} a_i X_i$.

The Cayley-Hamilton Theorem

This theorem provides an alternative method for finding the inverse of a matrix A. Also any positive integral power of A can be expressed, using this theorem, as a linear combination of those of lower degree.

Every square matrix satisfied its own characteristic equation. That means that, if

$$a_0 \lambda^n + a_1 \lambda^{n-1} + ... + a_{n-1} \lambda + a_n = 0$$

is the characteristic equation of a square matrix A of order n, then

$$a_0 A^n + a_1 A^{n-1} + ... + a_{n-1} A + a_n I = 0$$

Note: When λ is replaced by A in the characteristic equation, then constant term a_n should be replaced by $a_n I$ to get the result of Cayley-Hamilton theorem, where I is the unit matrix of order n. Also 0 in the R.H.S is a null matrix of order n.